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Multipliers of AW^*-Algebras

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Given a unital C^*-algebra B, when is it true that for each (non-unital, non-degenerate) C^*-subalgebra A of B, the idealizer of A in B coincides with the set of double centralizers of A? To formulate the problem more precisely, recall from [3] (or [5, 3.12]) that a double centralizer is a pair (ρ_r, ρ_l) of (necessarily) bounded linear operators on A such that

$$\rho_r(xy) = x\rho_r(y), \qquad \rho_l(xy) = \rho_l(x)\,y, \qquad \rho_r(x)\,y = x\rho_l(y)$$

for all x, y in A. Each element in the idealizer

$$M = \{b \in B \mid bA + Ab \subset A\}$$

determines a double centralizer (ρ_r, ρ_l), viz. $\rho_r(x) = xb$, $\rho_l(x) = bx$, and we ask whether this map $b \to (\rho_r, \rho_l)$ is surjective. If B is a von Neumann algebra the affirmative answer is well-known, see e.g. [5, 3.12.3], and it can also easily be established for C^*-algebras that are monotone complete (in the sense that each bounded, monotone increasing net of self-adjoint elements has a least upper bound in the algebra). In [6] Reid noticed that an affirmative answer (for every $A \subset B$) implies that B is necessarily an AW^*-algebra (which by definition means that every maximal abelian C^*-subalgebra of B is monotone complete, see [1, § 7]), and Johnson then proved in [4] that the AW^*-condition suffices. We present below a simple proof of Johnson's result. Since his theorem immediately gives the main result in [2], a revival of this circle of ideas may not be amiss. We shall use the monographs [1] and [5] as references for the theories of AW^*-algebras and C^*-algebras.

Theorem. *Let A be a C^*-subalgebra of an AW^*-algebra B. Assume further that the annihilator of A in B is zero. Then the set M of (two-sided) multipliers of A in B is isometrically *-isomorphic to the set of double centralizers of A (i.e. the set $M(A)$ of multipliers of A in its enveloping von Neumann algebra A'') via an isomorphism that extends the identity map on A.*

Proof. Identifying multipliers of A with double centralizers, cf. [5, 3.12.3], we obtain an isometric *-isomorphism π of M into $M(A)$ that extends the identity map on A. It only remains to show that π is surjective.

Assume first that p is a projection in $M(A)$. Thus $L=pA$ and $K=(1-p)A$ are closed right ideals with $L^* K=0$ and $L+K=A$. Let (x_λ) and (y_μ) be approximate units for $L\cap L^*$ and $K\cap K^*$, respectively, [5, 1.4.2]. Thus $x_\lambda y_\mu=0$ for all λ and μ. Working now inside the AW^*-algebra B we note that the range projections $[x_\lambda]$ and $[y_\mu]$ exist in B and are pairwise orthogonal for all λ and μ. Since the projections in B form a complete lattice, we have $q=\bigvee[x_\lambda]$ and $r=\bigvee[y_\mu]$ in B, with $qr=0$. As $x_\lambda\leqq q$ for every λ it follows that q is a left unit for L. Similarly r is a left unit for K. Since $A=L+K$, we see that $1-(q+r)$ annihilates A, whence $q+r=1$ by assumption. Furthermore, the decomposition $A=L+K$ shows that q is a left multiplier of A. Since $q=q^*$ it follows that $q\in M$. Now $qA=L$ and $(1-q)A=K$ and we conclude that $\pi(q)=p$.

To handle the general case we consider the 2×2-matrix algebras

$$\tilde{A}=A\otimes\mathbb{M}_2,\qquad \tilde{M}=M\otimes\mathbb{M}_2,\qquad \tilde{B}=B\otimes\mathbb{M}_2.$$

By a non-trivial result of Berberian, cf. [1, § 62], $\tilde{B}$ is an AW^*-algebra. Note that $M(\tilde{A})=M(A)\otimes\mathbb{M}_2$ and that $\tilde{M}$ is the set of multipliers of $\tilde{A}$ in $\tilde{B}$. Furthermore, the map $\tilde{\pi}\colon \tilde{M}\to M(\tilde{A})$ given by $(\tilde{\pi}(x))_{ij}=\pi(x_{ij})$, for $1\leqq i,j\leqq 2$, is the canonical isometry from $\tilde{M}$ into $M(\tilde{A})$ mentioned before. For each x in $M(A)$ with $0\leqq x\leqq 1$ we define the projection p in $M(A)$ by

$$p=\begin{pmatrix} x & (x-x^2)^{\frac{1}{2}} \\ (x-x^2)^{\frac{1}{2}} & 1-x \end{pmatrix}.$$

As the first part of the proof showed, there is then a projection q in $\tilde{M}$ such that $\tilde{\pi}(q)=p$. Thus with $y=q_{11}$ we have an element in M with $0\leqq y\leqq 1$, such that $\pi(y)=x$. Consequently $\pi(M_+)=M(A)_+$, whence π is surjective as claimed.

Corollary 1. *Let A be an essential, closed ideal in an AW^*-algebra M. Then $M=M(A)$.*

Corollary 2 [2]. *Each isomorphism $\pi\colon A_1\to A_2$ between closed essential ideals of AW^*-algebras M_1 and M_2, respectively, extends to an isomorphism $\pi\colon M_1\to M_2$.*

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