MORREY jr., C. B.
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Second Order Elliptic Equations in Several Variables and Hölder Continuity

In memoriam LEON LICHTENSTEIN

By

CHARLES B. MORREY jr.

1. *Introduction.* In this paper, we discuss the Hölder continuity of the solutions of equations of the form

$$(1.1)\quad \int_G [v_{,\alpha}(a^{\alpha\beta}u_{,\beta}+b^\alpha u+e^\alpha)+v(c^\alpha u_{,\alpha}+du+f)]\,dx=0 \quad \text{for all } v\in H^1_{2,0}$$

on G where we assume that the $a^{\alpha\beta}$ are bounded and measurable and satisfy the uniform ellipticity condition (2.3), the coefficients b^α and $c^\alpha\in L_{2p}$ and $d\in L_p$ for some $p>\nu/2$ (ν being the number of variables x), the $e^\alpha\in L_2$ and $f\in L_1$, and e and f satisfy certain integral growth conditions (see §§ 3, 4). In the equations (1.1), the tensor summation convention is assumed, as will be done throughout, the Greek letters running from 1 to ν; the subscripts after the commas denote differentiation.

In 1938, the writer [6] studied the solutions of (1.1) in the case where $\nu=2$ and the coefficients b^α, c^α, d and f were zero and showed that they were Hölder continuous on the interior provided that the e^α satisfied the corresponding integral growth condition; some results concerning boundary behavior were also obtained. Later [7], the writer generalized the results for the interior to systems of equations of the type (1.1) in which all the terms were present but ν still was 2; the restrictions on the b^α, c^α, and d made in that work were somewhat weaker than those made in this paper. The motivation for the study of equations with such general coefficients was that of proving certain differentiability properties of the solutions of a class of variational problems.

The methods used by the writer in this work were peculiar to the case $\nu=2$ and neither he nor anyone else was able to generalize the results to cases where $\nu>2$ (except in some special cases) until DE GIORGI [2, 3] and NASH [10, 11] concurrently and independently showed that "a-harmonic" functions in any number of variables are Hölder continuous on compact subsets of their domains of definition; a function is *a-harmonic* on a domain G if and only if it is in H^1_2 and satisfies $I_0(u, v; E)=0$ (see (2.7)) for each domain E with $\bar{E}$ compact and $E\subset G$ and each $v\in H^1_{2,0}$ on E. The methods of NASH and DE GIORGI are completely unrelated- Nash obtained his results as a by-product of his work on parabolic equations, whereas DE GIORGI's method involves only elliptic equations. Nash confined himself to bounded solutions, whereas DE GIORGI allowed solutions merely in L_2; since functions in H_2^1 are not necessarily bounded, the starting point for our investigations is the paper of DE GIORGI [3] which we discuss in § 2.

Very recently, STAMPACCHIA [13] studied equations of the type (1.1) in which the $b^\alpha \equiv 0$, the c^α and d are bounded, and $d(x) \geqq 0$. He showed that the solutions in $H_{2,0}^1$ or those satisfying the "homogeneous Neumann" boundary conditions or even certain mixed boundary conditions are bounded on G, provided that G is of his "type S", i.e. if ∂G satisfies an interior "strong cone" condition. In this paper, we show, first of all, that the eigenvalue problem for equations (1.1) has discrete eigenvalues with the usual Fredholm alternative holding for any given parameter value (see Theorem 4.1), provided G is bounded and then that the solutions u are Hölder-continuous on compact subsets of G if e and f satisfy proper integral growth conditions (see §§ 3, 4). In case G is of type $S^*(\alpha, a)$ (see § 5) and e and f satisfy proper growth conditions across ∂G, then the solutions u in $H_{2,0}^1$ (see below) are Hölder-continuous on $\bar{G}$ and vanish in the ordinary sense on ∂G; the same result holds if G is a Lipschitz domain (see § 5). Thus our results are generalizations of those of NASH and DE GIORGI and are partial generalizations of those of STAMPACCHIA. It is interesting that the domains of class $S^*(\alpha, a)$ are identical with the types of domains for which NASH ([11] appendix) proved the Hölder continuity of a-harmonic functions on $\bar{G}$, assuming that the given boundary values are Hölder-continuous.

We now introduce our notations: If G is a domain, ∂G denotes its boundary. $B(x_0, r)$ denotes the ν-ball with the center at x_0 and radius r. If S is a set in ν-space, $|S|$ denotes its ν-measure; however, occasionally we consider subsets S of $\Sigma = \partial B(0, 1)$ in which case $|S|$ means its $(\nu-1)$-measure. We define $\gamma_\nu = |B(0, 1)|$, $\Gamma_\nu = |\partial B(0, 1)|$; clearly $\Gamma_\nu = \nu \gamma_\nu$. If u is a function (or vector function), ∇u denotes its gradient, $\nabla^2 u$ denotes its second gradient $u_{,\alpha\beta}$, etc. If φ is any function, vector, or tensor, $|\varphi|$ denotes its length (the square root of the sum of the squares of its components). All integrals are Lebesgue integrals. The spaces L_p over some set have their usual meaning. The space H_2^1 over the domain G is the closure of the space of C' functions on G according to the norm

$$(\|u\|_p^1)^p = \int_G [|u|^2 + |\nabla u|^2]^{p/2} \, dx.$$

The space $H_{p,0}^1$ is the subspace of H_p^1 obtained by closing the space of functions of class C' having compact support interior to G. For such functions, we define a topologically equivalent norm

$$(\|u\|_{p,0}^1)^p = \int_G |\nabla u|^p \, dx.$$

If $p=2$, the spaces H_2^1 and $H_{2,0}^1$ are Hilbert spaces and

$$(u, v)_2^1 = \int_G (\nabla u \cdot \nabla v + u v) \, dx, \qquad (u, v)_{2,0}^1 = \int_G \nabla u \cdot \nabla v \, dx.$$

We define

$$D(u, G) = \int_G |\nabla u|^2 \, dx, \qquad d(u, G) = [D(u, G)]^{1/2}.$$

There are many results which are inequalities involving a constant which may depend on various other constants; we denote all these constants by C but do not assume that all these constants are the same.

2. *The De Giorgi lemmas and theorems; a-harmonic functions.* In this section, we quote the lemmas and relevant theorems of DE GIORGI, which are proved in [3], and then prove important Dirichlet growth theorems for a-harmonic functions and H_2^1 functions in general. We begin with DE GIORGI's definition of his $\mathfrak{B}(E;\gamma)$ classes:

DEFINITION. Suppose E is a domain and $\gamma>0$. A function $u\in\mathfrak{B}(E;\gamma)$ if and only if $u\in L_2$ on E, $u\in H_2^1$ on domains $\bar{D}$ with $\bar{D}$ compact and $D\subset E$, and if

$$\int_{A(k)\cap B(x_0,\varrho_1)} |\nabla u|^2\,dx \leqq \frac{\gamma}{(\varrho_2-\varrho_1)^2}\int_{A(k)\cap B(x_0,\varrho_2)} [u(x)-k]^2\,dx$$

$$\int_{B(k)\cap B(x_0,\varrho_1)} |\nabla u|^2\,dx \leqq \frac{\gamma}{(\varrho_2-\varrho_1)^2}\int_{B(k)\cap B(x_0,\varrho_2)} [u(x)-k]^2\,dx$$

for all k, x_0, ϱ_1, and ϱ_2 for which $0<\varrho_1<\varrho_2$ and $B(x_0,\varrho_2)\subset E$; here $A(k)$ is the set where $u(x)>k$ and $B(k)$ is the set where $u(x)<k$.

LEMMA I. *Each class $\mathfrak{B}(E;\gamma)$ is closed with respect to strong convergence in L_2.*

LEMMA II. *There is a constant $\beta_1(\nu)>0$ such that*

$$\beta_1\int_{A(k;\varrho)-A(\lambda;\varrho)}|\nabla u|\,dx \geqq (\lambda-k)\,[\tau(k,\lambda;\varrho)]^{(\nu-1)/\nu} \tag{2.1}$$

whenever $u\in H_2^1$ on $B(x_0,\varrho)$ and $\lambda>k$. Here $\tau(k,\lambda;\varrho)$ indicates the smaller of $|A(\lambda;\varrho)|$ and $|B(x_0,\varrho)-A(k;\varrho)|$ and $A(k;\varrho)=A(k)\cap B(x_0;\varrho)$, etc.

LEMMA III. *There is a constant $\beta_2(\nu)>0$ such that*

$$\int_{A(k;\varrho)}[u(x)-k]^2\,dx \leqq \beta_2|A(k;\varrho)|^{2/\nu}\int_{A(k;\varrho)}|\nabla u|^2\,dx \tag{2.2}$$

whenever $u\in H_2^1$ on $B(x_0,\varrho)$ and $|A(k;\varrho)|\leqq|B(x_0;\varrho)|/2$.

LEMMA IV. *There is a function $\vartheta(\sigma;\nu,\gamma)$ which is defined and positive for $0<\sigma<1$ such that if $u\in\mathfrak{B}(E;\gamma)$, $B(x_0;\varrho)\subset E$, and for some k we have*

$$|A(k;\varrho)|<\varrho^\nu\vartheta(\sigma;\nu,\gamma)$$

then

$$|A(k+\sigma c;\varrho-\sigma\varrho)|=0,$$

where

$$c\geqq 0 \quad \text{and} \quad c^2=(\varrho^\nu\vartheta)^{-1}\int_{A(k;\varrho)}[u(x)-k]^2\,dx.$$

REMARK. From Lemma IV, it follows that if $u\in\mathfrak{B}(E;\gamma)$ then u is (essentially) bounded on compact subsets of E.

LEMMA V. *There exists a number $\eta=\eta(\nu,\gamma)>0$ such that if $u\in\mathfrak{B}(E;\gamma)$ and $B(x_0;4\varrho)\subset E$, then*

$$\operatorname{osc}(u;\varrho)\leqq(1-\eta)\operatorname{osc}(u;4\varrho),$$

where $\operatorname{osc}(u;r)$ *denotes the* (essential) *oscillation on* $B(x_0;r)$.

THEOREM I. *Any function* $u \in \mathfrak{B}(E;\gamma)$ *satisfies a uniform Hölder condition with exponent* $\lambda(\nu,\gamma) > 0$ *on any compact subset of* E.

THEOREM II. *If the coefficients* $a^{\alpha\beta}$ *are bounded and measurable on* E *and satisfy* $a^{\alpha\beta} = a^{\beta\alpha}$ *and*

$$(1-h)\,|\xi|^2 \leqq a^{\alpha\beta}(x)\,\xi^\alpha \xi^\beta \leqq (1+h)\,|\xi|^2, \qquad x \in E,\ 0 < h < 1, \tag{2.3}$$

and all ξ, *then any a-harmonic function in* L_2 *on* $E \in \mathfrak{B}(E,\gamma)$ *where*

$$\gamma = (1+h)^2(1-h)^{-2}. \tag{2.4}$$

We now prove

THEOREM 2.1. *If* $u \in L_2$ *and is a-harmonic on* $B(x_0, a)$ *then*

$$d[u; B(x_0; r)] \leqq (1+h)(1-h)^{-1}(a-r)^{-1}\|u\|_2^0, \qquad 0 \leqq r < a. \tag{2.5}$$

If u *is also in* H_2^1 *on* $B(x_0, a)$, *then*

$$d[u; B(x_0; r)] \leqq C(\nu, h) \cdot d[u; B(x_0; a)] \cdot (r/a)^{\tau - 1 + \lambda_0} \tag{2.6}$$

where

$$\lambda_0 = \lambda[\nu, (1+h)(1-h)^{-1}], \qquad \tau = \nu/2.$$

If $u^* \in H_2^1$ *on a bounded domain* E, *there is a unique a-harmonic function* u *on* E *such that* $u - u^* \in H_{2,0}^1$ *on* E.

Proof. The first statement follows from the definitions and the last is proved by the usual lower-semicontinuity argument for minimizing the integral $I_0(u, u; E)$, where

$$I_0(u, v; E) = \int_E v_{,\alpha} a^{\alpha\beta} u_{,\beta}\, dx. \tag{2.7}$$

Or one can set $u = u^* + U$ and look for a solution U in $H_{2,0}^1$ of the non-homogeneous equation as in the proof of Theorem 3.2.

Next, let $\{u_n\}$ be any sequence of a_n-harmonic functions on $B(0, 1)$ with $d[u_n; B(0, 1)]$ uniformly bounded and where the a_n all satisfy (2.3) with the same h; we may as well assume that the average $\bar{u}_n$ of u_n over $B(0, 1)$ is zero. Then by RELLICH's theorem

$$\|u_n\|_2^0 \leqq C_2(\nu)\, d[u_n; B(0, 1)].$$

We may also assume that $u_n \to u$ in H_2^1 on $B(0, 1)$. Then u is also in $\mathfrak{B}(E, \gamma)$ where γ is given by (2.4) and $E = B(0, 1)$. Let us choose any ϱ and σ with $0 < \varrho, \sigma < 1$. There is a k such that $|A(k; \varrho)| < \varrho^\nu \vartheta(\sigma; \nu, \gamma)$ and $|B(-k, \varrho)| < \varrho^\nu \vartheta(\sigma; \nu, \gamma)$ for the limit function u. For those k, ϱ, and σ, we see from the strong convergence in L_2 of u_n to u that these inequalities hold for all sufficiently large n. Accordingly the u_n and u are all uniformly bounded on $B(0, \varrho\sigma)$ and so $u_n(0)$ and $u(0)$ are uniformly bounded. From Lemma V, it follows, then, that

$$|u(x)| \leqq C(\nu, h)\, d[u, B(0, 1)],$$

$$|u(x) - u(0)| \leqq C(\nu, h)\, d[u; B(0, 1)] \cdot |x|^{\lambda_0}, \qquad |x| \leqq \tfrac{1}{2}$$

for every a-harmonic function u where the $a^{\alpha\beta}$ satisfy (2.3).

From the closure theorem, we may assume that the $a^{\alpha\beta} \in C^\infty$ so the a-harmonic functions are. Thus we have (u a-harmonic)

$$(1-h)\, D[u, B(0,r)] \leqq I_0[u, u; B(0,r)] = \int_{\partial B(0,r)} (u-u_0)\, a^{\alpha\beta} n_\alpha u_{,\beta} dS$$
$$\leqq \Big[\int_{\partial B(0,r)} (u-u_0)^2 a^{\alpha\beta} n_\alpha n_\beta dS\Big]^{\frac{1}{2}} \Big[\int_{\partial B(0,r)} a^{\alpha\beta} u_{,\alpha} u_{,\beta} dS\Big]^{\frac{1}{2}} \qquad (\tau = \nu/2)$$
$$\leqq (1+h)\, \gamma_\nu^{\frac{1}{2}} r^{\tau+\lambda_0} \cdot [\varphi'(r)]^{\frac{1}{2}} [\varphi(1)]^{\frac{1}{2}}, \quad \varphi(r) = D[u, B(0,r)],$$
$$u_0 = u(0), \qquad 0 \leqq r \leqq \tfrac{1}{2}.$$

Hence

$$\varphi^{-2} \cdot \varphi' \geqq C(\nu, h)\, [\varphi(1)]^{-1} r^{-\tau-\lambda_0}, \qquad 0 \leqq r \leqq \tfrac{1}{2}$$
$$\varphi^{-1}(r) \geqq \varphi^{-1}(\tfrac{1}{2})\, [1 + \varphi(\tfrac{1}{2})\, C\, \varphi^{-1}(1)\, (r^{1-\tau-\lambda_0} - 2^{\tau+\lambda_0-1})]$$
$$\therefore \varphi(r) \leqq C(\nu, h)\, \varphi(1) \cdot r^{\tau+\lambda_0-1}, \qquad 0 \leqq r \leqq 1.$$

But now, if u is a-harmonic on a sphere $B(x_0, R)$, the function u' defined by $u'(y) = u(x_0 + Ry)$ is $'a$-harmonic on $B(0,1)$, where $'a^{\alpha\beta}(y) = a^{\alpha\beta}(x_0 + Ry)$ and the $'a^{\alpha\beta}$ clearly satisfy (2.3) with the same h. The result (2.6) then follows using homogeneity argument.

The following "Dirichlet growth" theorem is well known (see, for instance [8], p. 111 or [7], pp. 12, 13).

THEOREM 2.2. *If $u \in H_2^1$ on G and satisfies*

$$d[u, B(x_0, r)] \leqq L(r/\delta)^{\tau-1+\lambda}, \qquad 0 < \lambda < 1,\ 0 \leqq r \leqq \delta$$

for each x_0 in G, δ being the distance of x_0 from ∂G, then u satisfies a uniform Hölder condition with exponent λ on any compact subset of G and satisfies

$$|u(x) - u(x_0)| \leqq C(\nu, \lambda) \cdot L \cdot \delta^{1-\tau-\lambda} \cdot |x - x_0|^\lambda \quad \text{if} \quad |x - x_0| \leqq \delta/2,$$
$$B(x_0, \delta) \subset G.$$

3. *A special case.* In this section, we study the solutions of equations of the form

$$\int_G [v_{,\alpha}(a^{\alpha\beta} u_{,\beta} + e^\alpha) + v f]\, dx = 0 \qquad \text{for all } v \in H_{2,0}^1 \text{ on } G, \tag{3.1}$$

where we assume that the $a^{\alpha\beta}$ are bounded and measurable and satisfy (2.3) for some h.

LEMMA 3.1. *If S is any set of finite measure and $\sigma > 0$, then*

$$\int_S |\xi - x|^{\sigma-\nu} d\xi \leqq \Gamma \sigma^{-1} s^\sigma \quad \textit{where} \quad \gamma_\nu s^\nu = |S|.$$

Proof. For obviously

$$\int_S |\xi - x|^{\sigma-\nu} d\xi \leqq \int_{B(x,s)} |\xi - x|^{\sigma-\nu} d\xi.$$

LEMMA 3.2. *If $u \in H_{p,0}^1$ on the bounded domain G for some $p \geqq 1$, then*

$$u(x) = -\Gamma_\nu^{-1} \int_G |\xi - x|^{-\nu} (\xi^\alpha - x^\alpha)\, u_{,\alpha}(\xi)\, d\xi \tag{3.2}$$

for almost all x in G.

Proof. If u is of class C' and has support in G, $x \in G$, and r is the distance from x, then

$$u(x) = -\int_0^R u_r[x + r\zeta(\vartheta)]\,dr \qquad (\zeta \in \partial B(0,1)) \tag{3.3}$$

where we assume $B(x, R) \supset G$ and $u(\xi) = 0$ for $\xi \in B(x, R) - G$. Then (3.2) follows by averaging over $\partial B(0, 1)$. From (3.2), we see that

$$|u(x)| \leqq \Gamma_\nu^{-1} \int_G |\xi - x|^{1-\nu} \cdot |\nabla u(\xi)|\,d\xi \tag{3.4}$$

(u of class C' as above). From the Hölder inequality we see that

$$|u(x)|^p \leqq \Gamma_\nu^{-1} \Big[\Gamma_\nu^{-1} \int_G |\eta - x|^{1-\nu}\,d\eta\Big]^{p-1} \int_G |\xi - x|^{1-\nu} |\nabla u(\xi)|^p\,d\xi. \tag{3.5}$$

By integrating over G, we find that

$$\int_G |u(x)|^p\,dx \leqq g^p \cdot \int_G |\nabla u(\xi)|^p\,d\xi \quad \text{where} \quad \gamma_\nu g^\nu = |G|. \tag{3.6}$$

Thus the formula (3.2) holds in general by approximations.

Definition. Suppose $f \in L_1$ on G. We define its *potential* by

$$V(x) = -(\nu - 2)^{-1} \Gamma_\nu^{-1} \int_G |\xi - x|^{2-\nu} f(\xi)\,d\xi. \tag{3.7}$$

Remark. The following theorem enables us to reduce the general equations (3.1) to ones where $f \equiv 0$, provided the original f satisfies (3.8).

Theorem 3.1 *). *Suppose that $f \in L_1$ on G and satisfies*

$$\int_{G \cap B(x_0, r)} |f(\xi)|\,d\xi \leqq L r^{\nu-2+\lambda}, \qquad 0 < \lambda < 1, \tag{3.8}$$

for each sphere $B(x_0, r)$, suppose V is the potential of f, and suppose $v \in H^1_{2,0}$ on G. Then $V \in H^1_2$ on any bounded domain and satisfies

$$\int_{B(x_0, r)} |\nabla V(x)|^2\,dx \leqq C^2(\nu, \lambda) \cdot L^2 r^{\nu-2+2\lambda} \qquad x_0 \in G, \qquad 0 \leqq r \leqq R \tag{3.9}$$

where R is the diameter of G. Moreover

$$\int_G v(x) f(x)\,dx = -\int_G v_{,\alpha}(x) V_{,\alpha}(x)\,dx. \tag{3.10}$$

Proof. It is straightforward to show that $V \in L_1$ on any cell and (by integrating the expression for $V_{,\alpha}(x)$, using Fubini's theorem, etc.) is absolutely continuous along almost all lines in each coordinate direction with partial derivatives given by

$$V_{,\alpha}(x) = -\Gamma_\nu^{-1} \int_G |\xi - x|^{-\nu} (\xi^\alpha - x^\alpha) f(\xi)\,d\xi \tag{3.11}$$

almost everywhere.

*) Compare [7], pp. 61–64.

Now, we select $x_0 \in G$ and we write

$$f(\xi) = f_1(\xi) + f_2(\xi) \quad \text{where} \quad f_1(\xi) = f(\xi) \text{ in } B(x_0, 2r)$$

and $f_1(\xi) = 0$ elsewhere and let V_k be the potential of f_k. Let

$$\varphi_2(\varrho; x) = \int_{B(x,\varrho)\cap[G-B(x_0,2r)]} |f(\xi)|\, d\xi. \tag{3.12}$$

Then, from (3.11) for V_2, we have

$$\begin{aligned} |\nabla V_2(x)| &\leq \Gamma_\nu^{-1} \int_{G-B(x_0,2r)} |\xi - x|^{1-\nu} |f(\xi)|\, d\xi \\ &= \Gamma_\nu^{-1} \int_r^R \varrho^{1-\nu} \varphi_2'(\varrho; x)\, d\varrho \\ &\leq L\,\Gamma_\nu^{-1} \Big[R^{\lambda-1} + (\nu-1) \int_r^R \varrho^{\lambda-2}\, d\varrho\Big] \\ &\leq (\nu-1)\,\Gamma_\nu^{-1}(1-\lambda)^{-1} L\, r^{\lambda-1}, \qquad x \in B(x_0; r) \end{aligned}$$

since, obviously

$$\varphi_2(\varrho; x) \leq \begin{cases} 0, & 0 \leq \varrho \leq r \\ L\,\varrho^{\nu-2+\lambda} & r \leq \varrho \leq R \end{cases} \quad \big(x \in B(x_0; r)\big).$$

Accordingly

$$\int_{B(x_0,r)} |\nabla V_2(x)|^2\, dx \leq C(\nu, \lambda)\, L^2 r^{\nu-2+2\lambda}. \tag{3.13}$$

From the Schwarz inequality, we obtain

$$|\nabla V_1(x)|^2 \leq \Gamma_\nu^{-1} I_1 I_2, \quad \text{where} \quad 0 \leq \sigma < \lambda \quad \text{and} \tag{3.14}$$

$$I_1 = \Gamma_\nu^{-1} \int_{B(x_0,2r)} |\xi - x|^{\sigma-\nu} |f(\xi)|\, d\xi,$$

$$I_2 = \int_{B(x_0,2r)} |\xi - x|^{2-\nu-\sigma} |f(\xi)|\, d\xi$$

since $f_1(\xi) = 0$ outside $B(x_0, 2r)$. In order to evaluate I_2 define

$$\varphi_1(\varrho; x) = \int_{B(x,\varrho)\cap B(x_0,2r)\cap G} |f(\xi)|\, d\xi.$$

Then we see that

$$\varphi_1(\varrho; x) \leq \begin{cases} L\,\varrho^{\nu-2+\lambda}, & 0 \leq \varrho \leq r \\ L(2r)^{\nu-2+\lambda} & \varrho \geq r, \end{cases}, \quad x \in B(x_0; r).$$

Proceeding as with φ_2, we see that

$$I_2 \leq [(\nu - 2 - \sigma)(\lambda - \sigma)^{-1} + 2^{\nu-2+\lambda}]\, L\, r^{\lambda-\sigma}. \tag{3.15}$$

Integrating (3.14) over $B(x_0, r)$ and using (3.15), we see that

$$\int_{B(x_0,r)} |\nabla V_1(x)|^2\, dx \leq \sigma^{-1}(2r)^\sigma L(2r)^{\nu-2+\lambda} C(\nu, \lambda, \sigma)\, L\, r^{\lambda-\sigma}.$$

Since this holds for any σ with $0<\sigma<\lambda$, we see that

(3.16) $$\int_{B(x_0,r)} |\nabla V_1(x)|^2 dx \leqq C(\nu,\lambda)\, L^2 r^{\nu-2+2\lambda}.$$

Using (3.11), we see that

$$-\int_G v_{,\alpha}(x)\, V_{,\alpha}(x)\, dx = \Gamma_\nu^{-1} \int_G\int_G |\xi - x|^{-\nu} (\xi^\alpha - x^\alpha)\, v_{,\alpha}(x)\, f(\xi)\, dx\, d\xi$$
$$= \int_G v(\xi)\, f(\xi)\, d\xi$$

using Lemma 3.2 (with x and ξ interchanged).

THEOREM 3.2. *If G is a domain $\subset B(x_0,R)$, $e\in L_2$ on G, and f satisfies (3.8) on G, there exists a unique solution u in $H^1_{2,0}$ on G of (3.1). Moreover*

(3.17) $$\|u\|^1_{2,0} \leqq (1-h)^{-1} \left(\|e\|^0_2 + C_2 L R^{\tau-1+\lambda}\right).$$

Proof. On account of (2.3), we see that the inner product $I_0(u,v:G)$ [see (2.7)] leads to a norm which is topologically equivalent to $\|u\|^1_{2,0}$. Since

(3.18) $$\int_G (e^\alpha v_{,\alpha} + f v)\, dx = \int_G v_{,\alpha} (e^\alpha - V_{,\alpha})\, dx$$

is a linear functional in $H^1_{2,0}$, we conclude the existence of a unique solution u of (3.1) from Hilbert space theory. Setting $v=u$ in (3.1) and using (2.3), (3.18), etc., we obtain

$$(1-h)\,(\|u\|^1_{2,0})^2 \leqq \left|\int_G u_{,\alpha}(e^\alpha - V_{,\alpha})\, dx\right| \leqq \|u\|^1_{2,0} \left(\|e\|^0_2 + \|\nabla V\|^0_2\right)$$

from which (3.17) follows, on account of Theorem 3.1.

THEOREM 3.3. *Suppose $u\in H^1_{2,0}$ and satisfies (3.1) on $G=B(x_0,a)$ where $f=0$ and $e\in L_2$ on $B(x_0,a)$ with*

(3.19) $$\int_{B(x_0,r)} |e(x)|^2 dx \leqq L^2 (r/a)^{\nu-2+2\lambda}, \qquad 0<\lambda<\lambda_0,\ 0\leqq r\leqq a.$$

Then

(3.20) $$\int_{B(x_0,r)} |\nabla u(x)|^2 dx \leqq C(\nu,h,\lambda)\cdot L^2 (r/a)^{\nu-2+2\lambda} \qquad 0\leqq r\leqq a.$$

If $u\in H^1_2$ on $B(x_0;a)$ and e satisfies (3.19), then (3.20) holds with L replaced by $L+d[u;B(x_0,a)]$.

Proof. The last statement follows from the first, since we may write $u=U+H$ where H is the a-harmonic function coinciding with u on $\partial B(x_0,a)$. Then $U\in H^1_{2,0}$ and is a solution of (3.1) with the same e and f, so U satisfies (3.20). Moreover

(3.21) $$I_0[u,u;B(x_0,a)] = I_0[U,U;B(x_0,a)] + I_0[H,H;B(x_0,a)].$$

The result for u then follows from Theorem 2.1.

In order to prove the first statement*), let

$$\varphi(s) = \sup L^{-1} d[u, B(x_0, Rs)]$$

*) Cf. The proofs of Lemma 2.1 and Theorem 2.2 in [8].

for all e satisfying (3.19) with a replaced by R, where $0<R\leqq a$ and u is the solution of (3.1) ($f=0$) in $H^1_{2,0}$ on $B(x_0, R)$. Then, choose any e satisfying (3.19) and $0<r<R<a$ and write $u=U+H$ on $B(x_0, R)$ where H is the a-harmonic function $=u$ on $\partial B(x_0, R)$. Then (3.21) holds, so that

$$d[H; B(x_0, R)] \leqq C(h)\, d[u; B(x_0; R)] \leqq C L\, \varphi(R/a)$$

by the definition of φ. Also e satisfies

$$\int_{B(x_0,r)} |e(x)|^2 dx \leqq L^2 (R/a)^{\nu-2+2\lambda} \cdot (r/R)^{\nu-2+2\lambda}.$$

Hence, if we apply Theorem 2.1 and the definition of φ, we obtain

$$\begin{aligned} d[u, B(x_0, r)] &\leqq d[H; B(x_0, r)] + d[U; B(x_0, r)] \\ &\leqq C L\, \varphi(R/a)\, (r/R)^{\tau-1+\lambda_0} + L(R/a)^{\tau-1+\lambda} \varphi(r/R). \end{aligned}$$

Since e was arbitrary, we see that

$$(3.22) \qquad \varphi(s) \leqq t^{\tau-1+\lambda} \varphi(s/t) + C(\nu, h)\, \varphi(t)\, (s/t)^{\tau-1+\lambda_0} \quad 0<s\leqq t\leqq 1.$$

Now it is clear from Theorem 3.2 that φ is non-decreasing for $0<s\leqq 1$ with

$$\varphi(1) \leqq (1-h)^{-1}.$$

Next, choose σ with $0<\sigma<1$. Then, clearly

$$\varphi(s) \leqq S_0 s^{\tau-1+\lambda} \quad \text{for} \quad \sigma\leqq s\leqq 1 \quad \text{if} \quad S_0 = \varphi(1)\, \sigma^{1-\tau-\lambda}.$$

Applying (3.22) with $\sigma^2\leqq s\leqq\sigma$ and $t=\sigma^{-1}s$, we see that

$$(3.23) \qquad \varphi(s) \leqq S_1 s^{\tau-1+\lambda} \quad \text{where} \quad S_1 = S_0(1+C\omega), \quad \omega = \sigma^{\lambda_0-\lambda}.$$

Since $S_1 \geqq S_0$, we see that (3.23) holds for $\sigma^2\leqq s\leqq 1$. Applying (3.22) with $\sigma^4\leqq s\leqq\sigma^2$ and $t=\sigma^{-2}s$, we obtain

$$\varphi(s) \leqq S_2 s^{\tau-1+\lambda}, \quad S_2 = S_0(1+C\omega)(1+C\omega^2), \quad \sigma^4\leqq s\leqq 1.$$

By repeating the process, we find that

$$\varphi(s) \equiv S s^{\tau-1+\lambda}, \quad S = S_0(1+C\omega)(1+C\omega^2)(1+C\omega^4)\ldots, \quad 0<s\leqq 1.$$

The result follows.

4. *Existence theory and interior estimates for the general equations.* In this section, we study the solutions of the general equations (1.1) in which the $a^{\alpha\beta}$ satisfy (2.3), the b^α and $c^\alpha \in L_{2p}$ and $d \in L_p$ on G for some $p>\nu/2$; we call these *general conditions.*

LEMMA 4.1. *If $u\in H^1_2$ on $B(x_0; a)$, there exists a function $U\in H^1_{2,0}$ on $B(x_0, 2a)$ such that $U(x)=u(x)$ on $B(x_0, a)$ and*

$$(4.1) \qquad (\|U\|^1_{2,0})^2 \leqq C^2(\nu)\, (\|\nabla u\|^0_2)^2 + a^{-2}(\|u\|^0_2)^2 \equiv C^2(\nu)\, ({}'\|u\|^1_2)^2.$$

Proof. U is defined $=u$ on $B(x_0, a)$ and U is the harmonic function in $B(x_0, 2a) - B(x_0, a)$ coinciding with u on $\partial B(x_0, a)$ and vanishing on $\partial B(x_0, 2a)$.

One verifies the result by introducing spherical harmonics (as in [4], for instance), and computing the result.

LEMMA 4.2. *If* $u \in H^1_{r,0}$ *for some* r, $0<r<\nu$, *on some bounded domain* G, *or* $\in H^1_r$ *over the whole space then* $u \in L_{r'}$ *on* G, *where*

$$r' = \frac{\nu r}{\nu - r} \quad \text{and} \quad \|u\|^0_{r'} \leqq r \cdot \frac{\nu-1}{\nu-r} \prod_\alpha \left(\|u_{,\alpha}\|^0_r\right)^{1/\nu} \leqq r\nu^{-1}(\nu-1)(\nu-r)^{-1}\|u\|^1_{r,0}.$$

Proof. This follows from the representation (3.2) in the first case and by a limit process in the second and from the theorems in [12].

LEMMA 4.3. *If* $u \in H^1_{2,0}$ *on a bounded domain* G *or if* $u \in H^1_2$ *on a sphere* $G = B(x_0, a)$, *then* $u \in L_s$ *and*

$$(4.2) \qquad \|u\|^0_s \leqq C(\nu)\|u\|^1_{2,0} \quad \textit{or} \quad C(\nu)'\|u\|^1_2, \quad \textit{respectively, where} \quad s = 2\nu/(\nu-2).$$

Proof. This follows from Lemma 4.2 for $u \in H^1_{2,0}$ and the second follows using Lemma 4.1.

LEMMA 4.4. *If* $u \in H^1_{2,0}$ *on the bounded domain* G *and* $d \in L_p$ *on* G *for some* $p>\nu/2$, *then* $(du^2) \in L_t$, *where*

$$(4.3) \qquad t = p\nu/(p\nu + \nu - 2p),$$

$$(4.4) \qquad \|du^2\|^0_t \leqq \|d\|^0_p \cdot (\|u\|^0_s)^2 \leqq C(\nu)\|d\|^0_p(\|u\|^1_{2,0})^2,$$

$$(4.5) \qquad \int_{B(x_0,r)\cap G} |du^2|\,dx \leqq C(\nu)\|d\|^0_p(\|u\|^1_{2,0})^2 r^{2\mu}, \quad \mu = 1 - \nu/2p.$$

Proof. For, suppose p', $q'>1$, $(p')^{-1}+(q')^{-1}=1$. From the Hölder inequality we obtain

$$\|(du^2)\|^0_t \leqq \|d\|^0_{tp'} \cdot (\|u\|^0_{2tq'})^2.$$

The results follow by setting $tp'=p$ and $2tq'=s$ and then using Lemma 4.2.

LEMMA 4.5. *If* $f \in L_q$ *on a bounded domain* G, *where*

$$(4.6) \qquad q = 2\nu/(\nu+2)$$

then its potential $V \in H^1_2$ *on any bounded domain* Γ *with*

$$(4.7) \qquad \int_\Gamma |\nabla V)^2\,dx \leqq C^2(\nu)\left(\|f\|^0_q\right)^2.$$

If, also f *satisfies*

$$(4.8) \qquad \int_{B(x_1,r)} |f(x)|\,dx \leqq L\,\delta^{\tau-1}(r/\delta)^{\nu-2+\lambda}, \quad 0 \leqq r \leqq \delta,$$
$$B(x_1,\delta) \subset G, \quad L \geqq \|f\|^0_q \cdot \gamma_\nu^{1/s}$$

for all x_1 *in* G, *then*

$$(4.9) \qquad \int_{B(x_1,r)} |\nabla V|^2\,dx \leqq C(\nu,\lambda)\,L^2(r/\delta)^{\nu-2+\lambda}, \quad 0 \leqq r \leqq \delta.$$

Proof. From the results of CALDERON and ZYGMUND [1], it follows that $\nabla^2 V \in L_q$ over the whole space. From Lemma 4.2, the result (4.7) follows.

To prove the second result, choose a point $x_1 \in G$ and let δ be the distance of x_1 from ∂G. Write

$$f(x) = f_1(x) + f_2(x)$$

where $f_2(x) = 0$ in $B(x_1, \delta/2)$ and $f_1(x) = 0$ in $G - B(x_1, \delta/2)$ and let V_k be the potential of f_k. Clearly $f_2 \in L_q$ with $\|f_2\|_q^0 \leqq \|f\|_q^0$ so its potential $V_2 \in H_2^1$ on any bounded Γ. But since V_2 is harmonic on $B(x_1, \delta/2)$, it satisfies a condition (4.9) with $\lambda = 1$ and $L = \|f\|_q^0$. Moreover, it is easy to see that f_1 satisfies the condition of Theorem 3.1 with that L replaced by $C(\nu, \lambda) L \delta^{1-\tau-\lambda}$ for every sphere $B(x_0, r)$, so that V_1 satisfies (4.9). The result follows.

We now define

$$(4.10)\quad \begin{cases} I(u, v; G) = \int_G [v_{,\alpha}(a^{\alpha\beta} u_{,\beta} + b^\alpha u) + v(c^\alpha u_{,\alpha} + d u)]\, dx, \\ J(u, v; G) = \int_G [v_{,\alpha} b^\alpha u + v(c^\alpha u_{,\alpha} + d u)]\, dx = I(u, v; G) - I_0(u, v; G) \\ K(u, v; G) = \int_G u v\, dx. \end{cases}$$

We shall first prove an existence theorem for equations of the form

$$(1.1')\quad \begin{cases} I(u, v; G) + \lambda K(u, v; G) = L(v) \quad \text{for all } v \in H_{2,0}^1, \quad \text{where} \\ \qquad - L(v) = \int_G [e^\alpha v_{,\alpha} + f v]\, dx. \end{cases}$$

For $\lambda = 0$, the equations (1.1′) reduce to (1.1). Let us define the transformations T_0, T_1, T_2 and the function w by

$$(T_0 u, v)_{2,0}^1 = I_0(u, v), \qquad (T_1 u, v)_{2,0}^1 = J(u, v), \qquad (T_2 u, v)_{2,0}^1 = K(u, v).$$
$$(w, v)_{2,0}^1 = L(v)$$

where we have assumed G fixed and bounded. Then the equation (1.1)′ is equivalent to

$$(4.11)\qquad T_0 u + T_1 u + \lambda T_2 u = w.$$

THEOREM 4.1. *If G is a bounded domain and the coefficients satisfy our general conditions, then the equation (1.1)′ has a unique solution u in $H_{2,0}^1$ for any e in L_2 and any f whose potential V is in H_2^1, provided that λ does not belong to an isolated set of characteristic values. If λ is characteristic, the homogeneous equation ($e^\alpha = f = 0$) has solutions $u \not\equiv 0$, the manifold of these being finite dimensional. For any λ, the Fredholm alternative holds.*

Proof. It is well-known (see [7] or [9], for instance) that $K(u, v)$ is completely continuous (i.e. continuous with respect to weak convergence in u and v). Next, suppose $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ in $H_{2,0}^1$ on G. Then, by Lemma 4.4 and our hypotheses, we see that $\|b u_n\|_{2t}$, $\|c v_n\|_{2t}$ and $\|d u_n\|_t$ are uniformly bounded. A simple argument involving subsequences of subsequences shows that $J(u, v)$ is also completely continuous. Hence T_1 and T_2 are completely continuous.

Next we see that there is a λ_0 such that

$$I_0(u, u) + J(u, u) + \lambda K(u, u) \geqq 2^{-1}(1-h)\left(\|u\|_{2,0}^1\right)^2 \qquad \text{if } \lambda \geqq \lambda_0.$$

For, if not, there is a sequence u_n with

$$(4.12) \qquad \|u_n\| = 1, \qquad I_0(u_n, u_n) + J(u_n, u_n) + n\,K(u_n, u_n) < 2^{-1}(1-h).$$

A subsequence of $\{u_n\}$ converges weakly to some u and J_n and K_n converge to their values for that u. But since I_{0n}, J_n, and K_n are bounded, we see that $K_n \to 0$ so $u = 0$. Hence $J_n \to 0$. But, since $I_{0n} \geqq (1-h)$, we have a contradiction. Thus for $\lambda = \lambda_0$, $(T_0 + T_1 + \lambda_0 T_2)$ has a bounded inverse W, say. Then (4.11) is equivalent to

$$u + (\lambda - \lambda_0)\,U(u) = W(w), \qquad U = W\,T_2$$

where U is completely continuous. The results follow.

THEOREM 4.2. *There is an $a_0 > 0$, depending only on h, p, $\|b\|_{2p}^0 + \|c\|_{2p}^0$, and $\|d\|_p^0$ such that if $0 < a \leqq a_0$ and the coefficients satisfy our general conditions on $B(x_0, a)$, then*

$$I[u, u; B(x_0, a)] \geqq \frac{(1-h)}{2} D[u, B(x_0, a)] = \frac{(1-h)}{2}\left(\|u\|_{2,0}^1\right)^2$$

for every $u \in H_{2,0}^1$ on $B(x_0, a)$.

Proof. For

$$I[u, u; B(x_0, a)] = I_0[u, u; B(x_0, a)] + \int_{B(x_0, a)} [(b^\alpha + c^\alpha)\, u\, u_{,\alpha} + d u^2]\, dx.$$

Since

$$\Big|\int_{B(x_0,a)} (b^\alpha + c^\alpha)\, u\, u_{,\alpha}\, dx\Big| \leqq \|u\|_{2,0}^1 \Big\{\int_{B(x_0,a)} |b(x) + c(x)|^2 \cdot |u(x)|^2\, dx\Big\}^{\frac{1}{2}}$$

and $|b + c|^2 \in L_p$, the result follows from Lemma 4.4.

THEOREM 4.3. *If $0 < a \leqq a_0$, $a^{\alpha\beta}$, b^α, c^α, and d satisfy the general conditions on $B(x_0, a)$, $e \in L_2$ there, and f satisfies the first condition of Lemma 4.5 then there exists a unique solution u of (1.1) in $H_{2,0}^1$ on $B(x_0, a)$. Moreover*

$$(4.13) \qquad d[u, B(x_0, a)] \leqq 2(1-h)^{-1}\left[\|e\|_2^0 + C(\nu, \mu)\, \|f\|_q^0\right].$$

A corresponding result holds if f satisfies the condition of Theorem 3.1.

Proof. The proof parallels that of Theorems 3.2 and 4.1 since Theorem 4.2 holds.

THEOREM 4.4. *Suppose $u \in L_2$ on $B(x_0, a)$ where $0 < a \leqq a_0$ and the coefficients satisfy the conditions of Theorem 4.3 on $B(x_0, a)$. Suppose also that $u \in H_2^1$ and satisfies (1.1) on each $B(x_0, R)$ with $0 < R < a$. Then*

$$(4.14) \quad d[u, B(x_0, r)] \leqq C_3(h)\left\{\|e\|_2^0 + C(\nu, \mu)\, \|f\|_q^0 + (a-r)^{-1}\|u\|_2^0\right\}, \quad 0 \leqq r < a.$$

A corresponding result holds if f satisfies the condition of Theorem 3.1.

Proof. Let $h(s)$ be a fixed function of class C' with $h(s) = 1$ for $s \leqq 0$, $h(s) = 0$ for $s \geqq 1$, and $h'(s) \leqq 0$. Choose R so $r < R < a$ and define

$$\zeta(x) = h[(|x - x_0| - r)/(R - r)], \qquad v = \zeta^2 u, \;\; U = \zeta u.$$

Then v and $U \in H^1_{2,0}$ on $B(x_0, R)$. Substituting in (1.1), we obtain

$$(4.15)\quad \begin{cases} 0 = I[U, U; B(x_0, R)] + \int\limits_{B(x_0,R)} [\zeta e^\alpha U_{,\alpha} + \zeta U f + U(b^\alpha - c^\alpha)\zeta_{,\alpha} u + \\ + \zeta e^\alpha \zeta_{,\alpha} u - a^{\alpha\beta}\zeta_{,\alpha}\zeta_{,\beta} u^2]\, dx \geqq \frac{(1-h)}{2} (\|U\|^1_{2,0})^2 - \\ - \|U\|^1_{2,0}[\|e\| + C(\mu,\nu) L R^{\tau-1+2\mu} + C h_1 (R-r)^{-1} \|b-c\|^0_{2p} \cdot R^\lambda \|u\|^0_2] - \\ - h_1 (R-r)^{-1} \|e\|^0_2 \|u\|^0_2 - (1+h)(R-r)^{-2} h_1^2 (\|u\|^0_2)^2 {}^{*)}. \end{cases}$$

The result follows.

We now define the spaces S_λ and $S_{\lambda 0}$ for $0 < \lambda < 1$ as follows: $u \in S_\lambda$ on $B(x_0, a)$ if and only if $u \in H^1_2$ there and there is an L such that

$$(4.16)\qquad d[u, B(x_1, r)] \leqq L(r/\delta)^{\tau-1+\lambda}, \quad 0 \leqq r \leqq \delta = a - |x_1 - x_0|$$

for every x in $B(x_0, a)$; if $u \in S_\lambda$ we define $|||u|||_\lambda$ as the larger of $'\|u\|^1_2$ (see (4.1)) and the smallest L which satisfies (4.16)). The space $S_{\lambda 0}$ is the subspace of S_λ for which $u \in H^1_{2,0}$ and $|||u|||_{\lambda,0}$ is the larger of $\|u\|^1_{2,0}$ and the smallest L as above.

LEMMA 4.6. *Suppose the b^α, c^α and d satisfy our general conditions and suppose $u \in S_\lambda$ on $B(x_0, a)$. Then $c^\alpha u_{,\alpha} \in L_r$, $d u \in L_{r'}$, and $b^\alpha u \in L_2$ on $B(x_0, a)$, where*

$$r = \frac{2p}{p+1} > q \quad and \quad r' = \frac{2\nu p}{p\nu - 2p + 2} > q, \quad \mu = 1 - \frac{\nu}{2p}$$

and

$$(4.17)\quad \begin{cases} \int\limits_{B(x_1,\varrho)} |b(x)|^2 |u(x)|^2 dx \leqq (C(\nu,\lambda)\, |||u|||_\lambda \cdot \|b\|^0_{2p})^2 \delta^{2\mu} (\varrho/\delta)^{\nu-2+2\mu} \\ \int\limits_{B(x_1,\varrho)} |c^\alpha(x) u_{,\alpha}(x)|\, dx \leqq C(\nu)\, |||u|||_\lambda \cdot \|c\|^0_{2p}\, \delta^{\tau-1+\mu} (\varrho/\delta)^{\nu-2+\lambda+\mu} \\ \int\limits_{B(x_1,\varrho)} |d(x) u(x)|\, dx \leqq C(\nu,\lambda) \cdot |||u|||_\lambda \cdot \|d\|^0_p\, \delta^{\tau-1+2\mu} (\varrho/\delta)^{\nu-2+2\mu}, \quad 0 \leqq \varrho \leqq \delta \end{cases}$$

for each $x_1 \in B(x_0, a)$, where $\delta = a - |x_1 - x_0|$.

Proof. The first results follow from Lemma 4.3 and the Hölder inequality. From Theorem 2.2 and the definition of $|||u|||_\lambda$, we see that

$$|u(x) - u(x_1)| \leqq C(\nu,\lambda)\, |||u|||_\lambda\, \delta^{1-\tau-\lambda} |x - x_1|^\lambda, \qquad 0 \leqq |x - x_1| \leqq \delta/2$$
$$\int\limits_{B(x_1,\delta)} |u(x)|^2 dx \leqq \delta^2 \cdot |||u|||^2_\lambda.$$

From these facts, it is easy to conclude that

$$|u(x)| \leqq C(\nu,\lambda)\, |||u|||_\lambda \cdot \delta^{1-\tau} \qquad \text{for } 0 \leqq |x - x_1| \leqq \delta/2.$$

The results (4.17) follow easily from this and the Hölder inequality.

DEFINITION. For u in $H^1_{2,0}$ or in any $S_{\lambda,0}$, we define the linear operator T by: $Tu = U$ where U is the solution in $H^1_{2,0}$ of

$$\int\limits_{B(x_0,a)} [v_{,\alpha}(a^{\alpha\beta} U_{,\beta} + b^\alpha u) + v(c^\alpha u_{,\alpha} + d u)]\, dx.$$

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*) If we did not assume $a^{\beta\alpha} = a^{\alpha\beta}$, the integral would contain also the terms $(a^{\alpha\beta} - a^{\beta\alpha})\zeta_{,\alpha} u U_{,\beta}$ and the bracket multiplied by $\|U\|^1_{2,0}$ would have another term $Ch_1(R-r)^{-1}\|u\|^0_2$.

THEOREM 4.5. *There is a number* a_1 *with* $0<a_1\leqq a_0$ *which depends only on* h, ν, p, λ, *and the norms of* b, c, *and* d *such that if* $0<a\leqq a_1$, *and* $0<\lambda\leqq\mu$, $Tu\in S_{0\lambda}$ *and* $\|T\|\leqq\frac{1}{2}$.

This follows immediately from the preceding lemma and from Theorems 3.1 and 3.3. Accordingly we have the following Theorem:

THEOREM 4.6. *If* $0<a\leqq a_1$ *and if* e *satisfies* (3.19) *and* f *satisfies all the conditions of Lemma* 4.5 *with* $0<\lambda\leqq\mu$ *and* $\lambda<\lambda_0$, *then the solution of* (1.1) *which is in* $H^1_{2,0}$ *is also in* $S_{\lambda 0}$ *and*

$$|||u|||^1_{\lambda,0}\leqq C(\nu,p,\lambda)\,L.$$

A corresponding result holds if f *satisfies* (3.8) *with* $\lambda\leqq\mu$ *and* $\lambda<\lambda_0$.

Finally, we have the following final result on interior continuity:

THEOREM 4.7. *Suppose* $u\in L_2$ *on* G *and* $u\in H^1_2$ *on any domain* D *with* $\bar D$ *compact and* $\bar D\subset G$. *Suppose also that* u *satisfies* (1.1) *on each such domain where* e *and* f *satisfy the conditions of Theorem* 4.5 *for each* $B(x_0,r)\subset G$ *with* $\lambda\leqq\mu$ *and* $\lambda<\lambda_0$. *Then* $u\in S_\mu$ *on each sphere interior to* G, *where its norm depends only on* L, h, μ, λ, p, *the norms of* b, c, d, *and the distance of* $\overline{B(x_0,a)}$ *from* ∂G, *provided* $a\leqq a_1$. *Thus* u *satisfies a uniform Hölder condition on each such domain* D *which depends on* D *and the quantities above.*

Proof. Choose $\overline{B(x_0,a_1)}\subset G$. Then $u\in H^1_2$ on $B(x_0,a_1)$ and, in fact, its norm $'\|u\|^1_2$, as defined in (3.1), is bounded as indicated in Theorem 4.3. Then, let H be the a-harmonic function coinciding with u on $\partial B(x_0,a_1)$ and define U by

$$u=U+H.$$

Then $U\in H^1_{2,0}$ on $B(x_0,a_1)$ and satisfies (1.1) with e^α and f replaced by E^α and F, respectively, where

$$E^\alpha=e^\alpha+b^\alpha H,\qquad F=f+c^\alpha H_{,\alpha}+dH.$$

But now $H\in S_{\lambda_0}$ on $B(x_0,a_1)$ with

$$|||H|||_{\lambda_0}\leqq C(\nu,h)\,'\|u\|^1_2$$

by Theorem 2.1, since $I_0(u,u;B)=I_0(U,U;B)+I_0(H,H;B)$ $(B=B(x_0,a_1))$. But then it follows from Lemma 4.6 that E and F satisfy the conditions of Theorem 4.6, where L now involves $'\|u\|^1_2$ linearly. The results follow from Theorems 4.6 and 2.2.

5. *Hölder continuity at the boundary*. In this section, we prove our results about Hölder continuity at the boundary. We begin by defining domains of class $S^*(\alpha,a)$.

DEFINITION. A domain G is of class $S^*(\alpha,a)$, $0<\alpha<1$, if and only if it is bounded and

$$|B(x_0,r)-G|\geqq\alpha|B(x_0,r)|,\qquad\text{for all } r \text{ with } 0<r\leqq a$$

and any x_0 not in G.

We begin by generalizing some of DE GIORGI's lemmas and theorems.

DEFINITION. A function $u \in \mathfrak{B}^*(E, \gamma)$ iff $u \in L_2$ on E, $u \in H_2^1$ on domains D with $\bar{D}$ compact and $\bar{D} \subset E$, and if

$$(5.1)\quad \begin{cases} \int\limits_{A(k,\varrho_1)} |\nabla u|^2 dx \leqq \dfrac{\gamma}{(\varrho_2-\varrho_1)^2} \int\limits_{A(k,\varrho_2)} |u(x)-k|^2 dx & \text{for all } k \geqq 0, \\ \int\limits_{B(k,\varrho_1)} |\nabla u|^2 dx \leqq \dfrac{\gamma}{(\varrho_2-\varrho_1)^2} \int\limits_{B(k,\varrho_2)} |u(x)-k|^2 dx & \text{for all } k \leqq 0, \end{cases}$$

the notations being those of § 2.

It is easy to see that DE GIORGI's Lemma I holds for such functions and (by repeating his proof) that his Lemma IV holds, provided $k \geqq 0$ and a corresponding result holds for the sets $B(k, \varrho)$ for $k \leqq 0$; we label these results as Lemmas I* and IV*. As in DE GIORGI's case, these lemmas imply that any function $u \in \mathfrak{B}^*(E, \gamma)$ is bounded on compact subsets of E. We now prove a lemma which permits us to generalize DE GIORGI's Lemmas II and III slightly:

LEMMA 5.1. *Suppose $u \in H_p^1$ on $B(x_0, p)$, suppose u vanishes on a subset B with $|B| \geqq \alpha |B(\varrho)|$* $(B(\varrho) = B(x_0, \varrho))$, $0 < \alpha < 1$. *Then*

$$(5.2)\quad \int\limits_e |u(x)|^p dx \leqq \beta_p(\nu, \alpha) \cdot |G|^{(p-1)/\nu} |e|^{1/\nu} \int\limits_{B(\varrho)} |\nabla u|^p dx \qquad (G = B(\varrho) - B).$$

Proof. It is known ([7] or [9]) that u is equivalent to a function which is absolutely continuous along almost all lines in any direction and continues to have this property in any coordinate system related to the original by a bi-Lipschitz transformation. So we assume u to have this property already. It is easy to show that if x_0 is not in a set Z of measure zero [for instance if $u(x_0)$ is the Lebesgue derivative of $\int_e u(x)\,dx$ at x_0 and if the Lebesgue derivatives of $\int_e u_{,\alpha}(x)\,dx$ all exist at x_0] then u is absolutely continuous for $r \geqq 0$ along almost all radial lines through x_0.

Suppose $0 < \varepsilon < \alpha$ and $\eta = \varepsilon |B(\varrho)|$. We may cover G with an open set $G' \subset B(\varrho)$ such that $|G'| < |B(\varrho)| - |B| + \eta$. Let $B' = B(\varrho) - G'$; then $|B'| > (\alpha - \varepsilon) \cdot |B(\varrho)|$. For each x in G', let $\Sigma(x)$ be the set of points ζ on $\partial B(0, 1)$ such that $x + r\zeta \in B'$ for at least one r; clearly $\Sigma(x)$ is the union of a countable family of closed sets and so is measurable. It is geometrically evident that

$$(5.3)\qquad |\Sigma(x)| \geqq [C(\alpha - \varepsilon, \nu)]^{-1} \Gamma_\nu, \qquad C > 0$$

since $|\Sigma(x)|$ would be smallest if B' were a cone with vertex at x and axis along a diameter of $B(\varrho)$.

For each x in G' and ζ in $\Sigma(x)$, let $r(x, \zeta)$ be the smallest value of r such that $x + r\zeta \in B'$ and let $G(x)$ denote the set of all $\xi = x + r\zeta$ for $0 \leqq r < r(x, \zeta)$. If $x \in G' - Z$ and if ζ does not belong to a set of measure zero, we have $u(x + r\zeta)$ absolutely continuous in r for $0 \leqq r \leqq r(x, \zeta)$, so that

$$(5.4)\qquad u(x) = -\int\limits_0^{r(x,\zeta)} u_r(x + r\zeta)\,dr \qquad (u_r = \zeta^\alpha u_{,\alpha}),\ \zeta \in \Sigma(x).$$

Averaging over $\Sigma(x)$, we obtain

$$u(x) = -|\Sigma(x)|^{-1} \int_{G(x)} |\xi - x|^{-\nu} (\xi^\alpha - x^\alpha) u_{,\alpha}(\xi)\, d\xi. \tag{5.5}$$

From (5.3) and (5.5), it follows that

$$|u(x)| \leqq C \Gamma_\nu^{-1} \int_{G'} |\xi - x|^{1-\nu} |\nabla u(\xi)|\, d\xi, \quad x \in G' - Z. \tag{5.6}$$

Since ε is arbitrary (5.6) holds with $C = C(\alpha, \nu)$ and G' replaced by G. If $p > 1$, we may use the Hölder inequality to obtain

$$\begin{cases} |u(x)|^p \leqq C^p \Gamma_\nu^{-p} \left[\int_G |\xi - x|^{1-\nu} d\xi\right]^{p-1} \cdot \int_G |\xi - x|^{1-\nu} \cdot |\nabla u(\xi)|^p d\xi \\ \qquad \leqq C |G|^{(p-1)/\nu} \gamma_\nu^{-(p-1)/\nu} \Gamma_\nu^{-1} \int_G |\xi - x|^{1-\nu} \cdot |\nabla u(\xi)|^p d\xi, \quad x \in G - Z. \end{cases} \tag{5.7}$$

If $p = 1$, (5.7) coincides with (5.6) (as modified). The result follows by integrating over e.

We now state and prove our slight generalizations of DE GIORGI's Lemmas II and III:

LEMMA II*. *If $u \in H_p^1$ with $p \geqq 1$ on $B(\varrho) = B(x_0, \varrho)$, $k < \lambda$, and*

$$|B(\varrho) - A(k, \varrho)| \geqq \alpha |B(\varrho)|, \quad 0 < \alpha < 1,$$

then

$$(\lambda - k) |A(\lambda; \varrho)|^{(\nu-1)/\nu} \leqq \beta_1(\nu, \alpha) \int_{A(k,\varrho) - A(\lambda,\varrho)} |\nabla u|\, dx.$$

Proof. This follows immediately from Lemma 5.1 by taking $p = 1$ and $e = A(\lambda; \varrho)$ and replacing u by the function $w(x) = 0$ when $u(x) \leqq k$, $w(x) = u(x) - k$ for x on $A(k; \varrho) - A(\lambda; \varrho)$, and $w(x) = (\lambda - k)$ on $A(\lambda; \varrho)$.

LEMMA III*. *If $u \in H_2^1$ on $B(\varrho)$ and if*

$$|B(\varrho) - A(k; \varrho)| \geqq \alpha |B(\varrho)|, \quad \alpha > 0$$

then

$$\int_{A(k;\varrho)} |u(x) - k|^2 dx \leqq \beta_2(\nu, \alpha) |A(k; \varrho)|^{2/\nu} \int_{A(k;\varrho)} |\nabla u|^2 dx.$$

Proof. This follows from Lemma 5.1 by setting $p = 2$ and $e = A(k; \varrho) = G$ and replacing u by w where $w(x) = 0$ if $u(x) \leqq k$ and $w(x) = u(x) - k$ if $u(x) > k$.

By repeating DE GIORGI's proof of his Lemma V, using the modified lemmas above, we obtain the following lemma:

LEMMA V*. *Suppose $u \in \mathfrak{B}^*(E, \gamma)$. Then there is a number $\eta(\nu, \gamma, \alpha) > 0$ such that if $B(x_1, 4\varrho) \subset E$, $\mu > 0$, $\omega > 0$, $\mu - 2\omega \geqq 0$, $u(x) \leqq \mu$ on $B(x_1, 4\varrho)$, and $|A(\mu - 2\omega; 2\varrho)| \leqq (1 - \alpha) |B(x_1; 2\varrho)|$, then $u(x) \leqq \mu - \eta\omega$ on $B(x_1; \varrho)$.*

REMARK. Results corresponding to the lemmas above hold if u is replaced by $-u$.

We now prove an analog of DE GIORGI's Theorem II and extend our Theorem 2.1; Theorem II* has been proved by NASH in [11].

THEOREM II*. *Suppose the part* $B(x_0, a) \cap G$ *of the domain* G *is of type* $S^*(\alpha, a)$, *suppose* $u \in L_2$ *on* $B(x_0, a)$ *and* $u \in H_2^1$ *on each* $B(x_0, R)$ *with* $0 < R < a$, *suppose* u *is a-harmonic on* $G \cap B(x_0, a)$, *and suppose* $u(x) = 0$ *for* $x \in B(x_0, a) - G$. *Then* u *satisfies a uniform Hölder condition with exponent* $\lambda_0^*(h, \nu, \alpha)$ *on each* $B(x_0, R)$ *with* $0 < R < a$ *which depends only on* $h, \nu, \alpha, a - R$, *and* $\|u\|_2^0$ *and* $\|u\|_2^1$ *is finite on each such* $B(x_0, R)$.

THEOREM 5.1. *If* G *and* u *satisfy the hypotheses of Theorem* II* *and if* $u \in H_2^1$ *on* $B(x_0, a)$, *then*

$$d[u, B(x_1, r)] \leqq C(\nu, h, \alpha)\, d[u, B(x_1, \delta)] \cdot (r/\delta)^{\nu - 1 + \lambda_0^*}, \quad 0 \leqq r \leqq \delta = a - |x_1 - x_0|$$

for any $x_1 \in B(x_0, a)$.

Proofs. Suppose, first, that $x_1 \in B(x_0, a) - G$. Let $h(r)$ be an arbitrary function of class C' for $0 \leqq r \leqq \delta$ with $h(r) = 0$ near $r = \delta$. Let v be any function $\in H_2^1$ on $B(x_1, \delta)$ which is the strong limit in H_2^1 of functions of class C' on $\overline{B(x_1, \delta)}$ which vanish on $\overline{B(x_1, \delta)} - G$ and define

$$V(x) = h(|x - x_1|) \cdot v(x).$$

Then $V \in H_{2,0}^1$ on $B(x_1, \delta)$ and on $B(x_1, \delta) \cap G$ and V vanishes on $B(x_1, \delta) - G$ so that

$$\begin{aligned} \int_{B(x_1, \delta)} V_{,\alpha} a^{\alpha\beta} u_{,\beta}\, dx &= \int_{B(x_1, \delta) \cap G} V_{,\alpha} a^{\alpha\beta} u_{,\beta}\, dx = 0 \\ &= \int_0^R \left\{ h(r) \int_{\partial B(x_1, r)} v_{,\alpha} a^{\alpha\beta} u_{,\beta}\, dS + h'(r) \int_{\partial B(x_1, r)} v\, a^{\alpha\beta} n_{,\alpha} u_{,\beta}\, dS \right\} dr \\ &= \int_0^R h'(r) \left\{ \int_{\partial B(x_1, r)} v\, a^{\alpha\beta} n_{,\alpha} u_{,\beta}\, dS - \int_{B(x_1, r)} v_{,\alpha} a^{\alpha\beta} u_{,\beta}\, dx \right\} dr \end{aligned}$$

since u is a-harmonic on $B(x_1, \delta) \cap G$. Since this is true for any $h(r)$ as stated, we see that

$$\int_{B(x_1, r)} v_{,\alpha} a^{\alpha\beta} u_{,\beta}\, dx = \int_{\partial B(x_1, r)} v\, a^{\alpha\beta} n_\alpha u_{,\beta}\, dS, \quad 0 \leqq r \leqq R, \; r \notin \overset{*}{Z}. \tag{5.8}$$

If we apply (5.8) with $v = \text{pos}(u - k)$ where $k \geqq 0$, we see that (5.8) holds with u replaced by v. By following the procedure in the proof of the Leray-Cacciopoli lemma ([5], p. 153) we see that

$$\int_{A(k; \varrho_1)} |\nabla u|^2\, dx \leqq \frac{\gamma}{(\varrho_2 - \varrho_1)^2} \int_{A(k; \varrho_2)} |u(x) - k|^2\, dx,$$

where

$$0 < \varrho_1 < \varrho_2 \leqq \delta, \quad k \geqq 0, \quad A(k; \varrho) = A(k) \cap B(x_1; \varrho)$$

where the γ here is the same as in Theorem II. A similar result holds for the sets $B(k; \varrho)$. Accordingly $u \in \mathfrak{B}^*(E, \gamma)$ on $B(x_1, \delta)$.

Moreover, since $|B(x_1, r) - G| \geqq \alpha |B(x_1, r)|$ for $0 \leqq r \leqq \delta$, we see immediately from Lemma V* that u satisfies a Hölder condition with exponent $\lambda_0^*(\nu, \alpha, h)$ and constant depending only on those numbers, $\|u\|_2^0$, and δ with

$u(x_1)=0$. Now, from our previous results each point x_1 in $B(x_0, a)\cap G$ is also the center of a sphere $B(x_1, 5\varrho/2)$ in which

$$|u(x)-u(x_1)| \leqq K(x_1)\cdot|x-x_1|^{\lambda_0} \leqq K^*(x_1)\cdot|x-x_1|^{\lambda_0^*}, \qquad x\in B(x_1, 5\varrho/2).$$

If $0<R<a$, $\overline{B(x_0, R)}$ can be covered by a finite number of the spheres $B(x_1, \varrho)\big(\subset B(x_1, 5\varrho/2)\big)$. Thus u satisfies a Hölder condition of exponent λ_0^* on $B(x_0, R)$. The homogeneity argument in the proof of Theorem 2.1 may be repeated after which Theorem 5.1 follows as before.

We can now sketch how to prove the results concerning the boundary behavior of the solutions of equations (1.1) which were stated in the introduction for domains G of type $S^*(\alpha, a)$: We first consider solutions u in $H^1_{2,0}$ on G of equations (3.1) where we assume that e and f satisfy

$$(5.9)\quad \begin{cases} \displaystyle\int_{B(x_0,r)\cap G} |e|^2\,dx \leqq L^2(r/a)^{\nu-2+2\lambda}, \quad 0\leqq r\leqq a, \\ \displaystyle\int_{B(x_0,r)\cap G} |f|\,dx \leqq K L a^{1-\nu}(r/a)^{\nu-2+\lambda} \quad \text{for all } r, \end{cases} \qquad 0<\lambda<\lambda_{0,\mu}$$

for any x_0; the restriction on f guarantees that the gradient of the potential of f restricted to $B(x_0, a)$ satisfies a condition like that for e. First, we notice that if u is such a solution and we subtract off the function H_0 which is a-harmonic on $G\cap B(x_0, a)$ and coincides on $\partial[G\cap B(x_0, a)]$ with u and vanishes on $B(x_0, a)-G$, then $u_1=u-H_0$ vanishes on $\partial B(x_0, a)$ and is a solution of (3.1) with the same e and f. Then the argument of Theorem 3.3 can be repeated to obtain the result of that theorem for u_1, the only difference being that when one writes $u_1=U+H$ on $B(x_0, R)$, the function H is that function which is a-harmonic on $G\cap B(x_0, R)$ and coincides with u_1 on $\partial[G\cap B(x_0, R)]$ and is 0 on $B(x_0, R)-G$ as are u, u_1, and U; the argument works on account of Theorems II* and 5.1.

It is now clear that the relevant arguments of section 4 will go through for spheres $B(x_0, a)$ if we restrict ourselves throughout to the space $^*H^1_2$ consisting of all $u\in H^1_2$ on $\overline{B(x_0, a)}$ which are limits in H^1_2 of functions of class C' which vanish in a neighborhood of $[\overline{B(x_0, a)}-G]$ and define $^*H^1_{2,0}={}^*H^1_{2,0}\cap H^1_{2,0}$ and $^*S_\lambda={}^*H^2_1\cap S_\lambda$, etc.

In case G is a Lipschitz domain, i.e. one such that each boundary point is in a neighborhood N on $G\cup\partial G$ which can be mapped in a bi-Lipschitz way on a hemisphere in such a way that $N\cap\partial G$ corresponds to the flat boundary, and if e and f satisfy (5.9) and the coefficients satisfy our general conditions, we see that any solution u of (1.1) satisfies a uniform Hölder condition on $\overline{G}$ as follows: Let x_0 be any point of ∂G and let T be a bi-Lipschitz map as above. Then the transform of u satisfies an equation of the same form on the hemisphere but, of course, the norms of the new b^α, c^β, d, e, and f will be different and the new $a^{\alpha\beta}$ will satisfy (2.3) with a different h. But now $u=0$ along the flat boundary. If we extend u by reflection (as we would a harmonic function) and extend the coefficients and the e^α and f properly, the extended u will satisfy the extended equations on the full sphere and our results follow from our previous results for the interior.

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Dept. of Math., University of California, Berkeley, Cal. (U.S.A.)

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