Retrial queues with collision arising from unslotted *CSMA/CD* protocol*

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We consider a retrial queueing model with collision arising from the specific communication protocol *CSMA/CD*. Under the retrial control policy in which the retrial rate is inversely proportional to the number of customers in the retrial group, we derive the generating function of the limiting distribution of the number of customers in the retrial group at the moment when the channel is free. Using the theory of Markov regenerative processes, we also obtain the limiting distribution of the number of customers in the system at arbitrary time points.

Keywords: Retrial queues with collision, *CSMA/CD* protocol, retrial control policy, embedded Markov chains, Markov regenerative processes.

1. Introduction and model description

We consider a single channel queueing system with customer conducting retrials. Customers arrive from outside the system according to a Poisson process with rate λ. The service time of each customer consists of two consecutive phases. The first phase of service time is constant time α and the second phase of service time is a random variable S with distribution function $F(\cdot)$ and its LST (Laplace–Stieltjes transform $\tilde{F}(\cdot)$ defined by

$$\tilde{F}(u) = \int_0^\infty e^{-ut} dF(t),$$

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finite mean $\bar{s}$ and finite second moment $\overline{s^2}$. An arriving customer, either from outside the system or from the retrial group, to the service channel first checks the channel state. On arrival to the channel, if the channel is free, the customer immediately occupies it. On arrival to the channel, if the channel is busy with the first phase of a customer's service time, then a collision occurs. On the other hand, if the channel is busy with the second phase of a customer's service time, then a collision does not occur and the arriving customer joins the retrial group. If a collision occurs, then it takes a random time C, called the collision recovery time, for the channel to be free. Let $G(\cdot)$ be the distribution function of C, let $\tilde{G}(\cdot)$ be the LST of $G(\cdot)$, and let $\bar{c}$ be the mean of C and $\overline{c^2}$ the second moment of C. If a collision takes place, then all the customers in the service channel remain in the channel during the collision recovery time and then join the group at the end of collision recovery time. If no one arrives within the first phase of service time of a customer, then the customer in the service channel will be served successfully. In this case, it takes service time $S + \alpha$ for the channel to be free. The access of input stream to the channel is controlled by the retrial control policy described below. If the number of customers in the retrial group is n, then each customer in the retrial group independently attempts to occupy the channel after an exponential amount of time with parameter θ/n.

A major motivation for this model comes from a specific communication protocol *CSMA/CD* (carrier sense multiple access with collision detection). The first phase of service time, the second phase of service time, the collision recovery time, the customers in the retrial group and retrial control policy in the queueing terminology correspond to the propagation delay, transmission time, the collision recovery time, blocked terminals and retransmission control policy in *CSMA/CD* protocol terminology, respectively [11]. For the detailed description of *CSMA/CD* and other protocols in communication networks, see Tobagi and Hunt [12] or Hammond and O'Reilly [7]. It is known that the slotted *ALOHA* [6] and slotted *CSMA/CD* models [11] with infinite terminals are unstable in the absence of the channel control disciplines. For stabilized channels, Fayolle et al. [6] examined two classes of control policies: the retransmission control policy and the threshold control policy, and they gave sufficient conditions for the system to be stable under each control policy. Meditch and Lea [11] found a sufficient condition for the slotted nonpersistent *CSMA/CD* to be stable under the retransmission control policy, and they obtained the distribution of the number of blocked terminals for a stabilized channel at successful departure points when the service time is deterministic.

In this paper, we derive a sufficient condition for the system with a retrial control policy to be stable. Next, we obtain the generating function of the limiting distribution of the number of customers in the retrial group at the moment when the channel is free. We also obtain the limiting distribution of the number of customers in the system at arbitrary time. We follow the same technique, using the theory of Markov regenerative processes, as in Kulkarni and Choi [9]. As a related work, Falin [4] analyzed the retrial queueing model with collision, called the queue with double connections, in which, if an arriving customer finds the channel busy, then the channel becomes free immediately and both the arriving customer and the served customer join the retrial group. The analysis for the retrial queue with the same retrial control policy and $\alpha = 0$ was given by Choi and Park [3] and Farahmand [5].

The organization of this paper is as follows. In section 2, we investigate the embedded Markov chain for our model which provides the basis for the main results in the sequel. In section 3, we find the limiting distribution of the number of customers in the system at arbitrary time points.

2. Distribution of system size

Let T_n be the time when the channel is sensed idle for the nth time and $T_0 = 0$. We shall call $(T_{n-1}, T_n]$ the nth cycle. Let $Q(t)$ be the number of customers in the system at time t and $Q_n = Q(T_n + 0)$ $(n \geq 0)$. Since all customers in the system are in the retrial group just after T_n, Q_n represents the number of customers in the retrial group just after T_n. Next, we shall study the limiting distribution of $\{Q_n, n \geq 0\}$. First we note that the cycle $(T_{n-1}, T_n]$, $n \geq 1$ is one of the six different types as described below (fig. 1):

Type 1: An external arrival takes place before a retrial, a collision occurs by another external arrival.

Type 2: An external arrival takes place before a retrial, a collision occurs by a retrial.

Type 3: A retrial takes place before an external arrival, a collision occurs by an external arrival.

Type 4: A retrial takes place before an external arrival, a collision occurs by another retrial.

Type 5: An external arrival takes place before a retrial, service is completed without collision.

Type 6: A retrial takes place before an external arrival, service is completed without collision.

Let τ_b be the time when the first phase of service time ends in the cycles of types 5 and 6 (see fig. 1). Let $A_c(n)$ be the number of external arrivals during the period of a collision recovery time C in the nth cycle of types 1–4 and let $A_s(n)$ be the number of external arrivals during the time interval $(\tau_b, T_n]$, which is the same as the random variable S, in the nth cycle of types 5–6. Then $\{A_c(n), n \geq 1\}$ and $\{A_s(n), n \geq 1\}$ are sequences of i.i.d. random variables whose generic variables are denoted by A_c and A_s, respectively. It is well known that

$$E(z^{A_c}) = \tilde{G}(\lambda - G\lambda z), \quad E(z^{A_s}) = \tilde{F}(\lambda - \lambda z). \tag{2.1}$$

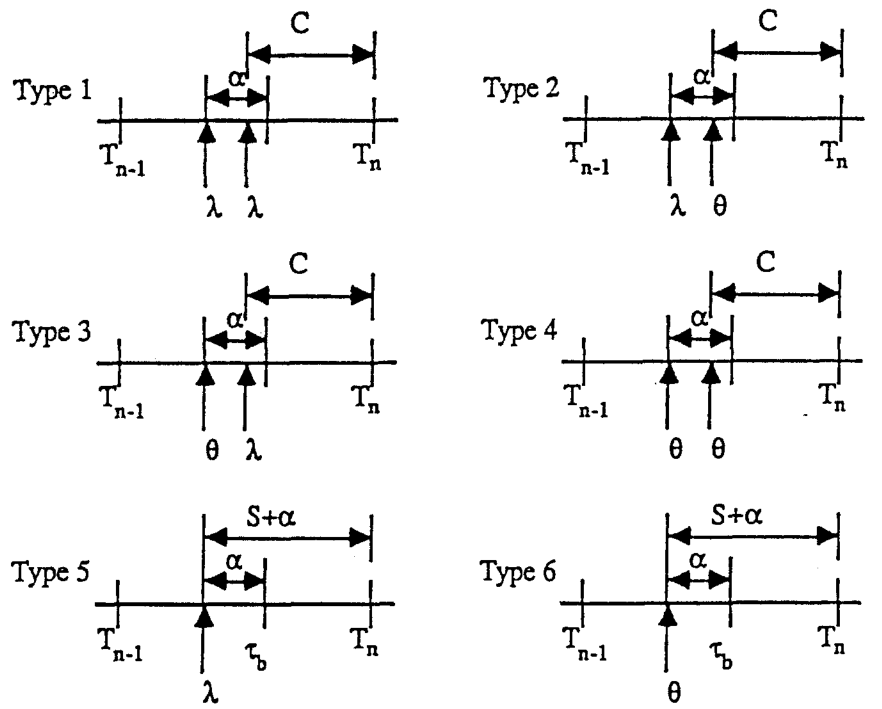

Fig. 1. Types of cycle (vertical arrows in the λ and θ represent external arrival times and retrial times to the channel, respectively).

Define $I_n = k$ if the nth cycle is of type k. From the above description of the model, we obtain the following:

$$Q_{n+1} = \begin{cases} Q_n + 2 + A_c(n+1), & \text{if } I_{n+1} = 1, \\ Q_n + 1 + A_c(n+1), & \text{if } I_{n+1} = 2, \\ Q_n + 1 + A_c(n+1), & \text{if } I_{n+1} = 3, \\ Q_n + A_c(n+1), & \text{if } I_{n+1} = 4, \\ Q_n + A_s(n+1), & \text{if } I_{n+1} = 5, \\ Q_n - 1 + A_s(n+1), & \text{if } I_{n+1} = 6. \end{cases} \tag{2.2}$$

From the independence of $(A_s(n+1), A_c(n+1))$ and $\{Q_k, 0 \le k \le n\}$, we see that $\{Q_n, n \ge 0\}$ is an embedded Markov chain. Clearly, $\{Q_n, n \ge 0\}$ is aperiodic and irreducible. From the definition of type I_{n+1} of the $(n+1)$st cycle, we have

$$P(I_{n+1} = 1 | Q_n = j) = \begin{cases} \dfrac{\lambda}{\lambda+\theta}(1 - e^{-(\lambda+\theta)\alpha})\dfrac{\lambda}{\lambda+\theta}, & \text{if } j \ge 1, \\ 1 - e^{-\lambda\alpha}, & \text{if } j = 0; \end{cases}$$

$$P(I_{n+1}=2\,|\,Q_n=j)=\begin{cases}\dfrac{\lambda}{\lambda+\theta}(1-\mathrm{e}^{-(\lambda+\theta)\alpha})\dfrac{\theta}{\lambda+\theta}, & \text{if } j\ge 1,\\ 0, & \text{if } j=0;\end{cases}$$

$$P(I_{n+1}=3\,|\,Q_n=j)=\begin{cases}\dfrac{\theta}{\lambda+\theta}(1-\mathrm{e}^{-(\lambda+\theta)\alpha})\dfrac{\lambda}{\lambda+\theta}, & \text{if } j\ge 2,\\ \dfrac{\theta}{\lambda+\theta}(1-\mathrm{e}^{-\lambda\alpha}), & \text{if } j=1,\\ 0, & \text{if } j=0;\end{cases}$$

$$P(I_{n+1}=4\,|\,Q_n=j)=\begin{cases}\dfrac{\theta}{\lambda+\theta}(1-\mathrm{e}^{-(\lambda+\theta)\alpha})\dfrac{\theta}{\lambda+\theta}, & \text{if } j\ge 2,\\ 0, & \text{if } j\le 1;\end{cases}$$

$$P(I_{n+1}=5\,|\,Q_n=j)=\begin{cases}\dfrac{\lambda}{\lambda+\theta}\,\mathrm{e}^{-(\lambda+\theta)\alpha}, & \text{if } j\ge 1,\\ \mathrm{e}^{-\lambda\alpha}, & \text{if } j=0;\end{cases}$$

$$P(I_{n+1}=6\,|\,Q_n=j)=\begin{cases}\dfrac{\theta}{\lambda+\theta}\,\mathrm{e}^{-(\lambda+\theta)\alpha}, & \text{if } j\ge 2,\\ \dfrac{\theta}{\lambda+\theta}\,\mathrm{e}^{-\lambda\alpha}, & \text{if } j=1,\\ 0, & \text{if } j=0.\end{cases}\tag{2.3}$$

Now we find a sufficient condition for the system to be stable by applying the following lemma [12].

PAKES' LEMMA

Let $\{X_n, n\ge 0\}$ be an irreducible and aperiodic Markov chain whose state space is the set of nonnegative integers. The following conditions are sufficient for the chain to be ergodic:

(a) $|E(X_{n+1}-X_n\,|\,X_n=i)|<\infty$ for all i;

(b) $\limsup_{i\to\infty} E(X_{n+1}-X_n\,|\,X_n=i)<0$.

Let $\bar{c} = E(C) < \infty$ and $\bar{s} = E(S) < \infty$. We have from (2.2) and (2.3), for $j \geq 2$,

$$\begin{aligned}
&E(Q_{n+1} - Q_n \mid Q_n = j) \\
&= \sum_{k=1}^{6} E(Q_{n+1} - Q_n \mid Q_n = j, I_{n+1} = k) P(I_{n+1} = k \mid Q_n = j) \\
&= \lambda \bar{c}(1 - \mathrm{e}^{-(\lambda+\theta)\alpha}) + \lambda \bar{s}\, \mathrm{e}^{-(\lambda+\theta)\alpha} + \frac{2\lambda}{\lambda+\theta} - \frac{2\lambda+\theta}{\lambda+\theta}\, \mathrm{e}^{-(\lambda+\theta)\alpha}.
\end{aligned}$$

Let $\eta = \lambda + \theta$ and

$$\rho = \frac{\lambda(2 - \mathrm{e}^{-\eta\alpha} + \bar{c}\eta(1 - \mathrm{e}^{-\eta\alpha}) + \bar{s}\eta\, \mathrm{e}^{-\eta\alpha})}{\eta\, \mathrm{e}^{-\eta\alpha}}.$$

Then η is the total access rate to the channel, and for $\{Q_n, n \geq 0\}$ to satisfy the condition (b) in the lemma, we must have

$$\rho < 1. \tag{2.4}$$

Thus, we have:

THEOREM 1

If (2.4) holds, then the embedded Markov chain $\{Q_n, n \geq 0\}$ is ergodic.

Remark

When $\alpha = 0$, our model becomes a retrial queue without collision, and with a retrial control policy in which the retrial rate is proportional to the inverse of the number of customers in the retrial group, condition (2.4) becomes $\lambda s < \theta/(\lambda + \theta)$, which is the necessary and sufficient condition for the system to be stable (see Choi and Park [3], Farahmand [5]).

In the remainder of this paper, we always assume that condition (2.4) holds. Then the limiting distribution $\{\pi_j\}_{j=0}^{\infty}$ of Markov chain $\{Q_n, n \geq 0\}$ exists and all π_j's are positive. Next, we will find the generating function of $\{\pi_j\}$. Using eqs. (2.1)–(2.3) and the formula

$$E(z^{Q_{n+1}} \mid Q_n = j) = \sum_{k=1}^{6} E(z^{Q_{n+1}} \mid Q_n = j, I_{n+1} = k) P(I_{n+1} = k \mid Q_n = j),$$

we obtain that

$$\begin{aligned}
E(z^{Q_{n+1}} \mid Q_n = j) = z^{j-1} \Bigg(& z\tilde{G}(\lambda - \lambda z) \left(\frac{\lambda z + \theta}{\lambda + \theta} \right)^2 (1 - \mathrm{e}^{-(\lambda+\theta)\alpha}) \\
& + \tilde{F}(\lambda - \lambda z) \frac{\lambda z + \theta}{\lambda + \theta}\, \mathrm{e}^{-(\lambda+\theta)\alpha} \Bigg), \qquad \text{for } j \geq 2,
\end{aligned} \tag{2.5}$$

$$E(z^{Q_{n+1}}\,|\,Q_n=1)=z^2\tilde{G}(\lambda-\lambda z)\left(\frac{\lambda(\lambda z+\theta)}{(\lambda+\theta)^2}(1-\mathrm{e}^{-(\lambda+\theta)\alpha})+\frac{\theta}{\lambda+\theta}(1-\mathrm{e}^{-\lambda\alpha})\right)$$
$$+\tilde{F}(\lambda-\lambda z)\,\frac{1}{\lambda+\theta}(\lambda z\,\mathrm{e}^{-(\lambda+\theta)\alpha}+\theta\,\mathrm{e}^{-\lambda\alpha}), \tag{2.6}$$

$$E(z^{Q_{n+1}}\,|\,Q_n=0)=z^2\tilde{G}(\lambda-\lambda z)(1-\mathrm{e}^{-\lambda\alpha})+\tilde{F}(\lambda-\lambda z)\mathrm{e}^{-\lambda\alpha}. \tag{2.7}$$

Let $\phi_n(z)=E(z^{Q_n})$ and $\phi(z)=\sum_{j=0}^{\infty}\pi_j z^j$. Then, under condition (2.4), $\phi_n(z)\to\phi(z)$. From eqs. (2.5), (2.6) and (2.7), we have that

$$\begin{aligned}
\phi_{n+1}(z)&=\sum_{j=0}^{\infty}E(z^{Q_{n+1}}\,|\,Q_n=j)P(Q_n=j)\\
&=\frac{1}{z}(\phi_n(z)-P(Q_n=0)-zP(Q_n=1))\\
&\quad\times\left(z\tilde{G}(\lambda-\lambda z)\left(\frac{\lambda z+\theta}{\lambda+\theta}\right)^2(1-\mathrm{e}^{-(\lambda+\theta)\alpha})+\tilde{F}(\lambda-\lambda z)\,\frac{\lambda z+\theta}{\lambda+\theta}\,\mathrm{e}^{-(\lambda+\theta)\alpha}\right)\\
&\quad+P(Q_n=1)\left(z^2\tilde{G}(\lambda-\lambda z)\left(\frac{\lambda(\lambda z+\theta)}{(\lambda+\theta)^2}(1-\mathrm{e}^{-(\lambda+\theta)\alpha})+\frac{\theta}{\lambda+\theta}(1-\mathrm{e}^{-\lambda\alpha})\right)\right.\\
&\quad\left.+\tilde{F}(\lambda-\alpha z)\,\frac{1}{\lambda+\theta}(\lambda z\,\mathrm{e}^{-(\lambda+\theta)\alpha}+\theta\,\mathrm{e}^{-\lambda\alpha})\right)\\
&\quad+P(Q_n=0)(z^2\tilde{G}(\lambda-\lambda z)(1-\mathrm{e}^{-\lambda\alpha})+\tilde{F}(\lambda-\lambda z)\mathrm{e}^{-\lambda\alpha}).
\end{aligned} \tag{2.8}$$

Letting $n\to\infty$ in (2.8), we obtain, after some algebraic manipulation,

$$\phi(z)=\pi_0A_0(z)+\pi_1A_1(z), \tag{2.9}$$

where $A_0(z)=a_0(z)/d(z)$, $A_1(z)=a_1(z)/d(z)$ and

$$d(z)=z\tilde{G}(\lambda-\lambda z)\left(\frac{\lambda z+\theta}{\lambda+\theta}\right)^2(1-\mathrm{e}^{-(\lambda+\theta)\alpha})+\tilde{F}(\lambda-\lambda z)\frac{\lambda z+\theta}{\lambda+\theta}\mathrm{e}^{-(\lambda+\theta)\alpha}-z,$$

$$a_0(z)=z\tilde{G}(\lambda-\lambda z)\left(\left(\frac{\lambda z+\theta}{\lambda+\theta}\right)^2(1-\mathrm{e}^{-(\lambda+\theta)\alpha})-z^2(1-\mathrm{e}^{-\lambda\alpha})\right)$$
$$+\tilde{F}(\lambda-\lambda z)\left(\frac{\lambda z+\theta}{\lambda+\theta}\,e^{-(\lambda+\theta)\alpha}-z\,\mathrm{e}^{-\lambda\alpha}\right),$$

$$a_1(z) = z^2\tilde{G}(\lambda-\lambda z)\left(\frac{\theta(\lambda z+\theta)}{(\lambda+\theta)^2}(1-e^{-(\lambda+\theta)\alpha}) - \frac{\theta z}{\lambda+\theta}(1-e^{-\lambda\alpha})\right)$$
$$+ z\tilde{F}(\lambda-\lambda z)\frac{\theta}{\lambda+\theta}(e^{-(\lambda+\theta)\alpha}-e^{-\lambda\alpha}).$$

It remains to determine π_0 and π_1. Using the facts $f(1)=1$ and $\phi'(0)=\pi_1$, we have from (2.9) a simultaneous linear equation:

$$1 = A_0(1)\pi_0 + A_1(1)\pi_1,$$
$$\pi_1 = A_0'(0)\pi_0 + A_1'(0)\pi_1.$$

This simultaneous linear equation yields

$$\pi_0 = \frac{1-A_1'(0)}{A_0'(0)A_1(1)+A_0(1)(1-A_1'(0))},$$
$$\pi_1 = \frac{A_0'(0)}{A_0'(0)A_1(1)+A_0(1)(1-A_1'(0))}. \tag{2.10}$$

Substituting π_0 and π_1 into (2.9), we obtain

$$\phi(z) = \frac{(1-A_1'(0))A_0(z)+A_0'(0)A_1(z)}{A_0'(0)A_1(1)+A_0(1)(1-A_1'(0))}. \tag{2.11}$$

Thus, we have the following theorem after a simple calculation:

THEOREM 2

Under condition (2.4), the Markov chain $\{Q_n, n \geq 0\}$ is ergodic and the generating function $\phi(z)$ of the limiting distribution of $\{Q_n, n \geq 0\}$ is given by (2.11), where

$$A_0'(0) = \frac{1-e^{-\lambda\alpha}\tilde{F}(\lambda)}{\frac{\theta}{\lambda+\theta}e^{-(\lambda+\theta)\alpha}\tilde{F}(\lambda)},$$

$$A_1'(0) = 1-e^{\theta\alpha}, \quad A_0(1) = \frac{a_0'(1)}{d'(1)} \quad \text{and} \quad A_1(1) = \frac{a_1'(1)}{d'(1)},$$

$$a_0'(1) = -\lambda(\bar{s}-\bar{c})(e^{-\lambda\alpha}-e^{-(\lambda+\theta)\alpha}) - \frac{2\theta}{\lambda+\theta} - \frac{2\lambda+\theta}{\lambda+\theta}e^{-(\lambda+\theta)\alpha} + 2e^{-\lambda\alpha},$$

$$a_1'(1) = -\lambda(\overline{s}-\overline{c})\,\frac{\theta}{\lambda+\theta}(e^{-\lambda\alpha}-e^{-(\lambda+\theta)\alpha}) + \frac{\lambda\theta}{(\lambda+\theta)^2}(1-e^{-(\lambda+\theta)\alpha})$$
$$+\ \frac{\theta}{\lambda+\theta}(2e^{-\lambda\alpha}-1-e^{-(\lambda+\theta)\alpha}),$$
$$d'(1) = \lambda\overline{c}(1-e^{-(\lambda+\theta)\alpha}) + \lambda\overline{s}\,e^{-(\lambda+\theta)\alpha} + \frac{2\lambda}{\lambda+\theta} - \frac{2\lambda+\theta}{\lambda+\theta}\,e^{-(\lambda+\theta)\alpha}.$$

COROLLARY 3

The mean number $\phi'(1)$ of customers in the system at embedding time is given by

$$\phi'(1) = \frac{d''(1)}{2d'(1)} + \frac{1}{2}\,\frac{A_0'(0)a_1''(1)+(1-A_1'(0))a_0''(1)}{A_0'(0)a_1'(1)+(1-A_1'(0))a_0'(1)},$$

where

$$d''(1) = 2\lambda\overline{c}\left(1+\frac{2\lambda}{\lambda+\theta}\right)(1-e^{-(\lambda+\theta)\alpha}) + \lambda^2\overline{c^2}(1-e^{-(\lambda+\theta)\alpha}) + 2\lambda\overline{s}\,\frac{\lambda}{\lambda+\theta}\,e^{-(\lambda+\theta)\alpha}$$
$$+\ \lambda^2\overline{s^2}e^{-(\lambda+\theta)\alpha} + \left(\frac{4\lambda}{\lambda+\theta} + \frac{2\lambda^2}{(\lambda+\theta)^2}\right)(1-e^{-(\lambda+\theta)\alpha}),$$

$$a_0''(1) = 2\lambda\overline{c}\left(-\frac{2\theta}{\lambda+\theta} + 3e^{-\lambda\alpha} - \frac{3\lambda+\theta}{\lambda+\theta}\,e^{-(\lambda+\theta)\alpha}\right)$$
$$+\ \lambda^2\overline{c^2}(e^{-\lambda\alpha}-e^{-(\lambda+\theta)\alpha}) + 2(1-e^{-(\lambda+\theta)\alpha})\left(\frac{2\lambda}{\lambda+\theta} + \left(\frac{\lambda}{\lambda+\theta}\right)^2\right)$$
$$-\ 6(1-e^{-\lambda\alpha}) + 2\lambda\overline{s}\left(\frac{\lambda}{\lambda+\theta}\,e^{-(\lambda+\theta)\alpha} - e^{-\lambda\alpha}\right) - \lambda^2\overline{s^2}(e^{-\lambda\alpha}-e^{-(\lambda+\theta)\alpha}),$$

$$a_1''(1) = 2\frac{\lambda\theta}{\lambda+\theta}\overline{c}\left(2(e^{-\lambda\alpha}-e^{-(\lambda+\theta)\alpha}) + \frac{\lambda}{\lambda+\theta}(1-e^{-(\lambda+\theta)\alpha}) - (1-e^{-\lambda\alpha})\right)$$
$$+\ \lambda^2\overline{c^2}\,\frac{\theta}{\lambda+\theta}(e^{-\lambda\alpha}-e^{-(\lambda+\theta)\alpha}) + \frac{2\theta}{\lambda+\theta}(e^{-\lambda\alpha}-e^{-(\lambda+\theta)\alpha})$$
$$+\ \frac{4\lambda\theta}{(\lambda+\theta)^2}(1-e^{-(\lambda+\theta)\alpha}) - \frac{4\theta}{\lambda+\theta}(1-e^{-\lambda\alpha})$$
$$-\ (2\lambda\overline{s}+\lambda^2\overline{s^2})\frac{\theta}{\lambda+\theta}(e^{-\lambda\alpha}-e^{-(\lambda+\theta)\alpha}).$$

3. Limiting distribution of the $\{Q(t), t \geq 0\}$ process

We define the idle period as the time period between the epoch when the channel is free and the epoch when the channel is occupied by one customer. Let $X(t)$ denote the state of the channel at time t as defined below (see. fig. 2).

$$X(t) = \begin{cases} 0 & \text{if the channel is in the idle period.} \\ 1 & \text{if the channel is in the period between the first arrival and the second arrival in the case of collision.} \\ 2 & \text{if the channel is in the collision recovery period.} \\ 3 & \text{if the channel is in the period of service time in the case of no collision.} \end{cases}$$

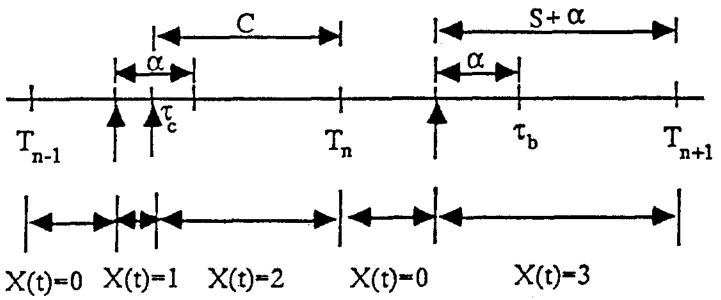

Fig. 2. The states of the channel.

In this section, we shall obtain the generating function of the limiting probabilities $P_{(j,k)} = \lim_{t\to\infty} P(Q(t) = j, X(t) = k \mid Q(0) = i, X(0) = 0)$ of the $(Q(t), X(t))$ process. Then, we easily obtain the generating function of the limiting probabilities $P_j = \lim_{t\to\infty} P(Q(t) = j \mid Q(0) = i)$ of the $Q(t)$ process by the relation $P_j = P_{(j,0)} + P_{(j,1)} + P_{(j,2)} + P_{(j,3)}$.

It is easily seen that $\{(Q_n, T_n), n \geq 0\}$ is a regular Markov renewal sequence and all conditional finite-dimensional distributions of $\{Q(t + T_n), t \geq 0\}$ given $\{Q(u), 0 \leq u \leq T_n, Q(T_n) = i\}$ are the same as those of $\{Q(t), t \geq 0\}$ given $Q(0) = i$. This shows that $\{Q(t), t \geq 0\}$ is a Markov regenerative process with embedded Markov renewal sequence $\{(Q_n, T_n), n \geq 0\}$. Let $\{\pi_i\}$ be the limiting distribution of $\{Q_n, n \geq 0\}$. Let $T_{(j,k)}$ be the time spent by the $(Q(t), X(t))$ process in state (j, k) during $[0, T_1)$. It is known from the property of Markov regenerative processes that the limiting probability of $(Q(t), X(t))$ as $t \to \infty$ exists and is given by

$$P_{(j,k)} = \frac{1}{M} \sum_{i=0}^{\infty} \pi_i E(T_{(j,k)} \mid Q(0) = i), \quad j = 0, 1, 2, \ldots, \quad k = 0, 1, 2, 3, \tag{3.1}$$

where M is a normalizing constant.

Let $P(z) = \sum_{j=0}^{\infty} P_j z^j$ and $P_k(z) = \sum_{j=0}^{\infty} P_{(j,k)} z^j$, $k = 0, 1, 2, 3$. Then $P(z) = \sum_{k=0}^{3} P_k(z)$.

THEOREM 4

The generating functions $P_k(z)$, $k = 0, 1, 2, 3$, are given by

$$P_0(z) = \phi(z) \frac{1}{M} \frac{1}{\lambda + \theta} + \pi_0 \frac{1}{M} \frac{\theta}{\lambda(\lambda + \theta)}, \tag{3.2}$$

$$\begin{aligned} P_1(z) = {} & \phi(z) \frac{1}{M} \frac{\lambda z + \theta}{(\lambda + \theta)^2} (1 - e^{-(\lambda+\theta)\alpha}) \left(2 - \lambda \frac{\alpha e^{-\lambda\alpha}}{1 - e^{-\lambda\alpha}} - \theta \frac{\alpha e^{-\theta\alpha}}{1 - e^{-\theta\alpha}} \right) \\ & - \pi_0 \frac{1}{M} \left(\frac{\lambda z + \theta}{(\lambda + \theta)^2} (1 - e^{-(\lambda+\theta)\alpha}) \left(2 - \lambda \frac{\alpha e^{-\lambda\alpha}}{1 - e^{-\lambda\alpha}} - \theta \frac{\alpha e^{-\theta\alpha}}{1 - e^{-\theta\alpha}} \right) \right. \\ & \left. - z(1 - e^{-\lambda\alpha}) \left(\frac{1}{\lambda} - \frac{\alpha e^{-\lambda\alpha}}{1 - e^{-\lambda\alpha}} \right) \right) \\ & - \pi_1 \frac{1}{M} \left(\frac{\theta z}{(\lambda + \theta)^2} (1 - e^{-(\lambda+\theta)\alpha}) \left(2 - \lambda \frac{\alpha e^{-\lambda\alpha}}{1 - e^{-\lambda\alpha}} - \theta \frac{\alpha e^{-\theta\alpha}}{1 - e^{-\theta\alpha}} \right) \right. \\ & \left. - \frac{\theta z}{\lambda + \theta} \left(\frac{1}{\lambda} - \frac{\alpha e^{-\lambda\alpha}}{1 - e^{-\lambda\alpha}} \right) (1 - e^{-\lambda\alpha}) \right), \end{aligned} \tag{3.3}$$

$$\begin{aligned} P_2(z) = {} & \frac{1}{M} \frac{1 - \tilde{G}(\lambda - \lambda z)}{\lambda - \lambda z} \left(\phi(z) \left(\frac{\lambda z + \theta}{\lambda + \theta} \right)^2 (1 - e^{-(\lambda+\theta)\alpha}) \right. \\ & - \pi_0 \left(\left(\frac{\lambda z + \theta}{\lambda + \theta} \right)^2 (1 - e^{-(\lambda+\theta)\alpha}) - z^2 (1 - e^{-\lambda\alpha}) \right) \\ & \left. - \pi_1 \left(\frac{(\lambda z + \theta)\theta z}{(\lambda + \theta)^2} (1 - e^{-(\lambda+\theta)\alpha}) - z^2 \frac{\theta}{\lambda + \theta} (1 - e^{-\lambda\alpha}) \right) \right), \end{aligned} \tag{3.4}$$

and

$$P_3(z)=\frac{1}{M}\left(\alpha+\frac{1-\tilde{F}(\lambda-\lambda z)}{\lambda-\lambda z}\right)\left(\phi(z)\frac{\lambda z+\theta}{\lambda+\theta}\mathrm{e}^{-(\lambda+\theta)\alpha}\right.$$
$$\left.-\ \pi_0\left(\frac{\lambda z+\theta}{\lambda+\theta}\mathrm{e}^{-(\lambda+\theta)\alpha}-z\,\mathrm{e}^{-\lambda\alpha}\right)-\pi_1\frac{\theta z}{\lambda+\theta}\left(\mathrm{e}^{-(\lambda+\theta)\alpha}-\mathrm{e}^{-\lambda\alpha}\right)\right),\quad(3.5)$$

where the normalizing constant M will be given by (3.19).

Proof

(1) Since the sample path of $Q(t)$ is independent of i during the time period $\{X(t)=0\}$ of the cycle under $Q(0)=i$, we have

$$E(T_{(j,0)}\,|\,Q(0)=i)=\begin{cases}0 & \text{if } j\neq i,\\ \dfrac{1}{\lambda+\theta} & \text{if } j=i\geq 1,\\ \dfrac{1}{\lambda} & \text{if } j=i=0.\end{cases}$$

Thus, from (3.1) we obtain

$$P_{(j,0)}=\begin{cases}\dfrac{1}{M}\dfrac{1}{\lambda+\theta}\pi_j & \text{if } j\geq 1,\\ \dfrac{1}{M}\dfrac{1}{\lambda}\pi_0 & \text{if } j=0,\end{cases}$$

and so we have (3.2).

(2) For simplicity of computation, let

$$P^k_{(j,l)}=\frac{1}{M}\sum_{i=0}^{\infty}\pi_i E(T_{(j,l)}\,|\,Q(0)=i,\ I=k)P(I=k\,|\,Q(0)=i))\tag{3.6}$$

and $P_l^k(z)=\sum_{j=0}^{\infty}P^k_{(j,l)}z^j$, $l=1,2,3$. Then for $l=1,2,3$ we have

$$P_l(z)=\sum_{k=1}^{6}P_l^k(z).\tag{3.7}$$

For $P_1(z)$, we will compute $P_1^k(z)$ for each k. In the case of type 1, an external arrival takes place before a retrial so that there are $i+1$ customers in the system when the channel starts service, and a collision has occurred by another external arrival. Thus, we have

$$E(T_{(j,1)}\,|\,Q(0)=i,\ I=1)=\begin{cases} E(A\,|\,A<\alpha)=\dfrac{1}{\lambda}-\dfrac{\alpha\,\mathrm{e}^{-\lambda\alpha}}{1-\mathrm{e}^{-\lambda\alpha}} & \text{if } j=i+1,\\ 0 & \text{if } j\neq i+1,\end{cases}$$

where A is the interarrival time. Hence, from (3.6) we obtain that

$$P^1_{(j,1)}=\frac{1}{M}\left(\frac{1}{\lambda}-\frac{\alpha\,\mathrm{e}^{-\lambda\alpha}}{1-\mathrm{e}^{-\lambda\alpha}}\right)P(I=1|Q(0)=j-1)\,\pi_{j-1}$$

$$=\begin{cases}\dfrac{1}{M}\left(\dfrac{1}{\lambda}-\dfrac{\alpha\,\mathrm{e}^{-\lambda\alpha}}{1-\mathrm{e}^{-\lambda\alpha}}\right)\left(\dfrac{\lambda}{\lambda+\theta}\right)^2(1-\mathrm{e}^{-(\lambda+\theta)\alpha})\pi_{j-1} & \text{if } j\geq 2,\\ \dfrac{1}{M}\left(\dfrac{1}{\lambda}-\dfrac{\alpha\,\mathrm{e}^{-\lambda\alpha}}{1-\mathrm{e}^{-\lambda\alpha}}\right)(1-\mathrm{e}^{-\lambda\alpha})\pi_0 & \text{if } j=1,\\ 0 & \text{if } j=0,\end{cases}$$

and so

$$P_1^1(z)=\phi(z)\frac{z}{M}\left(\frac{1}{\lambda}-\frac{\alpha\,\mathrm{e}^{-\lambda\alpha}}{1-\mathrm{e}^{-\lambda\alpha}}\right)\left(\frac{\lambda}{\lambda+\theta}\right)^2(1-\mathrm{e}^{-(\lambda+\theta)\alpha})$$

$$-\pi_0\frac{z}{M}\left(\frac{1}{\lambda}-\frac{\alpha\,\mathrm{e}^{-\lambda\alpha}}{1-\mathrm{e}^{-\lambda\alpha}}\right)\left(\left(\frac{\lambda}{\lambda+\theta}\right)^2(1-\mathrm{e}^{-(\lambda+\theta)\alpha})-1+\mathrm{e}^{-\lambda\alpha}\right). \tag{3.8}$$

In a like manner, one obtains

$$P_1^2(z)=(\phi(z)-\pi_0)\frac{z}{M}\left(\frac{1}{\theta}-\frac{\alpha\,\mathrm{e}^{-\theta\alpha}}{1-\mathrm{e}^{-\theta\alpha}}\right)\frac{\lambda\theta}{(\lambda+\theta)^2}(1-\mathrm{e}^{-(\lambda+\theta)\alpha}), \tag{3.9}$$

$$P_1^3(z)=(\phi(z)-\pi_0)\frac{1}{M}\left(\frac{1}{\lambda}-\frac{\alpha\,\mathrm{e}^{-\lambda\alpha}}{1-\mathrm{e}^{-\lambda\alpha}}\right)\frac{\lambda\theta}{(\lambda+\theta)^2}(1-\mathrm{e}^{-(\lambda+\theta)\alpha})$$

$$-\pi_1\frac{z}{M}\left(\frac{1}{\lambda}-\frac{\alpha\,\mathrm{e}^{-\lambda\alpha}}{1-\mathrm{e}^{-\lambda\alpha}}\right)\left(\frac{\lambda\theta}{(\lambda+\theta)^2}(1-\mathrm{e}^{-(\lambda+\theta)\alpha})-\frac{\theta}{\lambda+\theta}(1-\mathrm{e}^{-\lambda\alpha})\right), \tag{3.10}$$

$$P_1^4(z)=(\phi(z)-\pi_0-\pi_1 z)\frac{1}{M}\left(\frac{1}{\theta}-\frac{\alpha\,\mathrm{e}^{-\theta\alpha}}{1-\mathrm{e}^{-\theta\alpha}}\right)\left(\frac{\theta}{\lambda+\theta}\right)^2(1-\mathrm{e}^{-(\lambda+\theta)\alpha}). \tag{3.11}$$

In the case of types 5 and 6, a collision does not occur and hence $E(T_{(j,1)}|Q(0)=i, I=k), k=5, 6$. Combining the results (3.8)–(3.11) and using (3.7), we have the result (3.3).

(3) Let τ_c be the time when a collision occurs in the cycle of types 1–4 (see fig. 2). In the case of type 1, there are $i+2$ customers in the system at the moment when a collision occurs, so we have

$$E(T_{(j,2)}|Q(0)=i,\ I=1)=E\left(\int_0^\infty\int_0^t 1(Q(\tau_c+u)=j)\,\mathrm{d}u\,\mathrm{d}G(t)|Q(\tau_c)=i+2\right)$$

$$=\int_0^\infty\int_0^t \frac{(\lambda u)^{j-2-i}}{(j-2-i)!}\mathrm{e}^{-\lambda u}\mathrm{d}u\,\mathrm{d}G(t)1(j\ge i+2). \quad (3.12)$$

Thus, from (3.6) we have that

$$P^1_{(j,2)}=\frac{1}{M}\sum_{i=0}^{j-2}\int_0^\infty\int_0^t \frac{(\lambda u)^{j-2-i}}{(j-2-i)!}\mathrm{e}^{-\lambda u}\mathrm{d}u\,\mathrm{d}G(t)P(I=1|Q(0)=i)\pi_i$$

$$=\frac{1}{M}\int_0^\infty\int_0^t \frac{(\lambda u)^{j-2}}{(j-2)!}\mathrm{e}^{-\lambda u}\mathrm{d}u\,\mathrm{d}G(t)(1-\mathrm{e}^{-\lambda\alpha})\pi_0 1(j\ge 2)$$

$$+\frac{1}{M}\int_0^\infty\int_0^t \sum_{i=1}^{j-2}\pi_i\frac{(\lambda u)^{j-2-i}}{(j-2-i)!}\mathrm{e}^{-\lambda u}\mathrm{d}u\,\mathrm{d}G(t)\left(\frac{\lambda}{\lambda+\theta}\right)^2(1-\mathrm{e}^{-(\lambda+\theta)\alpha})1(j\ge 3),$$

and so

$$P^1_2(z)=\frac{z^2}{M}\int_0^\infty\int_0^t\sum_{j=2}^\infty \frac{(\lambda u z)^{j-2}}{(j-2)!}\mathrm{e}^{-\lambda u}\mathrm{d}u\,\mathrm{d}G(t)(1-\mathrm{e}^{-\lambda\alpha})\pi_0$$

$$+\frac{1}{M}\int_0^\infty\int_0^t\sum_{j=3}^\infty z^j\sum_{i=1}^{j-2} \frac{(\lambda u)^{j-2-i}}{(j-2-i)!}\mathrm{e}^{-\lambda u}\mathrm{d}u\,\mathrm{d}G(t)\left(\frac{\lambda}{\lambda+\theta}\right)^2(1-\mathrm{e}^{-(\lambda+\theta)\alpha})$$

$$=\phi(z)\frac{1-\tilde{G}(\lambda-\lambda z)}{\lambda-\lambda z}\frac{1}{M}\left(\frac{\lambda z}{\lambda+\theta}\right)^2(1-\mathrm{e}^{-(\lambda+\theta)\alpha})$$

$$-\pi_0\frac{1-\tilde{G}(\lambda-\lambda z)}{\lambda-\lambda z}\frac{1}{M}\left(\frac{\lambda z}{\lambda+\theta}\right)^2(\mathrm{e}^{-\lambda\alpha}-\mathrm{e}^{-(\lambda+\theta)\alpha}). \quad (3.13)$$

In a like manner, one obtains

$$P_2^2(z) = (\phi(z) - \pi_0)\frac{1-\tilde{G}(\lambda - \lambda z)}{\lambda - \lambda z}\frac{z}{M}\frac{\lambda\theta}{(\lambda+\theta)^2}(1 - \mathrm{e}^{-(\lambda+\theta)\alpha}), \tag{3.14}$$

$$\begin{aligned} P_2^3(z) &= (\phi(z) - \pi_0)\frac{1-\tilde{G}(\lambda - \lambda z)}{\lambda - \lambda z}\frac{z}{M}\frac{\lambda\theta}{(\lambda+\theta)^2}(1 - \mathrm{e}^{-(\lambda+\theta)\alpha}) \\ &\quad - \pi_1\frac{1-\tilde{G}(\lambda - \lambda z)}{\lambda - \lambda z}\frac{z^2}{M}\left(\frac{\lambda\theta}{(\lambda+\theta)^2}(1 - \mathrm{e}^{-(\lambda+\theta)\alpha}) - \frac{\theta}{\lambda+\theta}(1 - \mathrm{e}^{-\lambda\alpha})\right), \end{aligned} \tag{3.15}$$

$$P_2^4(z) = (\phi(z) - \pi_0 - \pi_1 z)\frac{1-\tilde{G}(\lambda - \lambda z)}{\lambda - \lambda z}\frac{1}{M}\left(\frac{\theta}{\lambda+\theta}\right)^2(1 - \mathrm{e}^{-(\lambda+\theta)\alpha}). \tag{3.16}$$

In the case of types 5 and 6, a collision does not occur, so we have $E(T_{(j,2)} | Q(0) = i, I = k) = 0$ for $k = 5, 6$. Combining the results (3.12)–(3.15), we have the result (3.4).

(4) The successful service completion can happen only in a cycle of types 5 and 6. Hence, we have $E(T_{(j,3)} | Q(0) = i, I = k) = 0$ for $k = 1, 2, 3, 4$. Let τ_b be the time when the first phase of service time ends in the cycles of types 5 and 6 (see fig. 2). In the case of type 5, an external arrival takes place before a retrial so that there are $i+1$ customers in the system at the moment when the period of propagation delay ends, and a collision does not occur. Thus, we have the following:

$$\begin{aligned} &E(T_{(j,3)} | Q(0) = i, I = 5) \\ &= \alpha\delta_{i+1,j} + E\left(\int_0^\infty \int_0^t 1(Q(\tau_b + u) = j)\,\mathrm{d}u\,\mathrm{d}F(t) | Q(\tau_b) = i+1\right) \\ &= \alpha\delta_{i+1,j} + \int_0^\infty \int_0^t \frac{(\lambda u)^{j-1-i}}{(j-1-i)!}\mathrm{e}^{-\lambda u}\,\mathrm{d}u\,\mathrm{d}F(t) 1(j \geq i+1), \end{aligned}$$

where $\delta_{k,l}$ is the Kronecker delta,

$$\begin{aligned} P_{(j,3)}^5 &= \frac{1}{M}\sum_{i=0}^\infty \pi_i E(T_{(j,3)} | Q(0) = i, I = 5) P(I = 5 | Q(0) = i) \\ &= \frac{1}{M}\left(\alpha\delta_{1,j} + \int_0^\infty \int_0^t \frac{(\lambda u)^{j-1}}{(j-1)!}\mathrm{e}^{-\lambda u}\,\mathrm{d}u\,\mathrm{d}F(t)\right)\mathrm{e}^{-\lambda\alpha}\pi_0 1(j \geq 1) \end{aligned}$$

$$+\frac{1}{M}\sum_{i=1}^{j-1}\pi_i\left(\alpha\delta_{i+1,j}+\int_0^{\infty}\int_0^{t}\frac{(\lambda u)^{j-1-i}}{(j-1-i)!}\mathrm{e}^{-\lambda u}\mathrm{d}u\,\mathrm{d}F(t)\right)\frac{\lambda}{\lambda+\theta}\mathrm{e}^{-(\lambda+\theta)\alpha}1(j\geq i+1)$$

and

$$P_3^5(z)=\frac{1}{M}\pi_0\left(\alpha z+\int_0^{\infty}\int_0^{t}\sum_{j=1}^{\infty}z^j\frac{(\lambda u)^{j-1}}{(j-1)!}\mathrm{e}^{-\lambda u}\mathrm{d}u\,\mathrm{d}F(t)\right)\mathrm{e}^{-\lambda\alpha}$$

$$+\frac{1}{M}\sum_{j=2}^{\infty}z^j\sum_{i=1}^{j-1}\pi_i\left(\alpha\delta_{i+1,j}+\int_0^{\infty}\int_0^{t}\frac{(\lambda u)^{j-1-i}}{(j-1-i)!}\mathrm{e}^{-\lambda u}\mathrm{d}u\,\mathrm{d}F(t)\right)\frac{\lambda}{\lambda+\theta}\mathrm{e}^{-(\lambda+\theta)\alpha}$$

$$=\pi_0\frac{z}{M}\left(\alpha+\frac{1-\tilde{F}(\lambda-\lambda z)}{\lambda-\lambda z}\right)\mathrm{e}^{-\lambda\alpha}+\frac{z}{M}(\phi(z)-\pi_0)\frac{\lambda\alpha}{\lambda+\theta}\mathrm{e}^{-(\lambda+\theta)\alpha}$$

$$+\frac{z}{M}(\phi(z)-\pi_0)\frac{1-\tilde{F}(\lambda-\lambda z)}{\lambda-\lambda z}\frac{\lambda}{\lambda+\theta}\mathrm{e}^{-(\lambda+\theta)\alpha}$$

$$=\phi(z)\left(\alpha+\frac{1-\tilde{F}(\lambda-\lambda z)}{\lambda-\lambda z}\right)\frac{z}{M}\frac{\lambda}{\lambda+\theta}\mathrm{e}^{-(\lambda+\theta)\alpha}$$

$$-\pi_0\left(\alpha+\frac{1-\tilde{F}(\lambda-\lambda z)}{\lambda-\lambda z}\right)\frac{z}{M}\left(\frac{\lambda}{\lambda+\theta}\mathrm{e}^{-(\lambda+\theta)\alpha}-\mathrm{e}^{-\lambda\alpha}\right). \tag{3.17}$$

Similarly, we obtain

$$P_3^6(z)=(\phi(z)-\pi_0)\left(\alpha+\frac{1-\tilde{F}(\lambda-\lambda z)}{\lambda-\lambda z}\right)\frac{1}{M}\frac{\theta}{\lambda+\theta}\mathrm{e}^{-(\lambda+\theta)\alpha}$$

$$-\pi_1\left(\alpha+\frac{1-\tilde{F}(\lambda-\lambda z)}{\lambda-\lambda z}\right)\frac{z}{M}\frac{\theta}{\lambda+\theta}(\mathrm{e}^{-(\lambda+\theta)\alpha}-\mathrm{e}^{-\lambda\alpha}). \tag{3.18}$$

Combining the results (3.17) and (3.18), we have the required result (3.5). Now we determine the normalizing constant M. Since $\phi(1)=P(1)=1$, we have

$$M=\frac{1}{\lambda+\theta}+K+\bar{c}(1-\mathrm{e}^{-(\lambda+\theta)\alpha})+(\alpha+\bar{s})\mathrm{e}^{-(\lambda+\theta)\alpha}$$

$$+\pi_0\left(\frac{\theta}{\lambda(\lambda+\theta)}+(1-\mathrm{e}^{-\lambda\alpha})\left(\frac{1}{\lambda}-\frac{\alpha\,\mathrm{e}^{-\lambda\alpha}}{1-\mathrm{e}^{-\lambda\alpha}}\right)+(\alpha+\bar{s}-\bar{c})(\mathrm{e}^{-\lambda\alpha}-\mathrm{e}^{-(\lambda+\theta)\alpha})-K\right)$$

$$+\pi_1\left(\frac{\theta}{\lambda+\theta}\left(\frac{1}{\lambda}-\frac{\alpha\,\mathrm{e}^{-\lambda\alpha}}{1-\mathrm{e}^{-\lambda\alpha}}\right)(1-\mathrm{e}^{-\lambda\alpha})\right.$$

$$\left.+\,(\alpha+\bar{s}-\bar{c})\frac{\theta}{\lambda+\theta}(\mathrm{e}^{-\lambda\alpha}-\mathrm{e}^{-(\lambda+\theta)\alpha})-K\frac{\theta}{\lambda+\theta}\right), \tag{3.19}$$

where

$$K=\frac{1}{\lambda+\theta}(1-\mathrm{e}^{-(\lambda+\theta)\alpha})\left(2-\lambda\frac{\alpha\,\mathrm{e}^{-\lambda\alpha}}{1-\mathrm{e}^{-\lambda\alpha}}-\theta\frac{\alpha\,\mathrm{e}^{-\theta\alpha}}{1-\mathrm{e}^{-\theta\alpha}}\right).$$

The proof of the theorem is complete. □

By noting $\lim_{t\to\infty} P(X(t)=k)=P_k(1)$, $k=0,1,2,3$, we have the limiting distribution of $X(t)$.

COROLLARY 5

The limiting distribution of $X(t)$ as $t\to\infty$ is given by

$$P(X=0)=\frac{1}{M}\left[\frac{1}{\lambda+\theta}+\pi_0\frac{\theta}{\lambda(\lambda+\theta)}\right],$$

$$P(X=1)=\frac{1}{M}\left[K-\pi_0\left(K-(1-\mathrm{e}^{-\lambda\alpha})\left(\frac{1}{\lambda}-\frac{\alpha\,\mathrm{e}^{-\lambda\alpha}}{1-\mathrm{e}^{-\lambda\alpha}}\right)\right)\right.$$

$$\left.-\pi_1\left(K\frac{\theta}{\lambda+\theta}-\frac{\theta}{\lambda+\theta}(1-\mathrm{e}^{-\lambda\alpha})\left(\frac{1}{\lambda}-\frac{\alpha\,\mathrm{e}^{-\lambda\alpha}}{1-\mathrm{e}^{-\lambda\alpha}}\right)\right)\right],$$

$$P(X=2)=\frac{1}{M}\left[\bar{c}(1-\mathrm{e}^{-(\lambda+\theta)\alpha})-\bar{c}\left(\pi_0+\frac{\theta}{\lambda+\theta}\pi_1\right)(\mathrm{e}^{-\lambda\alpha}-\mathrm{e}^{-(\lambda+\theta)\alpha})\right],$$

$$P(X=3)=\frac{1}{M}\left[(\alpha+\bar{s})\mathrm{e}^{-(\lambda+\theta)\alpha}+\left(\pi_0+\frac{\theta}{\lambda+\theta}\pi_1\right)(\alpha+\bar{s})(\mathrm{e}^{-\lambda\alpha}-\mathrm{e}^{-(\lambda+\theta)\alpha})\right],$$

where K and M are given in the proof of theorem 4.

COROLLARY 6

The mean number $P_i'(1)$ of customers in the system at channel state i is given by

$$P_0'(1)=\frac{1}{M}\,\frac{1}{\lambda+\theta}\phi'(1),$$

$$P_1'(1) = \left(\frac{\phi'(1)}{\lambda+\theta} + \frac{\lambda}{(\lambda+\theta)^2}\right)\frac{1}{M}(1-e^{-(\lambda+\theta)\alpha})\left(2-\lambda\frac{\alpha e^{-\lambda\alpha}}{1-e^{-\lambda\alpha}}-\theta\frac{\alpha e^{-\theta\alpha}}{1-e^{-\theta\alpha}}\right)$$

$$-\pi_0\frac{1}{M}\left(\frac{\lambda}{(\lambda+\theta)^2}(1-e^{-(\lambda+\theta)\alpha})\left(2-\lambda\frac{\alpha e^{-\lambda\alpha}}{1-e^{-\lambda\alpha}}-\theta\frac{\alpha e^{-\theta\alpha}}{1-e^{-\theta\alpha}}\right)\right.$$

$$\left.-(1-e^{-\lambda\alpha})\left(\frac{1}{\lambda}-\frac{\alpha e^{-\lambda\alpha}}{1-e^{-\lambda\alpha}}\right)\right)$$

$$-\pi_1\frac{1}{M}\left(\frac{\theta}{(\lambda+\theta)^2}(1-e^{-(\lambda+\theta)\alpha})\left(2-\lambda\frac{\alpha e^{-\lambda\alpha}}{1-e^{-\lambda\alpha}}-\theta\frac{\alpha e^{-\theta\alpha}}{1-e^{-\theta\alpha}}\right)\right.$$

$$\left.-\frac{\theta}{\lambda+\theta}(1-e^{-\lambda\alpha})\left(\frac{1}{\lambda}-\frac{\alpha e^{-\lambda\alpha}}{1-e^{-\lambda\alpha}}\right)\right),$$

$$P_2'(1) = \frac{1}{M}\frac{\lambda}{2}\overline{c^2}\left(1-e^{-(\lambda+\theta)\alpha}-\left(\pi_0+\frac{\theta}{\lambda+\theta}\pi_1\right)(e^{-\lambda\alpha}-e^{-(\lambda+\theta)\alpha})\right)$$

$$+\frac{1}{M}\bar{c}\left(\left(\phi'(1)+\frac{2\lambda}{\lambda+\theta}\right)(1-e^{-(\lambda+\theta)\alpha})\right.$$

$$-\pi_0\left(\frac{2\lambda}{\lambda+\theta}(1-e^{-(\lambda+\theta)\alpha})-2(1-e^{-\lambda\alpha})\right)$$

$$\left.-\pi_1\left(\frac{2\lambda\theta+\theta^2}{(\lambda+\theta)^2}(1-e^{-(\lambda+\theta)\alpha})-2\frac{\theta}{\lambda+\theta}(1-e^{-\lambda\alpha})\right)\right),$$

$$P_3'(1) = \frac{1}{M}\frac{\lambda}{2}\overline{s^2}\left(e^{-(\lambda+\theta)\alpha}-\left(\pi_0+\frac{\theta}{\lambda+\theta}\pi_1\right)(e^{-(\lambda+\theta)\alpha}-e^{-\lambda\alpha})\right)$$

$$+\frac{1}{M}(\alpha+\bar{s})\left(\left(\phi'(1)+\frac{\lambda}{\lambda+\theta}\right)e^{-(\lambda+\theta)\alpha}\right.$$

$$\left.-\pi_0\left(\frac{\lambda}{\lambda+\theta}e^{-(\lambda+\theta)\alpha}-e^{-\lambda\alpha}\right)-\pi_1\frac{\theta}{\lambda+\theta}(e^{-(\lambda+\theta)\alpha}-e^{-\lambda\alpha})\right).$$

4. Numerical examples

In this section, we present some numerical results for probability π_0 of system size being zero, mean $\phi'(1)$ of system size at embedding point, mean $P'(1)$ of

Table 1

Exponential service time with mean 1.0 ($\bar{s} = 1.0$, $\theta = 2.0$).

λ	α	c	ρ	$\phi'(1)$	$P'(1)$	π_0	x_0	x_1	x_2	x_3
0.30000	0.0020	0.1	0.43178	0.531	0.498	0.6533	0.6993	0.0001	0.0000	0.3006
		0.5	0.43233	0.532	0.374	0.6531	0.6992	0.0001	0.0002	0.3006
	0.0100	0.1	0.43720	0.549	0.507	0.6464	0.6966	0.0003	0.0002	0.3029
		0.5	0.43999	0.553	0.387	0.6456	0.6959	0.0003	0.0009	0.3029
	0.0500	0.1	0.46588	0.649	0.556	0.6102	0.6827	0.0017	0.0010	0.3145
		0.5	0.48051	0.676	0.459	0.6050	0.6785	0.0017	0.0052	0.3145
0.50000	0.0020	0.1	0.70226	1.687	1.668	0.3719	0.4988	0.0001	0.0001	0.5010
		0.5	0.70326	1.695	1.619	0.3712	0.4983	0.0001	0.0007	0.5010
	0.0100	0.1	0.71139	1.774	1.733	0.3592	0.4940	0.0005	0.0007	0.5048
		0.5	0.71645	1.816	1.719	0.3555	0.4912	0.0005	0.0036	0.5048
	0.0500	0.1	0.75992	2.328	2.168	0.2929	0.4690	0.0030	0.0041	0.5239
		0.5	0.78655	2.702	2.470	0.2711	0.4517	0.0029	0.0213	0.5240
0.642688	0.0020	0.1	0.88880	5.799	5.792	0.1467	0.3557	0.0001	0.0003	0.6439
		0.5	0.89016	5.882	5.880	0.1452	0.3546	0.0001	0.0014	0.6439
	0.0100	0.1	0.90063	6.572	6.531	0.1302	0.3493	0.0003	0.0014	0.6489
		0.5	0.90751	7.126	7.083	0.1226	0.3435	0.0003	0.0073	0.6489
	0.0500	0.1	0.96367	18.954	18.715	0.0459	0.3144	0.0026	0.0085	0.6744
		0.5	0.99998	46839.480	46839.160	2.1E − 5	0.2776	0.0022	0.0454	0.6748
0.64270	0.0020	0.1	0.88882	5.800	5.793	0.1467	0.3557	0.0001	0.0003	0.6439
		0.5	0.89018	5.882	5.881	0.1452	0.3546	0.0001	0.0014	0.6439
	0.0100	0.1	0.90064	6.573	6.532	0.1302	0.3493	0.0003	0.0014	0.6489
		0.5	0.90753	7.127	7.084	0.1226	0.3435	0.0003	0.0073	0.6490
	0.0500	0.1	0.96369	18.963	18.725	0.0459	0.3144	0.0026	0.0085	0.6745
		0.5	1.000003	−289155.4	−289155.7	−3.3E − 06				
0.65000	0.0020	0.1	0.89824	6.408	6.401	0.1346	0.3484	0.0001	0.0003	0.6513
		0.5	0.89962	6.508	6.508	0.1331	0.3472	0.0001	0.0014	0.6513
	0.0100	0.1	0.91020	7.352	7.310	0.1180	0.3419	0.0003	0.0015	0.6563
		0.5	0.91718	8.046	8.003	0.1101	0.3358	0.0003	0.0075	0.6563
	0.0500	0.1	0.97400	26.710	26.466	0.0329	0.3065	0.0026	0.0088	0.6822
		0.5	1.01083	−67.214	−67-536	−0.0146				

system size at arbitrary time point and the probability $x_i = P(X = i)$, $i = 0, 1, 2, 3$ of the channel states. In table 1, the numerical results are presented for the case of an exponential server with unit mean. The numerical results for hyperexponential service time with

$$\tilde{F}(u) = 0.5\left(\frac{1.5}{1.5+u}\right) + 0.5\left(\frac{0.75}{0.75+u}\right)$$

Table 2

Hyperexponential service time with mean 1.0 ($\bar{s} = 1.0$, $\theta = 2.0$).

λ	α	c	ρ	$\phi'(1)$	$P'(1)$	π_0	x_0	x_1	x_2	x_3
0.30000	0.0020	0.1	0.43178	0.549	0.515	0.6533	0.6993	0.0001	0.0000	0.3006
		0.5	0.43233	0.550	0.391	0.6531	0.6992	0.0001	0.0002	0.3006
	0.0100	0.1	0.43720	0.567	0.524	0.6467	0.6966	0.0003	0.0002	0.3029
		0.5	0.43999	0.571	0.404	0.6455	0.6959	0.0003	0.0009	0.3029
	0.0500	0.1	0.46588	0.669	0.575	0.6098	0.6827	0.0017	0.0010	0.3146
		0.5	0.48051	0.698	0.478	0.6045	0.6785	0.0017	0.0053	0.3146
0.50000	0.0020	0.1	0.70226	1.781	1.761	0.3718	0.4988	0.0001	0.0001	0.5010
		0.5	0.70326	1.789	1.707	0.3711	0.4983	0.0001	0.0007	0.5010
	0.0100	0.1	0.71139	1.871	1.829	0.3591	0.4940	0.0005	0.0007	0.5048
		0.5	0.71645	1.916	1.813	0.3554	0.4911	0.0005	0.0036	0.5048
	0.0500	0.1	0.75992	2.449	2.284	0.2925	0.4689	0.0029	0.0041	0.5240
		0.5	0.78655	2.838	2.598	0.2706	0.4516	0.0029	0.0215	0.5241
0.642688	0.0020	0.1	0.88880	6.212	6.204	0.1467	0.3557	0.0001	0.0003	0.6439
		0.5	0.89016	6.300	6.294	0.1452	0.3546	0.0001	0.0014	0.6439
	0.0100	0.1	0.90063	7.036	6.992	0.1302	0.3493	0.0003	0.0014	0.6489
		0.5	0.90751	7.624	7.575	0.1225	0.3434	0.0003	0.0073	0.6490
	0.0500	0.1	0.96367	20.226	19.980	0.0458	0.3144	0.0026	0.0085	0.6745
		0.5	0.99998	49820.210	49819.900	2.1E−5	0.2776	0.0022	0.0454	0.6748
0.64270	0.0020	0.1	0.88882	6.213	6.205	0.1467	0.3557	0.0001	0.0003	0.6439
		0.5	0.89018	6.301	6.295	0.1452	0.3546	0.0001	0.0014	0.6439
	0.0100	0.1	0.90064	7.037	6.993	0.1302	0.3493	0.0003	0.0014	0.6490
		0.5	0.90753	7.626	7.577	0.1225	0.3434	0.0003	0.0073	0.6490
	0.0500	0.1	0.96369	20.236	19.990	0.0458	0.3144	0.0026	0.0085	0.6745
		0.5	1.000003	−307556.8	−307557.1	−3.3E−06				
0.65000	0.0020	0.1	0.89824	6.869	6.862	0.1346	0.3484	0.0001	0.0003	0.6513
		0.5	0.89962	6.976	6.972	0.1331	0.3472	0.0001	0.0014	0.6513
	0.0100	0.1	0.91020	7.876	7.832	0.1179	0.3419	0.0003	0.0015	0.6563
		0.5	0.91718	8.615	8.567	0.1100	0.3358	0.0003	0.0075	0.6564
	0.0500	0.1	0.97400	28.524	28.272	0.0329	0.3064	0.0025	0.0088	0.6822
		0.5	1.01083	−71.536	−71.865	−0.0145				

are given in table 2. Finally, in table 3, we have the results for constant service time with unit mean. In tables 1–3, collision recovery time C is deterministic. As ρ increases to 1, we see that the mean of system size increases and π_0 decreases, as we expected. Also, when ρ is slightly greater than 1, π_0 is negative. This fact suggests that the condition $\rho < 1$ is also a necessary condition, which we could not prove.

Table 3

Deterministic service time with mean 1.0 ($\bar{s} = 1.0$, $\theta = 2.0$).

λ	α	c	ρ	$\phi'(1)$	$P'(1)$	π_0	x_0	x_1	x_2	x_3
0.30000	0.0020	0.1	0.43178	0.452	0.420	0.6534	0.6993	0.0001	0.0000	0.3006
		0.5	0.43233	0.452	0.301	0.6533	0.6992	0.0001	0.0002	0.3006
	0.0100	0.1	0.43720	0.467	0.428	0.6470	0.6966	0.0003	0.0002	0.3029
		0.5	0.43999	0.471	0.313	0.6462	0.6959	0.0003	0.0008	0.3029
	0.0500	0.1	0.46588	0.554	0.472	0.6129	0.6827	0.0019	0.0009	0.3145
		0.5	0.48051	0.578	0.377	0.6082	0.6789	0.0019	0.0047	0.3145
0.50000	0.0020	0.1	0.70226	1.266	1.255	0.3720	0.4988	0.0001	0.0001	0.5009
		0.5	0.70326	1.272	1.232	0.3714	0.4983	0.0001	0.0006	0.5009
	0.0100	0.1	0.71139	1.335	1.306	0.3599	0.4940	0.0006	0.0007	0.5047
		0.5	0.71645	1.369	1.309	0.3566	0.4914	0.0006	0.0033	0.5047
	0.0500	0.1	0.75992	1.779	1.648	0.2960	0.4690	0.0035	0.0038	0.5236
		0.5	0.78655	2.080	1.898	0.2753	0.4528	0.0034	0.0200	0.5237
0.642688	0.0020	0.1	0.88880	3.940	3.948	0.1468	0.3557	0.0001	0.0003	0.6439
		0.5	0.89016	3.999	4.040	0.1454	0.3546	0.0001	0.0013	0.6439
	0.0100	0.1	0.90063	4.485	4.458	0.1306	0.3493	0.0005	0.0014	0.6489
		0.5	0.90751	4.881	4.880	0.1232	0.3436	0.0004	0.0070	0.6489
	0.0500	0.1	0.96367	13.225	13.019	0.0466	0.3145	0.0028	0.0084	0.6743
		0.5	0.99998	33426.140	33425.850	2.1E − 5	0.2776	0.0022	0.0454	0.6748
0.64270	0.0020	0.1	0.88882	3.940	3.943	0.1468	0.3557	0.0001	0.0003	0.6439
		0.5	0.89018	3.999	4.041	0.1453	0.3546	0.0001	0.0013	0.6439
	0.0100	0.1	0.90064	4.485	4.458	0.1306	0.3493	0.0005	0.0014	0.6489
		0.5	0.90753	4.882	4.881	0.1231	0.3436	0.0004	0.0070	0.6489
	0.0500	0.1	0.96369	13.231	13.025	0.0466	0.3145	0.0028	0.0084	0.6743
		0.5	1.000003	−206349.1	−206349.4	−3.4E − 06				
0.65000	0.0020	0.1	0.89824	4.330	4.333	0.1347	0.3484	0.0001	0.0003	0.6512
		0.5	0.89962	4.401	4.443	0.1332	0.3473	0.0001	0.0014	0.6512
	0.0100	0.1	0.91020	4.990	4.962	0.1183	0.3419	0.0004	0.0014	0.6563
		0.5	0.91718	5.483	5.480	0.1106	0.3360	0.0004	0.0073	0.6563
	0.0500	0.1	0.97400	18.541	18.329	0.0334	0.3065	0.0027	0.0087	0.6821
		0.5	1.01083	−47.775	−48-076	−0.0149				

Remark

In this paper, we have studied the idealized retrial queueing model with collision, which is motivated by the communication protocol *CSMA/CD*. Thus, we have not dealt with the issue of implementing the control policy in the retrial group. Each customer in the retrial group needs to know the exact number n of customers in the retrial group to implement exponential retrial time with parameters θ/n. In a practical situation such as satellite communication, it is impossible for each blocked terminal (customer in the retrial group in queueing terminology) to know the total number of blocked terminals. However, statistical algorithms for the estimation of the number of blocked terminals of slotted *CSMA* protocol with deterministic transmission time were proposed (for example [1, p. 218]).

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