PROOF OF THE VAN DER WAERDEN CONJECTURE REGARDING THE PERMANENT OF A DOUBLY STOCHASTIC MATRIX

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In 1926 van der Waerden formulated the following conjecture [1-3]: the permanent of every double stochastic (n × n) matrix is not less than $n!/n^n$.

Since then various results have been obtained regarding the van der Waerden conjecture. In particular, it has been proved for $n \leqslant 5$. In [2] one has proved the following: if on the set of all doubly stochastic (n × n) matrices, the permanent attains its least value at a matrix without zero elements, then its permanent is equal to $n!/n^n$.

The field of real numbers will be denoted by $\mathbf{R}$. Let $X = (x_{ij})$ be a matrix of dimension (n × n) with elements from $\mathbf{R}$. If the conditions

$$x_{ij} \geqslant 0 \quad (1 \leq i,\ j \leqslant n),$$
$$x_{i1} + \ldots + x_{in} = 1 \quad (1 \leqslant i \leqslant n),$$
$$x_{1j} + \ldots + x_{nj} = 1 \quad (1 \leqslant j \leqslant n),$$

hold, then the matrix X is said to be doubly stochastic. The set of all doubly stochastic matrices of order n will be denoted by Ω_n. This is a closed subset of $\mathbf{R}^{n^2}$. Moreover, it is bounded: if $X = (x_{ij}) \in \Omega_n$, then $0 \leqslant x_{ij} \leqslant 1$ for $1 \leqslant i, j \leqslant n$. Consequently, Ω_n is compact. We consider now the set

$$\Omega_n^* = \{X = (x_{ij}) \in \Omega_n \mid x_{ij} \neq 0 \text{ for } 1 \leqslant i,\ j \leqslant n\}.$$

If $X = (x_{ij}) \in \Omega_n$ and $\delta > 0$, then

$$X_\delta = \left(\frac{x_{ij} + \delta}{1 + n\delta}\right) \in \Omega_n^*.$$

Since $X_\delta \to X$ for $\delta \to 0$, it follows that Ω_n^* is dense in Ω_n.

On the set of all (n × n) matrices with elements from $\mathbf{R}$ we define two numerical functions: the product and the permanent. Namely, let $X = (x_{ij})$ be an (n × n) matrix with elements from $\mathbf{R}$. Then

$$\Pi(X) = \prod_{1 \leqslant i,j \leqslant n} x_{ij} \text{ and } \operatorname{per}(X) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} x_{i\sigma(i)},$$

where S_n is the set of all n! permutations of the set $\{1, \ldots, n\}$. One can verify that the permanent is a symmetric semilinear function of the rows (columns).

<u>THEOREM 1.</u> If $X \in \Omega_n$, then $\operatorname{per}(X) \geqslant n!/n^n$.

In other words, the van der Waerden conjecture is true. The proof is based on a series of lemmas. First we define a certain family of functions.

We take $\varepsilon \in \mathbf{R}$. Let $X = (x_{ij})$ be an (n × n) matrix with elements from $\mathbf{R}$ such that $x_{ij} \neq 0$ for $1 \leqslant i, j \leqslant n$. We set

$$F_\varepsilon(X) = \operatorname{per}(X) + \varepsilon/\Pi(X).$$

Thus, the function F_ε is defined on the open set $(\mathbf{R} - 0)^{n^2} \subset \mathbf{R}^{n^2}$, and on this set it is continuous and differentiable. Since $(1/x)' = -1/x^2$ for $x \neq 0$, we have

$$\frac{\partial F_\varepsilon}{\partial x_{ij}} = \operatorname{per}(X_{ij}) - \frac{\varepsilon}{x_{ij}\,\Pi(X)}, \tag{1}$$

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where X_{ij} is the matrix obtained from the matrix X by deleting the i-th row and the j-th column.

LEMMA 1. Let $\varepsilon > 0$. Then there exists a point of minimum of the function F_ε in the set Ω_n^*.

Proof. We set $F = F_\varepsilon$. Since $\varepsilon > 0$, for all $X \in \Omega_n^*$ we have $F(X) > 0$. Therefore, there exists the infimum

$$a = \inf_{X \in \Omega_n^*} F(X), \quad a \geqslant 0.$$

We select a sequence $(A_m)_{m \geqslant 1}$ of elements of the set Ω_n^*, for which $F(A_m) \to a$ for $m \to +\infty$. Since $A_m \in \Omega_n^* \subset \Omega_n$, and the set Ω_n is compact, it follows, switching to a subsequence, that one can assume that $A_m \to A \in \Omega_n$ for $m \to +\infty$. If $A \notin \Omega_n^*$, then $\Pi(A) = 0$ and $\Pi(A_m) \to 0$. Then from the inequality

$$F(A_m) \geqslant \varepsilon / \Pi(A_m)$$

and from the condition $\varepsilon > 0$ we obtain the contradiction $F(A_m) \to +\infty$. Consequently, $A \in \Omega_n^*$, $F(A_m) \to F(A)$, $a = F(A)$. The lemma is proved.

LEMMA 2. Let $\varepsilon > 0$ and let $A = (a_{ij}) \in \Omega_n^*$ be a point of minimum of the function F_ε in the set Ω_n^*. Then $A = (1/n)$.

Proof. We set $p = \operatorname{per}(A)$, $c = \varepsilon/\Pi(A)$; $p_{ij} = \operatorname{per}(A_{ij})$, $d_{ij} = \left.\frac{\partial F_\varepsilon}{\partial x_{ij}}\right|_{X=A}$ $(1 \leqslant i, j \leqslant n)$. We fix $1 < i, j \leqslant n$ and we consider the matrix

$$A_h = \begin{pmatrix} a_{11}+h & \cdots & a_{1j}-h & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1}-h & \cdots & a_{ij}+h & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where $h \in \mathbf{R}$, which differs from the matrix A only by the elements with the indices (1, 1), (1, j), (i, 1), (i, j). We take $\rho = \min(a_{11}, a_{1j}, a_{i1}, a_{ij}) > 0$. If $|h| < \rho$, then $A_h \in \Omega_n^*$, so that

$$F_\varepsilon(A_h) = F_\varepsilon(A) + (d_{11} - d_{1j} - d_{i1} + d_{ij})h + o(h) \geqslant F_\varepsilon(A).$$

Consequently, $d_{11} - d_{1j} - d_{i1} + d_{ij} = 0$ for $1 < i, j \leqslant n$. If i = 1 or j = 1, then the last equality is automatically satisfied so that it is valid for all $1 \leqslant i, j \leqslant n$. Thus,

$$d_{ij} = d_{i1} + d_{1j} - d_{11} \quad (1 \leqslant i, j \leqslant n).$$

We set $\lambda_i = d_{i1} - d_{11}$ for $1 \leqslant i \leqslant n$, $\mu_j = d_{1j}$ for $1 \leqslant j \leqslant n$. Then $d_{ij} = \lambda_i + \mu_j$ for $1 \leqslant i, j \leqslant n$. We have obtained the Lagrange condition of the relative minimum.

Then, from (1) it follows that

$$p_{ij} - c/a_{ij} = \lambda_i + \mu_j \quad (1 \leqslant i, j \leqslant n). \tag{2}$$

We prove that $\lambda_1 = \ldots = \lambda_n$ and $\mu_1 = \ldots = \mu_n$, i.e., the right-hand side in (2) is a constant. We multiply both sides of (2) by a_{ij}:

$$a_{ij}p_{ij} - c = \lambda_i a_{ij} + \mu_j a_{ij} \quad (1 \leqslant i, j \leqslant n). \tag{3}$$

We consider $i\ (1 \leqslant i \leqslant n)$ fixed and we sum (3) with respect to j. Making use of the expansion of the permanent of the matrix A relative to the i-th row and taking into account the equality $\sum_{j=1}^n a_{ij} = 1$, we obtain

$$\sum_{j=1}^n a_{ij}p_{ij} - nc = p - nc = \lambda_i \sum_{j=1}^n a_{ij} + \sum_{j=1}^n \mu_j a_{ij} = \lambda_i + \sum_{j=1}^n \mu_j a_{ij}.$$

Setting $b = p - nc$, we have

$$\lambda_i = b - \sum_{j=1}^n \mu_j a_{ij} \quad (1 \leqslant i \leqslant n). \tag{4}$$

Similarly, we consider $j\ (1 \leqslant j \leqslant n)$ fixed and we sum (3) relative to i. Making use of the expansion of the permanent of the matrix A relative to the j-th column and taking into account the equality $\sum_{i=1}^n a_{ij} = 1$, we obtain

$$\sum_{i=1}^{n} a_{ij}p_{ij} - nc = p - nc = \sum_{i=1}^{n} \lambda_i a_{ij} + \mu_j \sum_{i=1}^{n} a_{ij} = \sum_{i=1}^{n} \lambda_i a_{ij} + \mu_j.$$

From here

$$\mu_j = b - \sum_{i=1}^{n} \lambda_i a_{ij} \quad (1 \leqslant j \leqslant n). \tag{5}$$

Now we express λ_i $(1 \leqslant i \leqslant n)$ in terms of $\lambda_1, \ldots, \lambda_n$. To this end, in the equality (4) we insert (5):

$$\begin{aligned} \lambda_i &= b - (a_{i1}\mu_1 + \ldots + a_{in}\mu_n) \\ &= b - [a_{i1}(b - a_{11}\lambda_1 - \ldots - a_{n1}\lambda_n) + \ldots + \\ &\quad + a_{in}(b - a_{1n}\lambda_1 - \ldots - a_{nn}\lambda_n)] = \\ &= (a_{i1}a_{11} + \ldots + a_{in}a_{1n})\lambda_1 + \ldots + (a_{i1}a_{n1} + \ldots + a_{in}a_{nn})\lambda_n. \end{aligned}$$

Thus, $\lambda_i = b_{i1}\lambda_1 + \ldots + b_{in}\lambda_n$, and $b_{i1}, \ldots, b_{in} > 0$ and $b_{i1} + \ldots + b_{in} = 1$. Then, taking $\lambda = \min(\lambda_1, \ldots, \lambda_n)$, we have $\lambda_1 \geqslant \lambda, \ldots, \lambda_n \geqslant \lambda$ and for some i we have $\lambda_i = \lambda$. For this i we write:

$$\lambda_i - \lambda = b_{i1}(\lambda_1 - \lambda) + \ldots + b_{in}(\lambda_n - \lambda) = 0,$$

and therefore $\lambda_1 = \lambda, \ldots, \lambda_n = \lambda$. Inserting these values into (5), we see that $\mu_j = b - \lambda$ for $1 \leqslant j \leqslant n$. Thus, (2) leads to the equality

$$p_{ij} = b + c/a_{ij} \quad (1 \leqslant i, j \leqslant n).$$

We have not used yet the condition $\varepsilon \geqslant 0$. From it we obtain that $c > 0$. Thus, in order to conclude the proof of Lemma 2, it is sufficient to show that we have

<u>LEMMA 3.</u> Assume that for the matrix $A = (a_{ij}) \in \Omega_n^*$ we have

$$\operatorname{per}(A_{ij}) = b + c/a_{ij} \quad (1 \leqslant i, j \leqslant n), \tag{6}$$

where $b, c \in \mathbf{R}$, $c \geqslant 0$. Then A = (1/n).

In turn, for the proof of Lemma 3 it is necessary to introduce and to investigate some symmetric bilinear forms on $\mathbf{R}^n$. This is done in the following two lemmas.

<u>LEMMA 4.</u> Let E be a vector space over $\mathbf{R}$ and let $f\colon E \times E \to \mathbf{R}$ be a symmetric bilinear form. We assume that there exists a vector $a \in E$ for which $f(a, a) = 0$ and, if $x \in E, x \notin \mathbf{R}a$, $f(x, a) = 0$, then $f(x, x) < 0$. In this case, if $t, s \in E$, $t \neq 0$, $f(t, s) = 0$, $f(s, s) > 0$, then $f(t, t) < 0$.

<u>Proof.</u> We note that if $x \in E$, $f(x, a) = 0$, then $f(x, x) \leqslant 0$. Indeed, either $x \notin \mathbf{R}a$, when $f(x, x) < 0$, or $x \in \mathbf{R}a$, when f(x, x) = 0. In particular, $f(s, a) \neq 0$.

We select a number $\eta \in \mathbf{R}$ for which $f(t + \eta s, a) = f(t, a) + \eta f(s, a) = 0$. Therefore $f(t + \eta s, t + \eta s) = f(t, t) + \eta^2 f(s, s) \leqslant 0$. If $\eta \neq 0$, then we obtain $f(t, t) \leqslant -\eta^2 f(s, s) < 0$. Let $\eta = 0$; consequently, $f(t, a) = 0$. If $t \notin \mathbf{R}a$, then $f(t, t) < 0$. We show that the inclusion $t \in \mathbf{R}a$ is not possible. Indeed, let $t = \tau a$, $\tau \in \mathbf{R}$. Then $f(t, s) = \tau f(a, s) = 0$, whence $\tau = 0$, $t = 0$, which is a contradiction. Thus, $f(t, t) < 0$. Lemma 4 is proved.

<u>LEMMA 5.</u> Let C = (c_{ij}) be an $(n - 2) \times n$ matrix with elements from $\mathbf{R}$ and such that $c_{ij} > 0$ for $1 \leqslant i \leqslant n - 2$, $1 \leqslant j \leqslant n$. We define a symmetric bilinear form $f\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ in the following manner:

$$f(x, y) = \operatorname{per}\begin{pmatrix} x_1 & \ldots & x_n \\ y_1 & \ldots & y_n \\ \ldots & \ldots & \ldots \\ c_{i1} & \ldots & c_{in} \\ \ldots & \ldots & \ldots \end{pmatrix}$$

for all $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, $y = (y_1, \ldots, y_n) \in \mathbf{R}^n$. In this case, if $t, s \in \mathbf{R}^n$, $t \neq 0$, $f(t, s) = 0$, $f(s, s) > 0$, then $f(t, t) < 0$.

<u>Proof.</u> We use induction on $n \geqslant 2$. For n = 2 we write $t = (t_1, t_2)$, $s = (s_1, s_2)$. We have $f(t, s) = t_1 s_2 + t_2 s_1 = 0$, $f(s, s) = 2 s_1 s_2 > 0$. In particular, $s_1 \neq 0$, $s_2 \neq 0$. If $t_1 = 0$, then $t_2 s_1 = 0$, $t_2 = 0$, $t = 0$, which is a contradiction. Thus, $t_1 \neq 0$. We multiply both sides of the equality $t_2 s_1 = -t_1 s_2$ by $t_1 s_2$. Then $t_1 t_2 s_1 s_2 = -t_1^2 s_2^2 < 0$, and, consequently, $t_1 t_2 < 0$. Thus, $f(t, t) = 2 t_1 t_2 < 0$, so that for n = 2 the lemma is proved.

Assume now that n > 2 and assume that for $n - 1$ the lemma holds. We select the vector $e_n = (0, \ldots, 0, 1) \in \mathbf{R}^n$. Then $f(e_n, e_n) = 0$. Let $x = (x_1, \ldots, x_{n-1}, x_n) \in \mathbf{R}^n$ and also $x \notin \mathbf{R}e_n$, $f(x, e_n) = 0$. We consider the vector $x' = (x_1, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}$. Since $x \notin \mathbf{R}e_n$, we have $x' \neq 0$, and since $f(x, e_n) = 0$, we have

$$\mathrm{per}\begin{pmatrix} x_1 & \dots & x_{n-1} & x_n \\ 0 & \dots & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot \\ c_{i1} & \dots & c_{i,n-1} & c_{in} \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \mathrm{per}\begin{pmatrix} x_1 & \dots & x_{n-1} \\ \cdot & \cdot & \cdot \\ c_{i1} & \dots & c_{i,n-1} \\ \cdot & \cdot & \cdot \end{pmatrix} = 0. \tag{7}$$

We write

$$f(x, x) = \mathrm{per}\begin{pmatrix} x_1 & \dots & x_{n-1} & x_n \\ x_1 & \dots & x_{n-1} & x_n \\ \cdot & \cdot & \cdot & \cdot \\ c_{i1} & \dots & c_{i,n-1} & c_{in} \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

and we expand the permanent with respect to the last column. Taking into account (7), we have

$$f(x, x) = \sum_{i=1}^{n-2} c_{in} f_i(x', x'), \tag{8}$$

where $f_i: \mathbf{R}^{n-1} \times \mathbf{R}^{n-1} \to \mathbf{R}$ is a form constructed over the matrix C_{in} obtained from the matrix C by the deletion of the i-th row and the n-th column. We consider the vector $c_i' = (c_{i1}, \dots, c_{i,n-1}) \in \mathbf{R}^{n-1}$. Equality (7) can be written as: $f_i(x', c'_i) = 0$. In addition, $f_i(c_i', c_i') > 0$, $x' \neq 0$. Therefore, by the inductive hypothesis, $f_i(x', x') < 0$ for $1 \leq i \leq n-2$. Now, from expansion (8) it follows that $f(x, x) < 0$.

Thus, the form $f: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ satisfies the conditions of Lemma 4 (for $\alpha = e_n$). Consequently, if $t, s \in \mathbf{R}^n$, $t \neq 0$, $f(t, s) = 0$, $f(s, s) > 0$, then $f(t, t) < 0$. Lemma 5 is proved.

Proof of Lemma 3. We select in the matrix A two rows $u = (u_1, \dots, u_n) \in \mathbf{R}^n$, $v = (v_1, \dots, v_n) \in \mathbf{R}^n$:

$$A = \begin{pmatrix} \cdot & \cdot & \cdot \\ u_1 & \dots & u_n \\ \cdot & \cdot & \cdot \\ v_1 & \dots & v_n \\ \cdot & \cdot & \cdot \end{pmatrix}.$$

Since $A \in \Omega_n^*$, we have $u_i > 0$, $v_i > 0$ for $1 \leq i \leq n$, $\sum_{i=1}^n u_i = 1$, $\sum_{i=1}^n v_i = 1$. We show that $u = v$.

In matrix A we consider all the remaining rows fixed and we define a symmetric bilinear form $f: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ in the following manner:

$$f(x, y) = \mathrm{per}\begin{pmatrix} \cdot & \cdot & \cdot \\ x_1 & \dots & x_n \\ \cdot & \cdot & \cdot \\ y_1 & \dots & y_n \\ \cdot & \cdot & \cdot \end{pmatrix}$$

for all $x = (x_1, \dots, x_n) \in \mathbf{R}^n$, $y = (y_1, \dots, y_n) \in \mathbf{R}^n$.

Thus, in the last matrix and in matrix A, the selected rows occupy the same places and all the remaining rows coincide.

Assume, as usual, that $e_1, \dots, e_n$ is the standard basis of the space $\mathbf{R}^n$ so that for every vector $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ we have $x = \sum_{i=1}^n x_i e_i$.

From conditions (6) it follows that

$$f(e_i, v) = b + c/u_i, \; f(u, e_i) = b + c/v_i \; (1 \leq i \leq n).$$

We consider the vector $t = (t_1, \dots, t_n) \in \mathbf{R}^n$, where $t_i = u_i - v_i$ for $1 \leq i \leq n$. Then $t = u - v$ and

$$f(t, e_i) = f(u, e_i) - f(v, e_i) = c\left(\frac{1}{v_i} - \frac{1}{u_i}\right) = \frac{c}{u_i v_i} t_i \; (1 \leq i \leq n).$$

Therefore

$$f(t, t) = \sum_{i=1}^n f(t, e_i) t_i = c \sum_{i=1}^n \frac{t_i^2}{u_i v_i} \geqslant 0.$$

We consider the vector $s = (s_1, \ldots, s_n) \in \mathbf{R}^n$, where $s_i = u_i v_i > 0$ for $1 \leqslant i \leqslant n$. We have

$$f(t, s) = \sum_{i=1}^{n} f(t, e_i) s_i = c \sum_{i=1}^{n} t_i = 0.$$

In addition, $f(s, s) > 0$. Since Lemma 5 can be applied to the form f, from the conditions $f(t, t) \geqslant 0$, $f(t, s) = 0$, $f(s, s) > 0$ there follows that t = 0, u = v.

Thus, all the rows of matrix A are mutually equal. Since their sum is the row (1, . . ., 1), we have A = (1/n). Lemma 3 and, simultaneously, Lemma 2 are proved.

Proof of Theorem 1. We select $\varepsilon > 0$. From Lemmas 1 and 2 it follows that the matrix $(1/n) \in \Omega_n^*$ is a point of minimum of the function F_ε in the set Ω_n^*. Thus, if $X \in \Omega_n^*$, then

$$\operatorname{per}(X) + \varepsilon / \Pi(X) \geqslant n!/n^n + \varepsilon n^{n^2}$$

for any $\varepsilon > 0$. From here for $\varepsilon \to 0$ we obtain that

$$\operatorname{per}(X) \geqslant n!/n^n$$

for all $X \in \Omega_n^*$. Since Ω_n^* is dense in Ω_n, the continuity of the permanent implies that the last inequality holds for all $X \in \Omega_n$. Theorem 1 is proved.

THEOREM 2. Let $A \in \Omega_n^*$ and $\operatorname{per}(A) = n!/n^n$. Then A = (1/n).

Proof. By virtue of Theorem 1, the point $A \in \Omega_n^*$ is a point of minimum of per in Ω_n^*. Therefore, from Lemma 2 for $\varepsilon = 0$ it follows that A = (1/n). Theorem 2 is proved.

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OBSTRUCTIONS TO LOCAL EQUIVALENCE OF DISTRIBUTIONS

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An n-dimensional distribution on the space $\mathbf{R}^{n+k}$ is a smooth field σ of n-dimensional tangential directions, i.e., a function that associates with each point $x \in \mathbf{R}^{n+k}$ an n-dimensional linear subspace σ_x of the tangent space $T_x\mathbf{R}^{n+k}$ [1]. Two n-dimensional distributions on $\mathbf{R}^{n+k}$ are said to be equivalent if there exists a diffeomorphism of the space $\mathbf{R}^{n+k}$ that transforms one of the distributions into the other one. In this note we indicate a natural local invariant of a distribution. It is proved that this invariant takes different values at different points for a quite general germ of an eight-dimensional distribution on $\mathbf{R}^{11}$.

1. Definitions. Let there be given an n-dimensional distribution σ on $\mathbf{R}^{n+k}$. Define a skew-symmetric bilinear function $\varphi_{\sigma,x}$ at each point $x \in \mathbf{R}^{n+k}$ on the linear space σ_x with values in $T_x\mathbf{R}^{n+k}/\sigma_x$. Let $u, v \in \sigma_x$ and suppose that U and V are arbitrary vector fields with the following properties: The values of these fields at each point of a certain neighborhood of x belong to a plane of the distribution σ, $U(x) = u$, $V(x) = v$. Let π_x: $T_x\mathbf{R}^{n+k} \to T_x\mathbf{R}^{n+k}/\sigma_x$ denote the quotient mapping. Set $\varphi_{\sigma,x}(u, v) = \pi_x([U, V](x))$, where $[\cdot, \cdot]$ is the commutator of vector fields. The independence of this expression from the choice of the fields U and V is proved by the following obvious lemma.

LEMMA 1. Let σ be an n-dimensional distribution on $\mathbf{R}^{n+k}$, and U and V be vector fields on $\mathbf{R}^{n+k}$ whose values belong to the planes of the distribution σ. Suppose that U(x) = 0 at a certain point $x \in \mathbf{R}^{n+k}$. Then $[U, V](x) \in \sigma_x$.

Proof. We call a collection of n vector fields, whose values at each point in a certain neighborhood of a point x form a basis of a plane of the distribution σ, the basic

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