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Linear Oblique Derivative Problems for the Uniformly Elliptic Hamilton-Jacobi-Bellman Equation

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1. Introduction

In this paper we are concerned with linear oblique boundary value problems for the Hamilton-Jacobi-Bellman equations of stochastic control theory (Fleming-Rishel [10], Krylov [13]). To formulate these problems we let Ω denote a bounded domain in Euclidean n space $\mathbb{R}^n$, and $\{L_k\}$, $\{f_k\}$ be sequences of linear elliptic operators and real functions on Ω, with L_k given by

$$L_k u = a_k^{ij} D_{ij} u + b_k^i D_i u + c_k u \tag{1.1}$$

where a_k^{ij}, b_k^i, c_k, $i,j=1,\dots,n$, $k=1,\dots,\infty$ are real functions on Ω, the matrices $[a_k^{ij}]$ being positive on Ω. The Hamilton-Jacobi-Bellman equation corresponding to the family $\{L_k, f_k\}$, namely

$$F[u] = \inf_{k\in\mathbb{N}} (L_k u - f_k) = 0, \tag{1.2}$$

is *uniformly elliptic* in Ω provided there exist positive constants λ, Λ such that

$$\lambda|\xi|^2 \leqq a_k^{ij}\xi_i\xi_j \leqq \Lambda|\xi|^2 \tag{1.3}$$

for all $\xi\in\mathbb{R}^n$, $x\in\Omega$, $k\in\mathbb{N}$. A linear boundary operator

$$Mu = \beta^i D_i u + \gamma u, \tag{1.4}$$

where β^i, γ $i=1,\dots,n$ are real functions on the boundary $\partial\Omega$ is called *oblique* if $\beta\cdot\nu>0$ on $\partial\Omega$, where ν denotes the unit inner normal to $\partial\Omega$, and *regularly oblique* if

$$\beta\cdot\nu \geqq \lambda \tag{1.5}$$

on $\partial\Omega$, for some positive constant λ. We shall treat here boundary value problems of the form

$$G[u] = Mu - g = 0 \quad \text{on } \partial\Omega, \tag{1.6}$$

which agree with the classical Neumann problem when $\beta=\nu$, $\gamma=0$. Concerning the coefficients of L_k, M, we shall assume that ν, β, γ and g have been extended to Ω, with (1.5) preserved and moreover $a_k^{ij}, b_k^i, c_k, f_k, \beta, \gamma, g \in C^{1,1}(\bar{\Omega})$ with norms independent of k, that is

$$(1.7) \qquad |a_k^{ij}, b_k^i, c_k, f_k, \beta, \gamma, g|_{1,1;\Omega} \leqq K, \quad i,j=1,\ldots,n, \quad k=1,\ldots,\infty,$$

for some fixed K, and

$$(1.8) \qquad c_k, \gamma \leqq 0 \quad \text{for all } k=1,\ldots,\infty.$$

The *Dirichlet problem* for the uniformly elliptic Bellman equation has been treated by Brezis and Evans [4], Evans and Friedman [8], Lions and Menaldi [23], Lions [18] and Evans and Lions [9] with *classical* solvability under conditions (1.3), (1.7), (1.8) being established independently by Evans [7] and Krylov [14]. An existence result embracing global regularity is the following recent theorem of Krylov [15] (see also L. Caffarelli, J.J. Kohn, L. Nirenberg and J. Spruck [6]).

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega \in C^{2,1}$ and $\phi \in C^{2,1}(\bar{\Omega})$. The the classical Dirichlet problem,*

$$(1.9) \qquad F[u]=0 \quad on\ \partial\Omega, \quad u=\phi \quad on\ \partial\Omega$$

is uniquely solvable, with solution $u \in C^{2,\alpha}(\bar{\Omega})$ for some $\alpha>0$ depending only on n and Λ/λ.

For the boundary value problem (1.2), (1.6) we shall prove the (almost) corresponding result.

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega \in C^{3,1}$ and suppose, in addition to* (1.3), (1.5), (1.7), (1.8) *we have*

$$(1.10) \qquad \sup_{k,\Omega} c_k + \sup_{\partial\Omega} \gamma < 0.$$

Then the classical boundary value problem,

$$(1.11) \qquad F[u]=0 \quad in\ \Omega, \quad Mu=g \quad on\ \partial\Omega,$$

is uniquely solvable with solution $u \in C^{1,1}(\bar{\Omega}) \cap C^{2,\alpha}(\Omega)$, for some $\alpha>0$ depending only on n and Λ/λ.

Boundary value problems of the type (1.9), (1.11) arise naturally in stochastic control theory and the reader is referred to the work of Lions [21, 22, 23] for a through treatment of the general control problem.

The plan of this paper is as follows. The equations (1.2) will be approximated by smooth equations of the general form

$$(1.12) \qquad F[u]=F(x,u,Du,D^2u)=0 \quad \text{in } \Omega$$

and the approximating boundary value problems (1.11) solved in the Hölder spaces $C^{2,\alpha}(\bar{\Omega})$ by the method of continuity. In Sect. 2, we derive the necessary first and second derivative bounds while in Sect. 3 we establish second derivative Hölder bounds. Theorem 1.2 is subsequently proved in Sect. 4 where we also indicate how its proof may be alternatively effected by the system approximation of [20]. We remark here that our second derivative bounds may be extended to equations (1.12) satisfying the natural structure conditions of [29]. An extension of this work to general *nonlinear* boundary value problems has been developed by Lieberman and Trudinger [16]. In a sequel [27] to this paper we shall treat $C^1(\bar{\Omega})\cap C^{1,1}(\Omega)$ solutions of Bellman-Signorini obstacle problems with application to the optimal stopping of reflected diffusion processes. In the final section of this paper we treat the case of degenerate operators, by combining the bounds of Sects. 2, 3 and the method of Lions [23], and indicate briefly the stochastic control interpretation of the results obtained here.

To conclude this introduction we remark that all notation in Sects. 2–4, unless otherwise indicated, follows the book [11].

2. Global Derivative Bounds

In this section we derive global first and second derivative bounds for solutions of the boundary value problems (1.11) where the operators F are modelled on smooth approximations to the Bellman operator (1.2). Specifically we assume that the function $F\in C^2(\Gamma)$ where $\Gamma=\Omega\times\mathbb{R}\times\mathbb{R}^n\times\$^n$, (here $\$^n$ denotes the space of symmetric $n\times n$ real matrices), and that F satisfies the following *structure* conditions:

$$\lambda I\leqq F_r(x,z,p,r);\qquad |F(x,z,p,r)|\leqq\mu_0(1+|r|); \tag{2.1}$$

$$|F_X(x,z,p,r)|\leqq\mu_1\{(1+|r|)|X'|+|X''|\}; \tag{2.2}$$

$$F_{XX}(x,z,p,r)\leqq\mu_2\{(1+|r|)|X'|+|X''|\}|X'|, \tag{2.3}$$

for all $x\in\Omega$, $|z|+|p|\leqq M_1$, $r\in\$^n$ and $X\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\times\$^n$, where $\lambda,\mu_0,\mu_1,\mu_2$, and M_1 are positive constants and X', X'' are given by

$$X'=(X_1,\dots,X_n,0,\dots,0),\qquad X''=X-X'.$$

F_X and F_{XX} denote the first and second derivatives respectively of F with respect to the vector X. Observe that (2.3) implies the concavity of F with respect to z,p,r.

Global and interior bounds for second derivatives of solutions of the Dirichlet problem for Eq. (1.10), under conditions similar to (2.1), (2.2), (2.3) are derived in [9], [7] (using the method of [15]) and by different methods in [14] and [29]. For the boundary condition (1.6), we shall adapt a key idea from [14] which involves treating pure second order directional derivatives of solutions as functions on $\Omega\times\mathbb{R}^n$. A similar application of this idea in [11, Second Edition] yielded one-sided third derivative estimates at the boundary for solutions of the Dirichlet problem; (see also [6]).

Accordingly let us now suppose that $u\in C^4(\Omega)$ satisfies (1.12) in Ω, with

$$|u|_{1;\Omega}=\sup_\Omega|u|+\sup_\Omega|Du|\leqq M_1,$$

and differentiate (1.10) twice with respect to a vector $\xi \in \mathbb{R}^n$, $|\xi| \leqq 1$. We obtain thus

(2.4) $$F_{ij} D_{ij\xi} u + F_{p_i} D_{i\xi} u + F_z D_\xi u + F_\xi = 0,$$

(2.5) $$F_{ij} D_{ij\xi\xi} u + F_{p_i} D_{i\xi\xi} u + F_z D_{\xi\xi} u + F_{\bar{X}\bar{X}} = 0,$$

where $F_{ij} = F_{r_{ij}}$ and $\bar{X} = (\xi, D_\xi u, D D_\xi u, D^2 D_\xi u)$. Using the structure conditions (2.2), (2.3), we can then estimate

(2.6) $$|F_{ij} D_{ij\xi} u| \leqq C(1 + |D^2 u|),$$

(2.7) $$F_{ij} D_{ij\xi\xi} u \geqq -C(1 + |D^2 u| + |D^2 D_\xi u|),$$

where C depends on n, M_1, μ_1 and μ_2. For local boundary estimates we first fix a point $y \in \partial\Omega$, which we can take as the origin, and flatten $\partial\Omega$ near y by means of a $C^{3,1}$ diffeomorphism ψ. The equation (1.10) and boundary condition (1.6) are transformed accordingly, the form of conditions (2.1), (2.2) and (2.3) being preserved with new constants λ, μ_1, μ_2 depending in addition on ψ. Indeed, letting $\psi = (\psi^1, \dots, \psi^n)$, $\tilde{u} = u \circ \psi^{-1}$, the transformed equation is given by $\tilde{F}[\tilde{u}] = 0$, where

$$\tilde{F}(x, z, p, r) = F(\tilde{x}, \tilde{z}, \tilde{p}, \tilde{r})$$

and

$$\tilde{x} = \psi^{-1}(x), \qquad \tilde{z} = z, \qquad \tilde{p}_i = \psi_i^k p_k,$$
$$\tilde{r}_{ij} = \psi_i^k \psi_j^l r_{kl} + \psi_{ij}^k p_k$$

Using the linearity of the above transformation with respect to z, p, r we obtain corresponding structure conditions with constants $\tilde{\lambda} = \tilde{\lambda}(\lambda, \psi)$, $\tilde{\mu}_0 = \tilde{\mu}_0(\mu_0, \psi, \mathrm{M}_1)$, $\tilde{\mu}_1 = \tilde{\mu}_1(\mu_1, \psi, M_1)$, $\tilde{\mu}_2 = \tilde{\mu}_2(\mu_1, \mu_2, \psi, M_1)$.

It therefore suffices to consider (1.12) in a half-ball

$$B_\delta^+ = \{x \in B_\delta(0) \mid x_n > 0\},$$

with boundary condition (1.6) holding on the flat boundary portion,

$$T = \{x \in B_\delta(0) \mid x_n = 0\}.$$

For $x \in B_\delta^+$, $\xi = (\xi_1, \dots, \xi_{n-1}, 0) \in \mathbb{R}^{n-1}$, $|\xi| \leqq 1$, we now consider the function w given by

(2.8) $$w(x, \xi) = \eta^2(x, \xi)(z(x, \xi) + A v')$$

where η is a smooth cut-off function to be later specified, $z(x, \xi) = D_{\xi\xi} u(x) = D_{ij} u(x) \xi_i \xi_j$, A is a constant and $v' = \sum_{i=1}^{n-1} |D_i u|^2$. To get a suitable differential inequality for w, we first observe that, by means of (2.6), (2.1) and the equation (1.12) itself, we may eliminate the derivatives $D_{nn\xi} u$ and $D_{nn} u$ from (2.7), thereby obtaining an inequality

(2.9) $$F_{ij} D_{ij} z + C_{ij} D_{ij\xi} u \geqq -C(1 + |D^2 u|')$$

with coefficients C_{ij} satisfying $C_{in}=0$, $|C_{ij}|\leqq C$ where $C=C(n,\lambda,\mu_0,\mu_1,\mu_2,M_1)$ and $|D^2 u|'=(\sum_{i+j\leqq 2n}|D_{ij}u|^2)^{\frac{1}{2}}$. Using the relations

$$D_{i\xi_j}z=2D_{ij\gamma}u, \qquad D_{\xi_i\xi_j}z=2D_{ij}u,$$

we then have for constants C_0 and C,

$$\sum_{i,j=1}^{2n-1}\tilde{F}_{ij}D_{ij}z\equiv F_{ij}D_{ij}z+\tfrac{1}{2}C_{ij}D_{i\xi_j}z+C_0\sum_{j=1}^{n-1}D_{\xi_j\xi_j}z \tag{2.10}$$
$$\geqq -C(1+|D^2u|'),$$

the extended matrix $[\tilde{F}_{ij}]$ being uniformly elliptic, (with minimum eigenvalue $\tilde{\lambda}\geqq\lambda/2$). Next by (2.6), we obtain

$$\sum_{k=1}^{n-1}F_{ij}D_{ik}u\,D_{jk}u+\tfrac{1}{2}F_{ij}D_{ij}v'\geqq -C(1+|D^2u|') \tag{2.11}$$

so that combining (2.10), (2.11) we arrive at the following differential inequality for w,

$$\eta^2\tilde{F}_{ij}D_{ij}w-2\tilde{F}_{ij}D_i\eta^2D_jw\geqq 2K\lambda(|D^2u|')^2\eta^4-6(\tilde{F}_{ij}D_i\eta D_j\eta)w \tag{2.12}$$
$$+2\eta(\tilde{F}_{ij}D_{ij}\eta)w-C(1+K)\eta^4(1+|D^2u|')\geqq A\lambda w^2-C_A$$

where C_A depends on $n,\lambda,\mu_0,\mu_1,\mu_2,M_1$ and $|\eta|_2$. By differentiating the boundary condition (1.6) twice, with respect to tangential directions, we obtain (fixing A) a corresponding inequality for w on T, namely

$$\beta_iD_iw+(\gamma-2\beta_iD_i\eta/\eta)w\geqq -C\eta^2$$

where C depends on $|\beta|_2$, $|\gamma|_2$, $|g|_2$ and M_1. At this point it is convenient to fix η by setting

$$\eta(x,\xi)=\{1-4[|x|'^2+(x_n-\varepsilon\delta)^2]/\delta^2-|\xi|^2\}^+ \tag{2.13}$$

where $\varepsilon=\kappa/\sqrt{1+\kappa^2}$, $\kappa=\sup_T|\beta|/\beta_n\leqq C$, and set $\mathcal{N}=\{(x,\xi)\in\mathbb{R}^{n-1}\,|\,\eta(x,\xi)>0\}$. It then follows that $\beta\cdot D\eta\geqq 0$ on $T\cap\partial\mathcal{N}$ and hence

$$\beta_iD_iw+\gamma w\geqq -C \tag{2.14}$$

on $T\cap\partial\mathcal{N}\cap\{w\geqq 0\}$. An upper bound for w in $\mathcal{N}$ may now be deduced from (2.12) and (2.14) by applying the maximum principle to the function $w+Cx_n/\lambda$. Consequently we obtain the one-sided tangential second derivative bound

$$D_{\xi\xi}u(0)\leqq C \tag{2.15}$$

for any $\xi=(\xi_1,\dots,\xi_{n-1},0)$, $|\xi|=1$, where C depends on η, λ, μ_0, μ_1, μ_2, M_1, Ω and $K_0=|\beta|_{1,1;\Omega}+|\gamma|_{1,1;\Omega}+|g|_{1,1;\Omega}$.

To estimate the remaining second derivatives we first observe from the inequality (2.5) that the function

$$G(x)=Mu(x)-g(x)$$

satisfies a differential inequality

$$|F_{ij} D_{ij} G| \leqq C(1+M_2), \quad (M_2 = \sup_{\Omega} |D^2 u|), \tag{2.16}$$

where C depends on n, M_1, μ_1 and K_0, and moreover by (1.7), G vanishes on $\partial\Omega$. Accordingly from [15, 111.2], G satisfies at 0 a bondary gradient estimate of the form

$$|DG(0)| \leqq C\sqrt{1+M_2}. \tag{2.17}$$

We remark here that such an estimate may also be seen by inspection of the constant dependence in the standard barrier argument [9, Theorem 14.1] or through a coordinate transformation of the form $x \to x/\sqrt{1+M_2}$. Using $(e_1, \ldots, e_{n-1}, \beta(0))$ as a basis in $\mathbb{R}^n$, instead of the canonical basis, $(e_1, \ldots, e_n)$, we thus extend the estimate (2.15) to arbitrary $\xi \in \mathbb{R}^n$. Finally by letting ξ range over the eigenvectors $(\xi_1, \ldots, \xi_n)$ of the coefficient matrix $A = [F_{ij}(0, u(0), Du(0), 0)]$ and using the Eq. (1.10) itself, together with (2.1) and the concavity of F with respect to r (2.3) we obtain

$$\begin{aligned}\sum^n \lambda_i D_{\xi_i \xi_i} u &= A \cdot D^2 u \\ &\geqq -F(0, u(0), Du(0), 0)\end{aligned}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. Hence we conclude, in the usual way, the full second

$$\sup_{\partial\Omega} |D^2 u| \leqq C \tag{2.18}$$

where C depends on the same quantities as in (2.15). Coupling (2.18) with the global Dirichlet problem bound, (see [8], [12] or [24]), now establishes a global bound

$$\sup_{\Omega} |D^2 u| \leqq C \tag{2.19}$$

where again C depends on the same quantities as in (2.15). We note here that the global and interior Dirichlet problem bounds are really implicit in the preceding considerations as is easily seen by letting B_δ be an arbitrary ball in $\mathbb{R}^n$, $\xi \in \mathbb{R}^n$ and $\varepsilon = 0$ in (2.13). This is essentially the approach of Krylov [14] and it yields for any solution $u \in C^4(\Omega) \cap C^2(\bar{\Omega})$ of (1.10), the estimate

$$|D^2 u(y)| \leqq C(1 + \sup_{B_\delta(y) \cap \partial\Omega} |D^2 u|) \tag{2.20}$$

for any $y \in \Omega$, $\delta > 0$, where C depends on n, Λ/λ, μ_0, μ_1, μ_2, δ and M_1.

For first derivative bounds the conditions (2.1), (2.2), (2.3) should be refined so that they resemble the Bellman operator with respect to p dependence. Accordingly we replace (2.11), (2.2), (2.3) by

$$(2.1)' \qquad \lambda I \leqq F_r(x,z,p,r); \qquad |F(x,z,p,r)| \leqq \mu_0(1+|p|+|r|);$$

$$(2.2)' \qquad |F_X(x,z,p,r)| \leqq \mu_1\{(1+|p|+|r|)|X'|+|X''|\};$$

$$(2.3)' \qquad F_{XX}(x,z,p,r) \leqq \mu_2\{(1+|p|+|r|)|X'|+|X''|\}|X'|,$$

for all $x \in \Omega$, $|z| \leqq M_0$, $r \in \mathbb{S}^n$. Setting $M_1 = |u|_{1;\Omega}$ and replacing u by u/M_1 we reduce to (2.1), (2.2), (2.3) with $M_1 = 1$ so that by (2.19) we have

$$\sup_\Omega |D^2 u| \leqq C(1+\sup_\Omega |Du|)$$

where now C depends on the same quantities as in (2.19) but with M_1 replaced by $M_0 = |u|_{0;\Omega}$. But then by interpolation (see [11, Lemma 6.35]), we deduce a bound for Du. Alternatively, we may proceed directly, using only conditions (2.1)′, (2.2)′ and a function w of the form

$$(2.21) \qquad w(x) = \eta^2(x,0)(v' + Bu^2),$$

in the half ball B_δ^+ where η is again given by (2.13), (see also [19]). Let us now formulate the resultant second derivative estimates.

Theorem 2.1. *Let $u \in C^4(\Omega) \cap C^3(\bar{\Omega})$ be a solution of the boundary value problem (1.10), (1.7) where $\partial\Omega \in C^{3,1}$ and F satisfies the structure conditions (2.1)′, (2.2)′, (2.3)′. Then we have the estimate*

$$(2.22) \qquad |u|_{2;\Omega} \leqq C$$

where C depends on n, λ, Λ, μ_0, μ_1, μ_2, $|u|_{0;\Omega}$, Ω and K_0.

To conclude this section we make the observation, which is important for our obstacle considerations in [27], that the one sided estimate (2.15) continues to be valid for non-linear boundary conditions of the form

$$(2.23) \qquad G[u] = G(x,u,Du) = 0 \quad \text{on } \partial\Omega$$

where $G \in C^2(\partial\Omega \times \mathbb{R} \times \mathbb{R}^n)$ satisfies

$$(2.24) \qquad G_p \cdot v \geqq \lambda$$

$$(2.25) \qquad |DG| \leqq \mu_1$$

$$(2.26) \qquad G_{XX} \leqq \mu_2 |X|^2$$

for all $x \in \Omega$, $|z|+|p| \leqq M_1$, $X \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$. We also point out that by appropriate modification of the function w, for example by taking

$$(2.27) \qquad w = \eta^2\{1 + A[v'(x) - v'(0)]\}\, z(x,\xi),$$

together with a reflection to all of B_δ, we may obtain the estimate (2.19) with

$$X' = (X_1, \dots, X_{2n+1}, 0, \dots, 0)$$

in (2.2), (2.3). As a result (2.22) will hold under the natural conditions, F1–F5 of [29].

3. Second Derivative Hölder Estimates

We come now to Hölder estimates for the second derivatives of solutions of the boundary value problem (1.11), where the function F is sufficiently smooth and concave with respect to the r variables. The corresponding interior estimates are due to Evans [7] and Krylov [13] and we shall make some use of the approach to them given in [11] and [27]. However the methods developed to handle the interior situation do not extend fully to our situation at the boundary and we invoke a divergence structure approach to overcome a crucial obstacle. To quantify our hypotheses we assume that F satisfies

(3.1) $$\lambda I \leqq F_r(x,z,p,r) \leqq \Lambda I, \quad \text{(Uniform ellipticity)};$$

(3.2) $$|DF(x,z,p,r)|, \quad |D^2 F(x,z,p,r)| \leqq \mu_2;$$

(3.3) $$F_{rr}(x,z,p,r) \leqq 0, \quad \text{(Concavity)},$$

for all $x \in \Omega$, $|z|+|p|+|r| \leqq M_2$ where λ, Λ and μ are positive constants. As in the preceding section, it suffices, by virtue of the interior estimates [7], [13], [11], and the usual flattening of $\partial\Omega$, to confine our attention to the equation (1.10) in the half-ball B_δ^+ with the boundary condition (1.7) holding on the flat boundary portion $T \subset \{x_n = 0\}$. Again we consider the tangential pure second order derivatives

$$z(x,\xi) = D_{\xi\xi} u$$

for $\xi \in \mathbb{R}^{n-1}$ but where now $|\xi|=1$ and u is normalized so that

$$0 \leqq z(x,\xi) \leqq 1, \quad \text{for all } |\xi|=1,$$

Following [11, Sect. 17.4], we introduce the functions

(3.4) $$w = w_\xi = z + \varepsilon \sum_{k=1}^{N} (D_{\xi_k \xi_k} u)^2, \quad 0 < \varepsilon < 1,$$

where $\xi_1, \dots, \xi_N$ range through a set of directions including the directions e_i, $(e_i \pm e_j)/\sqrt{2}$, $i,j = 1, \dots, n-1$, where $e_1, \dots, e_n$ denotes the canonical basis in $\mathbb{R}^n$. Using the once differentiated equation (2.4) to control third order derivatives of the form $D_{nn\xi} u$, together with the twice differentiated equation (2.5) and the concavity of F (3.3), we then obtain, as in [11, Sect. 17.4], differential inequalities of the form

(3.5) $$-\varepsilon\lambda |D^3 u|^2 + F_{ij} D_{ij} w \geqq -C_\varepsilon$$

provided $|u|_2 \leqq M_2$, where C_ε depends on n, μ_2 and K. Also by differentiation of the boundary condition (1.6) we obtain on T,

(3.6) $$|\beta_i D_i w| \leqq C_1$$

where C_1 depends on n, M_2 and $K_0 = |\beta|_{1,1} + |\gamma|_{1,1} + |g|_{1,1}$. Under hypotheses (1.5), we may further simplify (3.5) and (3.6) by setting

$$\tilde{w} = w + C_1 x_n/\lambda + C x_n^2/\lambda,$$

so that we have then

$$-\lambda\varepsilon|D^3u|^2+F_{ij}D_{ij}\tilde{w}\geqq 0 \text{ in } B_\delta^+, \quad \beta_i D_i\tilde{w}\geqq 0 \text{ on } T. \tag{3.7}$$

To convert (3.7) to a conormal divergence structure inequality (cf. [17], [18]) we first redefine $[F_{ij}]$ so that its symmetric part remains the same but now

$$F_{in}=\beta_i F_{nn}/\beta_n, \quad i=1,\dots,n. \tag{3.8}$$

By integration, we then obtain from (3.7),

$$\lambda\varepsilon\int_{B_\delta^+}|D^3u|^2\phi+\int_{B_\delta^+}F_{ij}D_i\tilde{w}D_j\phi\leqq\int_{B_\delta^+}(D_jF_{ij})\phi D_i\tilde{w}$$

for all $\phi\geqq 0$, $\in C_0^1(B_\delta)$, so that using the Schwartz inequality,

$$\int_{B_\delta^+}F_{ij}D_i\tilde{w}D_j\phi\leqq C\int_{B_\delta^+}|D\tilde{w}|^2\phi \tag{3.9}$$

where C depends on $n,\lambda,\mu,\varepsilon$ and M_2. Divergence structure results, such as the weak Harnack inequality [28, Theorem 2], may now be applied to $\tilde{w}$ but for our purposes here we need the following *projected* weak Harnack inequality in the boundary T, (cf. [31]).

Lemma 3.1. *Let* $v\geqq 0$, $\in W^{1,2}(B_\delta^+)\cap L^\infty(B_\delta^+)$ *satisfy the inequality,*

$$\int_{B_\delta^+}a^{ij}D_ivD_j\phi\geqq-\mu_0\int_{B_\delta^+}\phi|Dv|^2 \tag{3.10}$$

for all $\phi\geqq 0$, $\in C_0^1(B_\delta)$, *where* $[a^{ij}]$ *satisfies*

$$\lambda|\xi|^2\leqq a^{ij}\xi_i\xi_j\leqq\Lambda|\xi|^2$$

for all $\xi\in\mathbb{R}^n$, *and where* μ,λ,Λ *are positive constants. Then we have the estimate*

$$\delta^{1-n}\int_{T\cap B_{\delta/2}}v\leqq C\inf_{T\cap B_{\delta/2}}v \tag{3.11}$$

where C *depends on* n,λ,Λ, *and* $\exp(\mu_0\sup_{B_\delta^+}v)$.

Proof. We observe first of all that the conormal inequality (3.10) behaves with respect to test function arguments as an inequality in the full ball B_δ, with v extended as an even function of x_n. It follows then from the proof of the weak Harnack inequality [28, Theorem 2] that for any $p\in(0,\frac{1}{2})$

$$\{\delta^{-n}\int_{B_{\delta/2}^+}v^{2p}+\delta^2|Dv^p|^2\}^{\frac{1}{2}p}\leqq C\inf_{B_{\delta/2}^+}v$$

where C depends on the same quantities as in (3.11) as well as p. By the trace Sobolev inequality [1, Theorem 5.4] we thus have (for $n>2$)

$$(\delta^{1-n}\int_{T\cap B_{\delta/2}}v^\chi)^{1/\chi}\leqq C\inf_{B_{\delta/z}^+}v$$

for $\chi=2p(n-1)/(n-2)$ and hence (3.11) follows.

To use (3.10), we set $T_R = T \cap B_R$ for $R \leqq \delta$, and write for $R \leqq \delta/4$,

$$W_1 = \sup_{T_R} w, \qquad W_2 = \sup_{T_{2R}} w, \qquad \tilde{W}_2 = \sup_{B_{2R}^+} \tilde{w}.$$

Applying (3.10) to the functions $\tilde{W}_2 - \tilde{w}$ and using the estimate

$$\tilde{W}_2 - W_2 \leqq CR;$$

(see the proof of Theorem 17.26 in [9]), we thus obtain

$$R^{1-n} \int_{T_R} (W_2 - w) \leqq C(W_2 - W_1 + R + R^2) \tag{3.12}$$

where C depends on n, λ, Λ, μ, K_0, M_2 and ε.

Next we may use Krylov's boundary gradient Hölder estimate [15, Theorem 4.1] to extend (3.12) to arbitrary directions $\xi \in \mathbb{R}^n$. To see this we apply the Krylov estimate to the differential inequality (2.16) to get an estimate, for any $R \leqq \delta/2$

$$\operatorname*{osc}_{T_R} DG \leqq CR^{\alpha}$$

where $\alpha > 0$ depends only on n, λ, Λ and C depends also on μ, K_0, δ and M_2. Consequently by virtue of the Hölder continuity of $G_p = \beta$, we obtain

$$\operatorname*{osc}_{T_R} D(\beta(0) \cdot Du) \leqq CR^{\alpha}$$

and hence using $(e_1, \ldots, e_{n-1}, \beta(0))$ as a basis in $\mathbb{R}^n$, as in Sect. 2, we obtain from (3.12) the estimate

$$R^{1-n} \int_{T_R} (W_2 - w) \leqq C(W_2 - W_1 + R^{\alpha}) \tag{3.13}$$

where w is given by (3.4) for arbitrary $\xi \in \mathbb{R}^n$, $|\xi| = 1$ and C depends on n, λ, Λ, μ, K_0, M_2, δ and ε. With ε chosen sufficiently small, (for example $\varepsilon = 1/10n^2$ suffices), the argument of [11, Sect. 17.4] now applies in the $n-1$ dimensional balls T_R and we deduce finally, for any $R \leqq \delta$,

$$\operatorname*{osc}_{T_R} D^2 u \leqq CR^{\alpha}$$

where C and α are positive constants depending on n, λ, Λ, μ, K_0, M_2 and δ. A similar estimate then follows with T_R replaced by B_R^+; (see the proof of [11, Theorem 17.26]).

We therefore have the following global Hölder estimate for second derivatives.

Theorem 3.2. *Let $u \in C^4(\Omega) \cap C^3(\bar{\Omega})$ be a solution of the boundary value problem* (1.10), (1.7) *where $\partial\Omega \in C^{3,1}$ and F satisfies the structure conditions* (3.1), (3.2), (3.3) *with $M_2 = |u|_{2;\Omega}$. Then for any $\alpha \in (0,1)$ we have the estimate*

$$[D^2 u]_{\alpha;\Omega} \leqq C \tag{3.14}$$

where C depends on n, λ, Λ, μ, K_0, $|u|_{2;\Omega}$, Ω and α.

We remark that the arbitrariness of the Hölder exponent α follows from the linear L^p theory [2]. Also the proof of Theorem 3.1 clearly embraces the general nonlinear boundary conditions of the form (2.23), provided we adjoin to (3.1), (3.2), (3.3), the conditions

$$G_p \cdot v \geqq \lambda; \quad |DG|, |D^2 G| \leqq K_0. \tag{3.15}$$

4. Existence Theorems

Theorem 1.2 may be established by the method of continuity and the estimates of the preceding sections. Assuming the hypotheses of Theorem 1.2 we approximate the Hamilton-Jacobi-Bellman operator (1.2) by mollification as in the case of the Dirichlet problem [9, Sect. 17.5]. Let $\rho \geqq 0$, $\in C_0^\infty(\mathbb{R}^N)$ be a mollifier on $\mathbb{R}^N$, $N \geqq 1$ with $\int \rho = 1$, and set for $\varepsilon > 0$, $y \in \mathbb{R}^N$,

$$h_\varepsilon(y) = \varepsilon^{-n} \int_{\mathbb{R}^N} \rho\left(\frac{y-z}{\varepsilon}\right) \inf_{k=1,\ldots,N} z_k \, dz.$$

The operators F_ε, given by

$$F_\varepsilon[u] = h_\varepsilon(L^1 u, \ldots, L^N u),$$

will then satisfy the structure conditions (2.1)′, (2.2)′, (2.3)′ uniformly in ε, and the structure conditions (3.1), (3.2), (3.3) with μ_2 depending on ε (through the lower bound on F_{rr}!). Furthermore classical solutions of the boundary value problem,

$$F_\varepsilon[u] = 0 \text{ in } \Omega, \quad Mu - g = 0 \text{ on } \partial\Omega \tag{4.1}$$

will be uniformly bounded with respect to ε, by virtue of the condition (1.10). To see this we note that we can construct positive functions $w_1, w_2 \in C^2(\bar{\Omega})$ with $w_1 = c_1 - \exp(c_2 x_1)$ in Ω, $D w_2 = -c_3 v$ on $\partial\Omega$ so that $L_k w_1 \leqq -|f_k|$ in Ω, $M w_2 \leqq -|g|$ on $\partial\Omega$ for all $k = 1, \ldots, N$. Maximizing the functions $w_i \pm u$ over Ω, we obtain the bounds

$$\sup_\Omega |u| \leqq \sup \{|w_1| + |g - M w_1|/\gamma_0\} \quad \text{if } \gamma_0 = -\sup \gamma > 0, \tag{4.2}$$

$$\sup_\Omega |u| \leqq \sup_{\substack{\Omega \\ k=1,\ldots,N}} \{|w_2| + |f_k - L_k w_2|/c_0\} \quad \text{if } c_0 = -\sup c > 0. \tag{4.3}$$

It then follows from the method of continuity, as presented for example in [11, Theorem 17.28], that the boundary value problem (4.1) is uniquely solvable with solution $u \in C^3(\Omega) \cap C^{2,\alpha}(\bar{\Omega})$ for any $\alpha > 0$. But since the global C^2 bounds (Theorem 2.1) and the Evans-Krylov *interior* $C^{2,\beta}(\Omega)$ bounds for sufficiently small $\beta = \beta(n, \Lambda/\lambda)$, [11, Theorem 17.14] are independent of ε, we thus obtain by approximation, a solution $u \in C^{1,1}(\bar{\Omega}) \cap C^{2,\beta}$ of the boundary value problem,

$$\inf_{k=1,\ldots,N} (L_k u - f_k) = 0 \text{ in } \Omega, \quad Mu = g \text{ on } \partial\Omega. \tag{4.4}$$

Letting $N \to \infty$, yields the result asserted in Theorem 1.2.

As mentioned in the introduction, we could have alternatively proved Theorem 1.2 by approximation with a weakly coupled system of semi-linear equations,

$$L_k u = f_k + \beta_\varepsilon(u_k - u_{k+1}), \quad k = 1, \dots, N, \quad (u_{N+1} = u_1), \tag{4.5}$$

where $\beta_\varepsilon \in C^2(\mathbb{R})$ is a penalty function satisfying $\beta_\varepsilon(t) = 0$ for $t \leqq 0$, $\beta_\varepsilon(t) \to \infty$ as $\varepsilon \to 0$, for $t > 0$, β'_ε, $\beta''_\varepsilon \geqq 0$. The methods of Sect. 2 readily extend to yield global C^2 bounds for the solutions u_k which are independent of ε and k and a $C^{1,1}(\bar{\Omega})$ solution of our given problem (1.9) now results by the limit argument of [8] and [20]. The interior $C^{2,\alpha}$ estimates of Evans [7] and Krylov [13] imply that such a solution $u \in C^{2,\beta}(\Omega)$ again for some $\beta = \beta(n, \Lambda/\lambda)$.

Note that the above considerations also yield the existence of a unique solution $u \in C^{2,\alpha}(\bar{\Omega})$, $0 < \alpha < 1$, of the boundary value problem (1.12), (1.6) when $F \in C^2(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n)$ satisfies (2.1)′, (2.2)′, (2.3)′ together with $\sup F_z < 0$. By virtue of our remark at the end of Sect. 2 these conditions extend to embrace the natural structure conditions of [29].

5. Applications to Optimal Stochastic Control

We briefly sketch here the stochastic interpretation of Theorem 1.2. First we explain the control problem corresponding to (1.2), (1.6): let (Π, P) be a probability space endowed with a filtration $(\mathscr{F}, \mathscr{F}_t)$ satisfying the usual assumptions and a continuous adapted Brownian motion B_t in $\mathbb{R}^m$. We suppose that the state of the system we wish to control is given through the solution X_t of the following stochastic differential equation with reflecting boundary conditions

$$\begin{cases} dX_t = \sigma(X_t, \alpha_t)\, dB_t + b(X_t, \alpha_t)\, dt + \beta(X_t)\, dL_t, \\ X_0 = x,\ X_t \in \bar{\Omega} \text{ for all } t \geqq 0,\ X_t, L_t \text{ are continuous,} \\ \qquad \text{adapted, } L_t \text{ is nondecreasing} \\ \text{and } L_t = \int_0^t 1_{\partial\Omega}(X_s)\, dL_s; \end{cases} \tag{5.1}$$

where $\sigma^{ij}, b^i, \beta^i$ $(1 \leqq i \leqq n,\ 1 \leqq j \leqq m)$ satisfy the conditions listed below. Here, α_t is the control process that we assume to be some arbitrary progressively measurable process with values in a given separable metric space $\mathscr{A}$. Problems of the type (5.1) are treated in Ikeda and Watanabe [12], Bensoussan and Lions [3], Lions and Sznitman [26]....

We next consider the cost function

$$J(x, \alpha_t) = E \int_0^\infty f(X_t, \alpha_t)\, e^{-r_t}\, dt + E \int_0^\infty g(X_t)\, e^{-r_t}\, dL_t \tag{5.2}$$

where $r_t = \int_0^t c(X_s, \alpha_s)\, ds + \int_0^t \gamma(X_s)\, dL_s$, c, γ are nonnegative given functions on $\bar{\Omega} \times \mathscr{A}, \bar{\Omega}$ and f, g are given real functions on $\bar{\Omega} \times \mathscr{A}, \bar{\Omega}$. We assume that $\phi = \sigma^{ij}$,

b^i, β^i, f, g, c, γ $(1 \leqq i \leqq n, 1 \leqq j \leqq m)$ satisfy

$$|\phi(\cdot,\alpha)|_{1,1;\bar{\Omega}} \leqq K, \quad \alpha \in \mathscr{A}; \quad \phi(x,\alpha) \in C(\bar{\Omega} \times \mathscr{A}) \tag{5.3}$$

$$\beta(x) \cdot v \geqq \lambda > 0 \quad \text{on } \partial\Omega, \tag{5.4}$$

for some positive constant λ.

Finally, we introduce the value function

$$u(x) = \inf_{\alpha_t} J(x, \alpha_t) \tag{5.5}$$

where the infimum is taken over all controls α_t.

Dynamic programming arguments indicate (cf. [10, 13, 22]) that u should "solve"

$$\inf_{\alpha_t \in \mathscr{A}} (L_\alpha u + f_\alpha) = 0 \quad \text{in } \Omega \tag{5.6}$$

where

$$L_\alpha = a_\alpha^{ij} D_{ij} + b_\alpha^i D_i - c_\alpha$$

and $\phi_\alpha(\cdot) = \phi(\cdot, \alpha)$ for $\phi = a^{ij}, b^i, f, c$, $(1 \leqq i,j \leqq n)$, and where $a^{ij} = \frac{1}{2} \sigma^{ik} \sigma^{jk}$ $(1 \leqq i,j \leqq n)$. Furthermore, u should satisfy the boundary condition

$$\beta^i \cdot D_i u - \gamma u + g = 0 \quad \text{on } \partial\Omega \tag{5.7}$$

(at least if L_α is uniformly elliptic near $\partial\Omega$).

Observe that (5.6)–(5.7) is nothing but (1.2), (1.6) (choosing a dense family in $\mathscr{A}$). An immediate application of Theorem 1.2 yields the

Corollary 5.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega \in C^{3,1}$ and suppose* (5.3), (5.4) *and*

$$\inf_{\alpha,\Omega} c_\alpha + \inf_{\partial\Omega} \gamma > 0 \tag{5.8}$$

$$\lambda |\xi|^2 \leqq a_\alpha^{ij} \xi_i \xi_j \leqq \Lambda |\xi|^2, \quad \textit{for all } \xi \in \mathbb{R}^n,\ x \in \Omega,\ \alpha \in \mathscr{A} \tag{5.9}$$

for some positive constants λ, Λ. Then $u \in C^{1,1}(\bar{\Omega}) \cap C^{2,\theta}(\Omega)$ (for some $\theta \in]0,1[$ depending only on n and λ/Λ) and u is the unique solution of (5.6)–(5.7).

We skip the proof of Corollary 4.2 since, in view of Theorem 1.2, there exists a solution $\tilde{u}$ of (5.6)–(5.7) and one checks by an easy use of Itô's formula that $\hat{u} \equiv u$ (see for similar proofs the verification theorems in [24, 13, 10] ...).

We finally conclude by considering the case of degenerate operators $(L_\alpha)_{\alpha \in \mathscr{A}}$ and we assume:

$$\partial\Omega = \Gamma_+ \cup \Gamma_- \text{ with } \Gamma_+, \Gamma_- \text{ closed, disjoint, possibly empty;} \tag{5.10}$$

$$\lambda |\xi|^2 \leqq a_\alpha^{ij} \xi_i \xi_j \leqq \Lambda |\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^n,\ x \in \Gamma_+,\ \alpha \in \mathscr{A} \tag{5.11}$$

for some positive constants λ, Λ;

$$a_\alpha^{ij} v_i v_j = 0 \text{ on } \Gamma_-, \quad -b_\alpha^i v_i - a_\alpha^{ij} D_{ij} d \leqq 0 \text{ on } \Gamma_- \tag{5.12}$$

for all $\alpha \in \mathscr{A}$, where $d(x) = \operatorname{dist}(x, \partial\Omega)$; and

$$c_\alpha \geqq c > \lambda_0 \tag{5.13}$$

for some positive constant c, where λ_0 is a constant depending only on $D_x \sigma, D_x b$ (cf. [24, 26] for explicit formula).

In view of (5.3), (5.11) holds for $x \in \mathcal{O}$ (replacing possibly λ by $\lambda/2$, Λ by 2Λ) where $\mathcal{O}$ is some smooth open set included in Ω, such that $\Gamma_+ \subset \partial\mathcal{O}$. By easy approximation arguments, one shows using the bounds obtained in the preceding sections that $u \in C^{1,1}(\bar{\mathcal{O}}) \cap C^{2,\theta}(\mathcal{O})$ (for some $\theta \in (0,1)$). Then, using (5.12), (5.13), we follow the method of Lions [20] and we obtain the

Theorem 5.2. *We assume* (5.10), (5.11), (5.12), (5.13). *Then the value function u is the unique function in $C^{0,1}(\bar{\Omega})$ satisfying*

(5.14) $$u \in C^{1,1}(\bar{\mathcal{O}}) \cap C^{2,\theta}(\mathcal{O}), \quad \beta^i D_i u - \gamma u + g = 0 \ \text{on} \ \Gamma_+,$$

(5.15) *u is semiconcave in Ω* i.e. *$u - \frac{1}{2} c |x|^2$ is concave, for some $c > 0$,*

(5.16) $$L_\alpha u \in L^\infty(\Omega) \quad \text{and} \quad \sup_{\alpha \in \mathcal{A}} |L_\alpha u| \in L^\infty(\Omega),$$

(5.17) $$\inf_{\alpha \in \mathcal{A}} (L_\alpha u + f_\alpha) = 0 \quad \text{a.e. in } \Omega.$$

More general stochastic control problems involving optimal stopping and boundary controls are considered in [27]. Also the function u in Theorems 1.2, 5.1 is shown in [30] to belong to $C^{2,\theta}(\bar{\Omega})$; (see also [27] for a consequent alternative derivation via interpolation of the $C^{1,1}(\bar{\Omega})$ bounds).

References

1. Adams, R.A.: Sobolev Spaces, New York: Academic Press 1975
2. Agmon, S., Douglis, A., Nirenberg, L.: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. Commun. Pure Appl. Math. **12**, 623–727 (1959), II. **17**, 35–92 (1964)
3. Bensoussan, A., Lions, J.L.: Contrôle impulsionnel et inèquations quasi variationnelles. Paris: Dunod 1982
4. Brezis, H., Evans, L.C.: A variational inequality approach to the Bellman-Dirichlet equation for two elliptic operators. Arch. Rational. Mech. Anal. **71**, 1–13 (1979)
5. Caffarelli, L., Nirenberg, L., Spruck, J.: The Dirichlet problem for nonlinear second order elliptic equations. I. Monge Ampère equations. Commun. Pure Appl. Math. **37**, 369–402 (1984)
6. Caffarelli, L., Kohn, J.J., Nirenberg, L., Spruck, J.: The Dirichlet problem for nonlinear second order elliptic equations. II. Complex Monge Ampère and uniformly elliptic equations. Commun. Pure Appl. Math. **38**, 209–252 (1985)
7. Evans, L.C.: Classical solutions of the Hamilton-Jacobi-Bellman equation for uniformly elliptic operators. Trans. Am. Math. Soc. **275**, 245–255 (1983)
8. Evans, L.C., Friedman, A.: Optimal stochastic switching and the Dirichlet problem for the Bellman equation. Trans. Am. Math. Soc. **253**, 365–389 (1979)
9. Evans, L.C., Lions, P.-L.: Résolution des equations de Hamilton-Jacobi-Bellman pour des opérateurs uniformément elliptiques. C.R. Acad. Sci. Paris, **290**, 1049–1052 (1980)
10. Fleming, W.H., Rishel, R.: Deterministic and stochastic optimal control. Berlin Heidelberg New York: Springer 1975
11. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. First edition. Berlin-Heidelberg-New York: Springer 1977. Second edition: 1983
12. Ikeda, N., Watanabe, S.: Stochastic differential equations and diffusion processes. Amsterdam: North-Holland 1981
13. Krylov, N.V.: Controlled Diffusion Processes. Berlin-Heidelberg-New York: Springer 1980

14. Krylov, N.V.: Boundedly inhomogeneous elliptic and parabolic equations. Izv. Akad. Nauk. SSSR **46**, 487-523 (1982)
15. Krylov, N.V.: Boundedly inhomogenous elliptic and parabolic equations in a domain. Izv. Akad. Nauk. SSSR **47**, 75-108 (1983)
16. Lieberman, G.M., Trudinger, N.S.: Nonlinear oblique boundary value problems for nonlinear elliptic equations, to appear.
17. Lieberman, G.M.: The nonlinear oblique derivative problem for quasilinear elliptic equations. Nonlinear Anal. **8**, 49-65 (1984)
18. Lions, J.L.: Sur les problèmes aux limites du type derivées obliques. Ann. of Math. **64**, 207-239 (1959)
19. Lions, P.L.: Résolution de problèmes elliptiques quasilineaires. Arch. Rational Mech. Anal. **74**, 335-353 (1980)
20. Lions, P.L.: Résolution analytique des problèmes de Bellman-Dirichlet. Acta Math. **146**, 151-166 (1981)
21. Lions, P.L.: Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. I. Comm. Part. Diff. Equats. 8, 1101-1174 (1983), II. Commun. Partial. Differ. Equations **8**, 1229-1276 (1983)
22. Lions, P.L.: Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. III. In Nonlinear Partial Differential Equations and their applications. Collége de France Seminar, Vol. V, London: Pitman 1984
23. Lions, P.L.: Hamilton-Jacobi-Bellman equations and the optimal control of stochastic processes. Proc. Int. Cong. Math., Warsaw (1983)
24. Lions, P.L.: On the Hamilton-Jacobi-Bellman equations. Acta Applicandae **1**, 17-41 (1983)
25. Lions, P.L., Menaldi, J.L.: Problèmes de Bellman avec les contrôles dans les coefficients de plus haut degré. C.R. Acad. Sci. Paris **287**, 409-412 (1978)
26. Lions, P.L., Sznitman, A.S.: Stochastic differential equations with reflecting boundary conditions. Commun. Pure Appl. Math. **37**, 511-537 (1984)
27. Lions, P.L., Trudinger, N.S.: Optimal control of reflected diffusion processes with optimal stopping. To appear
28. Trudinger, N.S.: On Harnack type inequalities and their application to quasilinear elliptic equations. Commun. Pure Appl. Math. **20**, 721-747 (1967)
29. Trudinger, N.S.: Fully nonlinear, uniformly elliptic equations under natural structure conditions. Trans. Am. Math. Soc. **278**, 751-769 (1983)
30. Trudinger, N.S.: Some recent results on fully nonlinear elliptic equations. Proc. Centre for Math. Anal. Australian National Univ. **9**, 65-83 (1984)
31. Ziemer, W.P.: Mean values of subsolutions of elliptic and parabolic equations. Trans. Am. Math. Soc. **279**, 555-568 (1983)

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