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# Descent for the $K$-Theory of Polynomial Rings

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## Introduction

Let $A[X]$ be a polynomial ring over a commutative ring $A$ and let $\alpha \in K_n(A[X])$, $n \in \mathbb{Z}$. We say that $\alpha$ is extended (from $A$) if it lies in the image of $K_n(A)$. Our purpose is to prove statements of the following type: $\alpha$ is extended if and only if it is extended locally in the étale topology on $\operatorname{Spec}(A)$. We take as our starting point analogous results obtained by Vorst [16] in the Zariski topology. Of course the "locally extended is extended" theme goes back to Quillen's solution of Serre's problem [13]. As was pointed out by Lindel it would be of interest to have étale descent results for isomorphism classes of projective modules, not just for their stable isomorphism classes. To be more specific, one would hope that if $A$ is local and $P$ is a finitely generated projective $A[X]$ module which becomes extended (hence free) after base change to a henselization of $A$ or to a strict henselization, then $P$ must be extended to begin with. But our method applies only to the stable isomorphism classes, as it relies heavily on the module structure of $NK_n(A)$ over the ring of big Witt vectors $W(A)$, [5, 14, 17]. In order to understand how $NK_n(A)$ localizes in the étale topology we also investigate the behaviour of $W(A)$ and its truncations $W_t(A)$ under étale extensions. Here our results are similar to those obtained by Illusie for $p$-Witt vectors [9]. In the last section we describe $NK_n(B)$ in terms of $NK_n(A) \underset{W(A)}{\otimes} W(B)$ when $B$ is étale over $A$ and $A$, $n$ are as in Sect. 1.

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## § 1. Descent for $NK_n$

(1.1) All rings are commutative with unit. Let $n \in \mathbb{Z}$ and let $A$ be a ring. One puts $NK_n(A) = \operatorname{coker}\ (K_n(A) \to K_n(A[X]))$. Here, if $n < 0$ Bass' definition of negative $K$ groups is used.

(1.2) Assume that one of the following two conditions holds.

(i) $n \leqq 2$.

(ii) A is noetherian and each zero-divisor of $A$ is contained in a minimal prime ideal of $A$.

**Theorem.** *Given such $A$, $n$, let $B$ be étale and faithfully flat over $A$. Then the Amitsur complex* [10; p. 119]

$$0 \to NK_n(A) \to NK_n(B) \to NK_n(B \underset{A}{\otimes} B) \to NK_n(B \underset{A}{\otimes} B \underset{A}{\otimes} B) \to \dots$$

*is exact.* (Compare with [16; 1.7]).

(1.3) *Remark.* Saying that $B$ is étale and faithfully flat over $A$ is the same as saying that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a covering family - consisting of 1 element - in the étale topology. The Amitsur complex is the complex whose cohomology is taken when computing the Čech cohomology groups, with coefficients $NK_n$, of this covering family. We have augmented the complex with $NK_n(A)$.

(1.4) *Remark.* Instead of assuming that $B$ is unramified over $A$ one may make the weaker assumption that for each fiber the trace is surjective. (This surjectivity is automatic in characteristic 0.) In the terminology of EGA the conditions of the theorem then read: "For such $A, n$, let $B$ be faithfully flat, finitely presented, quasi-finite over $A$, such that for each maximal ideal $\mathfrak{m}$ of $A$ the trace $B/\mathfrak{m}B \to A/\mathfrak{m}$ is surjective". We will explain in 1.20 below how to modify the proof under these weaker assumptions.

(1.5) *Remark.* Consider for a noetherian ring $A$ the property $(P)$ Every zero-divisor is contained in a minimal prime ideal. If $A$ is reduced then it satisfies $(P)$. (Exercise.) If $B$ is finite and flat over $A$ and $A$ has $(P)$, then so does $B$ [11; 9D, 5E]. Further $A$ has $(P)$ if and only if all its local rings have it [11; 7C]. If $A$ has $(P)$ and $B$ is quasi-finite, flat, finitely presented over $A$, then it easily follows from Zariski's Main Theorem [8; IV, 8.12.6] and from [11; 9D, 5E, 7C] that $B$ has $(P)$. In particular, if $B$ is étale over $A$ and $A$ has $(P)$ then so does $B$. Property $(P)$ is relevant to us because of the following theorem of Vorst.

(1.6) As in Vorst [16] we write $[f]$ for the Teichmüller lifting to $W(A)$ of $f \in A$ (cf. 1.14 below). This $[f]$ acts on $NK_n(A)$ compatibly with the action on $K_n(A[X])$ that is induced by the substitution $X \mapsto fX$.

**Theorem** (Vorst). *Let $A$, $n$ be as in 1.2 and let $f \in A$. Then*

$$NK_n(A)_{[f]} \simeq NK_n(A_f).$$

*Proof.* Instead of Lemma 1.6 of [16] use Lemma 1.7 below. □

(1.7) **Lemma.** *Let $A$ have the property discussed in* 1.5. *For $f \in A$ there are $g \in A$ and $m \geqq 1$ so that $f^m g = 0$ and so that $f^m + g$ is a non-zero-divisor.*

*Proof.* Choose $\tilde{g}$ so that it lies exactly in those minimal prime ideals that do not contain $f$. Then choose $m$ so that $(f\tilde{g})^m = 0$ and put $g = \tilde{g}^m$. □

(1.8) Vorst's theorem allows us to go local. When doing this it is good to keep in mind that $f_1, \dots, f_m$ in $A$ generate the unit ideal if and only if $[f_1], \dots, [f_m]$ generate the unit ideal in $W(A)$. [16; 1.8]. In other words

$$\mathrm{Spec}(A_{f_1}), \dots, \mathrm{Spec}(A_{f_m}) \text{ cover } \mathrm{Spec}(A)$$

if and only if

$$\mathrm{Spec}(W(A)_{[f_1]}), \dots, \mathrm{Spec}(W(A)_{[f_m]}) \text{ cover } \mathrm{Spec}(W(A)).$$

Also observe that, by a limit argument, one may extend Theorem 1.6 to multiplicative systems: If $S$ is such a system in $A$ and $[S]$ is the system of the $[f] \in W(A)$ with $f \in S$, then $[S]^{-1} NK_n(A) \simeq NK_n(S^{-1}A)$, provided that $A$ satisfies the conditions of the theorem.

(1.9) We give some corollaries to Theorem 1.2.

**Corollary.** *Let $X$ be a reduced scheme, $\mathcal{N}\mathcal{K}_n$ the Zariski sheafification of $U \mapsto NK_n(\Gamma(U))$ on the étale site of $X$.*

(i) *$\mathcal{N}\mathcal{K}_n$ is an étale sheaf with*

$$H^i_{\text{ét}}(X, \mathcal{N}\mathcal{K}_n) = H^i_{\text{Zar}}(X, \mathcal{N}\mathcal{K}_n).$$

(ii) *If $X$ is affine, $X = \mathrm{Spec}(A)$, then*

$$H^i_{\text{ét}}(X, \mathcal{N}\mathcal{K}_n) = \begin{cases} NK_n(A) & \text{if } i=0 \\ 0 & \text{if } i>0. \end{cases}$$

*Proof.* Theorem 1.2 easily implies that $\mathcal{N}\mathcal{K}_n$ is an étale sheaf and that, if $X$ is affine, the étale Čech cohomology groups are like is suggested in (ii). (Use that a reduced ring is a filtered direct limit of noetherian reduced subrings and see [8; IV, 17.7.8]). By a criterion of Cartan [6; Chap. II, 5.9.2] part (ii) follows. (The proof of Cartan's criterion remains valid in the étale topology, cf. [12; Chap. III].) Now it is also clear that the higher direct images of $\mathcal{N}\mathcal{K}_n$ under $\pi: X_{\text{ét}} \to X_{\text{Zar}}$ vanish and (i) follows from the Leray spectral sequence for $\pi$ with coefficients $\mathcal{N}\mathcal{K}_n$. □

(1.10) If $A$ is a positively graded ring, $A = \bigoplus_{i \geqq 0} A_i$, we write $A^+$ for $\bigoplus_{i>0} A_i$. By a trick of Weibel [18] the relative $K$-groups $K_n(A, A^+)$ are naturally direct summands of the $NK_n(A, A^+)$ which in their turn are naturally direct summands of $NK_n(A)$. (Weibel uses the homomorphism $A \to A[X]$ that sends $a \in A_i$ to $aX^i$). Thus we get

**Corollary** (Weibel). *Let $A$, $n$, $B$ be as in 1.2 and assume moreover that $A \to B$ is a homomorphism of positively graded rings. Then the complex*

$$0 \to K_n(A, A^+) \to K_n(B, B^+) \to K_n(B \otimes_A B, (B \otimes_A B)^+) \to \dots$$

*is exact.* □

(1.11) **Corollary.** *Let $A$, $n$ be as in (1.2) and assume $A$ is local, essentially of finite type over a field or over an excellent discrete valuation ring. Then $NK_n(A) \to NK_n(\hat{A})$ is injective where $\hat{A}$ denotes the completion of $A$.*

*Proof.* As $K$-theory commutes with filtered direct limits and Theorem 1.2 tells that $NK_n(A) \to NK_n(B)$ is injective for any étale neighborhood $(B, \mathfrak{q})$ of $(A, \mathfrak{m})$, we get an injection $NK_n(A) \to NK_n(A^h)$, where $A^h$ is a henselization of $A$ [12]. By Artin approximation $NK_n(A^h) \to NK_n(\hat{A})$ is injective. (Apply (1.12) and (1.8) of [2] to the functor which assigns to an $A^h$ algebra $C$ the set of non-trivial elements in the kernel of $NK_n(A^h) \to NK_n(C)$.) □

(1.12) An obvious corollary of (1.2) is

**Corollary.** *Let* $A$, $n$ *be as in* (1.2) *and let* $B$ *be Galois over* $A$ *with Galois group* $G$ [12; Chap. I, 5.4]. *Then* $NK_n(A) \simeq H^0(G, NK_n(B))$.

(1.13) *Remark.* Arguing as in the proof of (1.2) one may also show that $H^i(G, NK_n(B))$ vanishes for $i > 0$.

(1.14) Before turning to the proof of Theorem 1.2 we briefly recall how $W(A)$ acts on $NK_n(A)$, so as to fix notation. (For details see [5, 1, 7, 14, 17]). By Bass' theory of contracted functors [3; Chap. XII, §7] we may assume $n \geqq 1$ and by the fundamental theorem we may then view $NK_n(A)$ as a summand $\tilde{K}_{n-1}(\mathcal{Nil}(A))$ of $K_{n-1}(\mathcal{Nil}(A))$, where $\mathcal{Nil}(A)$ is the category of pairs $(P, f)$ with $f$ a nilpotent endomorphism of the finitely generated projective $A$ module $P$. Using the product structure of algebraic $K$-theory and a tensor product functor one makes $K_{n-1}(\mathcal{Nil}(A))$ into a module over the ring $K_0(\mathcal{End}(A))$ where $\mathcal{End}(A)$ is the category of pairs $(P, f)$ with $f$ an endomorphism of the finitely generated projective $A$ module $P$. Let $[P, f]$ denote the class of $(P, f)$ and let $e$ be the idempotent $[A, 1] - [A, 0]$ of $K_0(\mathcal{End}(A))$. Then $\tilde{K}_{n-1}(\mathcal{Nil}(A)) = eK_{n-1}(\mathcal{Nil}(A))$ is a module over $\tilde{K}_0(\mathcal{End}(A)) = eK_0(\mathcal{End}(A))$. By a theorem of Almkvist there is an injective ring homomorphism

$$\chi\colon \tilde{K}_0(\mathcal{End}(A)) \to W(A)$$

with dense image, such that $\chi(e[P, f]) = \omega(\det(1 - Tf)^{-1})$ where

$$\omega\colon (1 + TA[[T]])^\times \to W(A)$$

is the usual isomorphism of topological abelian groups, here normalized as in Bloch [4]. (On $(1 + TA[[T]])^\times$ one puts the $T$-adic topology.) We shall identify $\tilde{K}_0(\mathcal{End}(A))$ with its image under $\chi$. If $f \in A$ we write $[f]$ for $e[A, f] = [A, f] - [A, 0] = \omega((1 - Tf)^{-1})$. An arbitrary element of $W(A)$ can be written uniquely as a convergent sum $\sum_{i \geqq 1} V_i[f_i]$ where the $V_i$ are the Verschiebung operators, characterized by $V_m(\omega(q(T))) = \omega(q(T^m))$. The continuous multiplication in $W(A)$ satisfies

$$(V_i[f])(V_j[g]) = d\, V_{ij/d}[f^{j/d} g^{i/d}]$$

where $d = gcd(i, j)$, $f, g \in A$.

We write $\mathrm{Filt}^m W(A)$ for $\omega((1 + T^{m+1} A[[T]])^\times)$, and $W_m(A)$ for the ring $W(A)/\mathrm{Filt}^m(A)$ of truncated big Witt vectors of length $m$. If $M$ is a $W(A)$ module we write $\mathrm{Filt}_m M$ for the submodule consisting of the element that are annihilated by $\mathrm{Filt}^m W(A)$. One can show [14] that $NK_n(A)$ has a module structure over the ring $W(A)$ such that $NK_n(A)$ is the union of the

$\mathrm{Filt}_m NK_n(A)$, $m \geqq 1$, and such that restriction of scalars to $\tilde{K}_0(\mathscr{E}nd(A))$ agrees with the module structure of $\tilde{K}_{n-1}(\mathscr{N}il(A))$ over $\tilde{K}_0(\mathscr{E}nd(A))$. As the image of $\chi$ is dense these properties determine the action of $W(A)$ on $NK_n(A)$.

(1.15) We now start proving Theorem 1.2. If $C$ is an $A$ algebra, write $H^j(C)$ for the $j$-th cohomology of the complex

$$0 \to NK_n(C) \to NK_n(C \underset{A}{\otimes} B) \to NK_n((C \underset{A}{\otimes} B) \underset{C}{\otimes} (C \underset{A}{\otimes} B)) \to \dots.$$

Consider $\alpha \in H^j(A)$. We have to show $\alpha$ vanishes. By a limit argument we may assume $A$ is noetherian [8; IV, 17.7.9] so that we may freely refer to [12] in the sequel. By (1.8) it suffices to show that for each maximal ideal $\mathfrak{m}$ of $A$ there is $f \in A$, $f \notin \mathfrak{m}$ with $[f]\alpha = 0$. A limit argument tells that we must show that the image of $\alpha$ in $H^j(A_\mathfrak{m})$ vanishes (see (1.8) again) and we further assume $A$ is local. First suppose $A$ is strictly henselian. Then $A \to B$ has a section [8; IV, 18.8.1] and this section provides a contracting homotopy of the Amitsur complex (Exercise, or see [12; Chap. III, 2.1 and 2.2]). Thus the theorem is trivially true in this case.

In general, if $A$ is local and $A^{sh}$ is a strict henselization of $A$, we now know that the image of $\alpha$ in $H^j(A^{sh})$ is zero. This implies that there is a standard étale neighborhood $(C_f, \mathfrak{q})$ of $(A, \mathfrak{m})$ [12; Chap. I, 3.14] such that the image of $\alpha$ in $H^j(C_f) = H^j(C)_{[f]}$ vanishes. Here $C$ is a finite $A$ algebra, free as an $A$ module, $f \in C$, $\mathfrak{q}$ is a maximal ideal of $C$ (lying over the maximal ideal $\mathfrak{m}$ of $A$) with $f \notin \mathfrak{q}$, $A \to C_f$ is étale. There is a power $f^m$ of $f$ so that $[f^m]\varphi_*(\alpha) = [f]^m \varphi_*(\alpha) = 0$, where $\varphi_*$ is induced by $\varphi: A \to C$, $\varphi_*(\alpha) \in H^j(C)$. Replacing $f$ by its power to simplify notation we record

$$[f]\,\varphi_*(\alpha) = 0. \tag{1.16}$$

(1.17) **Lemma.** *For $r \geqq 1$ the trace map $T_{C/A}: C \to A$ maps $f^r C$ onto $A$.*

*Proof.* It suffices to show that the image of $f^r C$ contains a unit. Go modulo $\mathfrak{m}$ and use that the trace form of the finite separable field extension $A/\mathfrak{m} \to C/\mathfrak{q}$ is non-degenerate. ($C/\mathfrak{q}$ is one of the factors of the artinian $A/\mathfrak{m}$ algebra $C/\mathfrak{m}C$.) □

(1.18) *End of Proof of 1.2.* Let $\varphi^*: \tilde{K}_0(\mathscr{E}nd(C)) \to \tilde{K}_0(\mathscr{E}nd(A))$ be induced by the forgetful functor $\mathscr{E}nd(C) \to \mathscr{E}nd(A)$. The Verschiebung operators $V_r$ are induced by the functors

$$(P, g) \mapsto \left(P^{\oplus r}, \begin{pmatrix} 0 & & g \\ 1 & & \\ & \ddots & \\ 0 & 1 & 0 \end{pmatrix}\right).$$

See [7]. Clearly they commute with $\varphi^*$. For $g \in C$ and $f$ as above we find $\varphi^*((V_r[g])[f]) = \varphi^*(V_r[f^r g]) = V_r \varphi^*[f^r g] = V_r \omega(\det(1 - T f^r g)^{-1})$ where in the last member $f^r g$ is viewed as an $A$ linear endomorphism of $C$. Therefore $\varphi^*((V_r[g])[f]) \equiv V_r[T_{C/A}(f^r g)] \bmod \mathrm{Filt}^r W(A)$ and with Lemma 1.17 this shows that $\varphi^*(\tilde{K}_0(\mathscr{End}(C))[f])$ is dense in $W(A)$. The functoriality of the $K$-theory product yields for $P \in \tilde{K}_0(\mathscr{End}(C))$ the projection formula

$$\varphi^*(P[f])\alpha = \varphi^*(P[f]\,\varphi_*(\alpha)).$$

The right hand side vanishes by (1.16). Choose $s$ so that $\alpha \in \mathrm{Filt}_s H^j(A)$ and choose $P$ so that $\varphi^*(P[f]) \equiv 1 \bmod \mathrm{Filt}^s W(A)$. Then the left hand side in the projection formula equals $\alpha$ so that $\alpha = 0$. □

(1.19) *Exercise.* Show that $\varphi^*$ extends to a continuous $W(A)$ linear map $W(C) \to W(A)$.

(1.20) Under the conditions indicated in Remark 1.4 the proof goes roughly like this. If $A$ is local the map $A^{sh} \to A^{sh} \otimes_A B$ does not quite split, but there is by [8; IV, 18.5.11] a direct factor $D$ of the tensor product that is finite over $A^{sh}$, free as an $A^{sh}$ module [11; 3G], such that the trace $D \to A^{sh}$ is surjective. As $D$ is a $B$ algebra the map $D \to D \otimes_A B$ splits, so that $H^j(D)$ vanishes. One derives from this as in (1.18) (with $f=1$) that $H^j(A^{sh})$ vanishes and from there one follows the old proof. □

(1.21) *Remark.* As is clear from the proof, Theorem 1.2 remains valid with $\mathbb{Z}/m$ coefficients, i.e. the complex

$$0 \to NK_n(A)/mNK_n(A) \to NK_n(B)/mNK_n(B) \to \dots$$

is also exact, and therefore the complex

$$0 \to NK_n(A; \mathbb{Z}/m) \to NK_n(B; \mathbb{Z}/m) \to \dots$$

is exact too.

## § 2. Big Witt Vectors and Étale Maps

(2.1) Big Witt vectors and truncated big Witt vectors behave rather badly. If $A$ is noetherian, $W_2(A)$ need not be noetherian, and if $B$ is finite over $A$, $W_2(B)$ need not be finite over $W_2(A)$. Nevertheless we will see that for étale maps good properties may be proved, cf. [9].

(2.2) If $A$ is of finite type over $\mathbb{Z}$, say generated by $x_1, \dots, x_{d-1}$ where $d \geqq 1$, and if $t \geqq 1$, then $W_t(A)$ is finite over the subring generated by $[x_1], \dots, [x_{d-1}]$. (Exercise. Also get used to the abuse of notation.) In this situation $W_t(A)$ is thus noetherian of dimension at most $d$. As most of our problems commute with filtered direct limits this observation allows us to reduce to the noetherian case when studying truncated big Witt vectors. For the Big Witt vectors themselves this does not work.

(2.3) **Lemma.** *Let* $t \geqq 1$, $f \in A$. *With notations as in* (1.8), (1.14), *we have* $W_t(A_f) \simeq W_t(A)_{[f]}$ *and, more generally, for a multiplicative system* $S$ *in* $A$: $W_t(S^{-1}A) \simeq [S]^{-1} W_t(A)$. □

(2.4) **Theorem** *Let $B$ be étale over $A$, $t \geqq 1$.*

(i) $W_t(A) \underset{W(A)}{\otimes} W_{t+1}(B) \to W_t(B)$ *is an isomorphism.*

(ii) $W_t(B)$ *is étale over* $W_t(A)$.

(iii) *If $C$ is another $A$ algebra and $B$ or $C$ is finite over $A$ then*

$$W_t(B) \underset{W_t(A)}{\otimes} W_t(C) \to W_t(B \underset{A}{\otimes} C)$$

*is an isomorphism. It also is an isomorphism if $B$ and $C$ are both étale over $A$.*

(iv) *If $B$ is étale and finite over $A$, of degree $r$, then $W(B)$ is étale and finite over $W(A)$, of degree $r$.*

(v) *If $B$ is étale and faithfully flat over $A$ and $M$ is a $W_t(A)$ module, then the "augmented Amitsur complex"*

$$0 \to M \to M \underset{W_t(A)}{\otimes} W_t(B) \to M \underset{W_t(A)}{\otimes} W_t(B \underset{A}{\otimes} B) \to \dots$$

*is exact.*

(2.5) To prove the theorem we start with a few lemmas.

**Lemma.** *Let $A$ be local of residue characteristic $p>0$ and let $B$ be étale over $A$. For $e \geqq 1$ the $p^e$-th powers in $B$ generate $B$ as an $A$ module.*

*Proof.* Define $\Delta: A[X] \to A[X]$ by $(\Delta f)(X) = f(X+1) - f(X)$. Take $f(X) = X^{p^e}$ and consider the $A$ module $M$ generated by $f(B)$. It contains $(\Delta^i f)(B)$ for $i \geqq 1$, in particular for $i = p^e - 1$. Thus $M$ contains $(p^e)!\,B$. Go modulo $(p^e)!$ and $p$ is nilpotent in $A$. By the Nakayama lemma we may assume $A$ has characteristic $p$. Then [9; 1.5.7.1] applies. □

(2.6) **Lemma.** *Let $A$ be local, $B$ finite étale over $A$, $t \geqq 1$, such that each residue field of $B$ has at least $t^2$ elements. There is a basis $b_1, \dots, b_n$ of the free $A$ module $B$ such that for each $i$ with $1 \leqq i \leqq t$ the sequence $b_1^i, \dots, b_n^i$ is also a basis. It follows that $[b_1], \dots, [b_n]$ is a basis of the $W_t(A)$ module $W_t(B)$.*

*Proof.* For the last statement use $[b]\, V_r[a] = V_r[b^r a]$ and proceed along the filtration Filt* $W_t(B)$. For the rest we may assume $A$ is a field. We may assume $B$ is a finite (separable) field extension of $A$. We will take $b_j$ of the form $b^j$ with $b$ chosen suitably. Clearly $b$ is suitable if and only if $b^i$ generates the field extension for each $i$ with $1 \leqq i \leqq t$. If $B$ is a finite field, a generator of its multiplicative group is a suitable choice (exercise). If $B$ is infinite we see from a general position argument that it suffices to show for fixed $i$ that $b^i$ can be taken outside each of the finitely many field extensions of $A$ that are properly contained in $B$. By the same kind of general position argument we need consider only one of the intermediate fields. We must show that this field $K$ does not contain all $i$-th powers. As $B$ is separable over $K$ we may assume $i$ is not a multiple of the characteristic. Choose $a_0, \dots, a_i$ distinct and non-zero in $A$ and recall that the Vandermonde determinant $\det(a_r^s)$ is invertible. Thus if $K$ contains $(a_r + b)^i = \sum_s a_r^s \binom{i}{s} b^{i-s}$ for all $r$, it contains $\binom{i}{s} b^{i-s}$. Take $s = i-1$. □

(2.7) **Lemma.** *Let $t \geqq 1$ and let $I$ be an ideal in $A$.*

(i) *If $B$ is étale over $A$ then $W_t(I)\,W_t(B) = W_t(IB)$.*

(ii) *If $C$ is an $A$ algebra and $I$ is finitely generated then the $W_t(I)\,W_t(C)$-adic completion of $W_t(C)$ equals $W_t(\hat{C})$ where $\hat{C}$ is the $IC$-adic completion of $C$.*

*Proof.* (i) By (2.3) and (1.8) we may assume $A$ is local. Use Lemma 2.5 and the formulas $(V_r[b])(V_{rp^e}[a]) = r\,V_{rp^e}[b^{p^e} a]$, proceeding – as always – along the filtration of $W_t(B)$.

(ii) Now use $V_s[a^{rs} c] = [a]^r\, V_s[c]$. □

(2.8) *Part* (i) *of Theorem* 2.4 is proved like part (i) of (2.7). □

(2.9) Recall the ghost maps $gh_n\colon W(A) \to A$ sending $\sum V_r[a_r]$ to $\sum_{d|n} d a_d^{n/d}$. They are homomorphisms [7]. Let Rad denote the Jacobson radical.

**Lemma.** *Let $A$ be local with residue field $k$ and let $t \geqq 1$.*

(i) $\mathrm{Rad}(W_t(A))$ *is the kernel of the direct product over $i$ not divisible by* $\mathrm{char}(k)$, $1 \leqq i \leqq t$, *of the composite homomorphisms $W_t(A) \underset{gh_i}{\to} A \to k$. This makes $W_t(A)/\mathrm{Rad}(W_t(A))$ isomorphic with $\prod_i k$. Thus $W_t(A)$ is semilocal.*

(ii) *If $B$ is finite étale over $A$ then* $\mathrm{Rad}(W_t(B)) = \mathrm{Rad}(W_t(A))\,W_t(B)$ *and $W_t(B)/\mathrm{Rad}(W_t(B))$ is isomorphic with $\prod_i (B \underset{A}{\otimes} k)$.*

*Proof.* (i) As $W(\mathrm{Rad}(A)) \subset \mathrm{Rad}(W(A))$ (Exercise) we may assume $A$ is a field. If $p = \mathrm{char}(A) \neq 0$ then $p$ is topologically nilpotent in $W(A)$ and therefore $V_{pi}[a]$ is topologically nilpotent for $i \geqq 1$, $a \in A$.

(ii) Use (2.7(i)) and (2.4(i)) and argue similarly. □

(2.10) *Proof of* (2.4(ii)). We may assume $A$, $B$, $W_t(A)$, $W_t(B)$ are noetherian (see (2.2)). Using [8; Chap. IV, 17.6.3], (2.3) and (1.8), we may change the problem somewhat and now assume $A$ local, $B$ essentially étale [8; Chap. IV, 18.6.1] over $A$. Let $\mathfrak{m}$ be the maximal ideal of $A$. As $W_t(\mathfrak{m}) \subset \mathrm{Rad}(W_t(A))$ and $W_t(\mathfrak{m})\,W_t(B) = W_t(\mathfrak{m}B) \subset \mathrm{Rad}(W_t(B))$ by (2.9), the new problem is to show that the $W_t(\mathfrak{m}B)$-adic completion of $W_t(B)$ is étale over the $W_t(\mathfrak{m})$-adic completion of $W_t(A)$ [8; Chap. IV, 17.6.3]. By (2.7) we may thus replace $A$ by $\hat{A}$, $B$ by $\hat{B}$. Now $B$ has become finite étale over $A$ and it is clear from (2.9) that $W_t(B)$ is unramified over $W_t(A)$. If the residue fields of $B$ have at least $t^2$ elements then (2.6) shows that $W_t(B)$ is free over $W_t(A)$ and we are done. If some residue fields are too small, choose $C$ faithfully flat and étale over $B$ so that its residue fields are sufficiently larger and observe that $W_t(C)$ is étale over $W_t(A)$, faithfully flat over $W_t(B)$. □

(2.11) *Proof of* (2.4(iii)). Surjectivity is proved like part (i) of (2.7). Remains to show our map is injective. We may assume $A$ is of finite type over $\mathbb{Z}$. Using standard étale maps and localization the case that both $B$ and $C$ are étale over $A$ is reduced to the case that $C$ is finite over $A$ and $B$ étale over $A$. As in (2.10) we change that case slightly and now consider the case that $A$ is local, $C$ finite over $A$, $B$ essentially étale over $A$, $A$ essentially of finite type over $\mathbb{Z}$. Check that $W_t(C)$ is finite over $W_t(A)$ (first pretend $A$ is of finite type over $\mathbb{Z}$, then localize) and that $W_t(A)$, $W_t(B)$, $W_t(C)$ are noetherian. To see that our map is injective, we may complete it with respect to the $W_t(\mathfrak{m})\,W_t(B)$-adic topology. Using (2.7) and the fact that $W_t(C)$ is a finitely presented $W_t(A)$ module we see that this amounts to replacing $A$, $B$, $C$ by their respective completions. This way we have reduced to the case that $A$ is local, $B$ is finite étale over $A$, which is thus the only case that remains. Finish as in (2.10), using Lemma 2.6. □

(2.12) *Remark.* Of course the restriction on the size of the residue fields in Lemma 2.6 was needed only to produce a basis of a specific form for $W_t(B)$ over $W_t(A)$. One may always construct a basis by lifting one of $W_t(B)/\mathrm{Rad}(W_t(B))$ over $W_t(A)/\mathrm{Rad}(W_t(A))$, when $B$ is finite étale over the local ring $A$ (see (2.9), (2.4(ii))).

(2.13) *Proof of* (2.4(iv)). We first assume $A$ is of finite type over $\mathbb{Z}$, generated by $d-1$ elements, $d \geqq 1$. We shall construct $d+r$ elements $b_1, \ldots, b_{d+r}$ of $W(B)$ that generate $W(B)$ as a $W(A)$-module. To see that they generate it suffices to check that they generate $W_t(B \otimes_A A_{\mathfrak{m}})$ over $W_t(A_{\mathfrak{m}})$ for each $t \geqq 1$ and each maximal ideal $\mathfrak{m}$ of $A$ (use (2.4(i)), (1.8), (2.3)). By Lemma 2.9 and the Nakayama lemma we must assure that for $t \geqq 1$ and for each $\mathfrak{m}$ whose residue characteristic does not divide $t$, the $gh_t(b_i)$ generate $B/\mathfrak{m}B$ over $A/\mathfrak{m}$. We construct the $b_i$ along the filtration, considering one $t$ at a time, and leaving the residues mod $\mathrm{Filt}^{t-1}(W(B))$ fixed when working on $t$. Fix $t$. Construct $b_i$ by induction on $i$, and using general position arguments as in [3; Chap. IV, proof of (2.8)] so that the set of $\mathfrak{m}$ with residue characteristic that does not divide $t$ and with $\dim_{A/\mathfrak{m}}$ (span of $gh_t(b_1), \ldots, gh_t(b_i)$ in $B/\mathfrak{m}B) \leqq \min(r-1,\ i-j)$, has dimension at most $d-j$ for $j \geqq 0$. (The dimension of the empty set is $-\infty$. For $i = d+r$, $j = d+1$ the condition says that $gh_t(b_1), \ldots, gh_t(b_{d+r})$ span $B/\mathfrak{m}B$ if $t \notin \mathfrak{m}$.) This construction yields a surjective $W(A)$ linear map $F \to W(B)$, where $F$ is a free $W(A)$ module on $d+r$ generators. We wish to find a splitting of this map. This splitting is constructed along the projective system $W(B) = \varprojlim W_t(B)$ using the fact that $F \otimes_{W(A)} W_t(A) \to W_t(B)$ splits, and that the freedom of choice in the splitting of $F \otimes_{W(A)} W_t(A) \to W_t(B)$ surjects onto the freedom of choice of splitting $F \otimes_{W(A)} W_{t-1}(A) \to W_{t-1}(B)$. Thus $W(B)$ is a finitely generated projective $W(A)$ module. To see that it is étale over $W(A)$ we must split $W(B) \otimes_{W(A)} W(B) \to W(B)$ [8; Chap. IV, 18.3.1]. This can be done for similar reasons. It is clear from Lemma 2.9 that $W_t(B)$ has degree $r$ over $W_t(A)$. Therefore the idempotent $E$ in $W(A)$ which defines the part of the spectrum over which $W(B)$ has degree $r$ is congruent to 1 modulo $\mathrm{Filt}^t\, W(A)$, for each $t$. We must have $E=1$. We have proved part (iv) if $A$ is of finite type over $\mathbb{Z}$. The general case is obtained by base change from this special case and one deals with it by applying base change at appropriate stages of the above proof. □

(2.14) *Remark.* Because $W(A)$ is rather unwieldy as an abstract ring, it may be better to consider it only with its topology. Thus part (ii) seems more relevant than part (iv).

(2.15) *Remark.* If $p$ is prime and all integers prime to $p$ are invertible in $A$, then one may want to pass to $p$-Witt vectors by multiplying with the appropriate idempotent in $W(A)$ [4; I(3.5)].

(2.16) *Proof of* (2.4(v)). Again we may assume $A$ is local noetherian. By part (ii) the map $W_t(A) \to W_t(A^{sh})$ is faithfully flat where $A^{sh}$ is a strict henselization of $A$. By part (iii) base change along this faithfully flat map reduces us to the case $A = A^{sh}$. As in (1.15) this case is trivial because of a contracting homotopy.

## § 3. Étale Localization of $NK_n$

(3.1) Let $M$ be a $W(A)$-module that is the union of its submodules $\mathrm{Filt}_t M$ (see (1.14) for the notation). If $B$ is étale over $A$, then part (i) of Theorem 2.4 allows us to form $\varinjlim_t (\mathrm{Filt}_t M \underset{W_t(A)}{\otimes} W_t(B))$. We call the limit $M \underset{W(A)}{\overline{\otimes}} W(B)$ and, for $m \in \mathrm{Filt}_t M$, $b \in W_t(B)$, we write $m \overline{\otimes} b$ for the corresponding element of $M \underset{W(A)}{\overline{\otimes}} W(B)$. There is an obvious map from $M \underset{W(A)}{\otimes} W(B)$ onto $M \underset{W(A)}{\overline{\otimes}} W(B)$ and its kernel is generated by the $m \otimes b$ for which there is a sequence $(b_n)$, converging to $b$, with $m \otimes b_n = 0$ in $M \underset{W(A)}{\otimes} W(B)$. (Exercise. First show that the kernel is generated by the $m \otimes b$ with $m \in \mathrm{Filt}_t M$, $b \in \mathrm{Filt}^t W(B)$ for some $t$).

*Remark* If $\mathrm{Filt}^t W(B) \subset (\mathrm{Filt}^t W(A))\, W(B)$, then clearly $M \underset{W(A)}{\overline{\otimes}} W(B)$ coincides with $M \underset{W(A)}{\otimes} W(B)$. Weibel has pointed out that this applies if $A$ contains the field of rational numbers. (Inspect the images of $\mathrm{Filt}^t W(A)$ and $\mathrm{Filt}^t W(B)$ under the isomorphism $gh$: $W(B) \to \prod_\infty B$ of [4, p. 195]). Another case where this applies is given in (3.2(ii)). But one may show that it does not apply when $A = k[X]$, $B = k[X, X^{-1}]$, where $k$ is a field of positive characteristic.

(3.2) **Theorem.** *Let $A$, $n$ be as in* (1.2) *and let $B$ be étale over $A$.*

(i) *The map $NK_n(A) \underset{W(A)}{\overline{\otimes}} W(B) \to NK_n(B)$ is an isomorphism.*

(ii) *If $B$ is finite étale over $A$ then $NK_n(A) \underset{W(A)}{\otimes} W(B) \to NK_n(B)$ is an isomorphism.*

(3.3) *Remark.* This theorem generalizes Theorem 1.6 above of Vorst and was suggested by him.

(3.4) *Proof of the Theorem.* (ii) By part (iv) of Theorem 2.4 the $W(A)$ module $W(B)$ is finitely presented so that $\mathrm{Filt}^t W(B) \subset (\mathrm{Filt}^t W(A))\, W(B)$. Thus (3.1) shows that part (ii) follows from part (i).

(i) For an $A$ algebra $C$ let $H^j(C)$ denote the $j$-th cohomology of the complex

$$0 \to NK_n(C) \underset{W(C)}{\overline{\otimes}} W(B \underset{A}{\otimes} C) \to NK_n(B \underset{A}{\otimes} C) \to 0,$$

and let $\alpha \in H^j(A)$. We have to show $\alpha$ vanishes. As this problem involves direct limits with truncated big Witt vectors in them, we can still make the usual reductions and assume $A$ is local noetherian. We arge as in the proof of

Theorem 1.2. If $A^{sh}$ is a strict henselization of $A$ then $H^j(A^{sh})$ vanishes because $B \underset{A}{\otimes} A^{sh}$ is a direct product of finitely many copies of $A^{sh}$ and possibly a factor whose degree over $A^{sh}$ is smaller than the degree of $B$ over $A$, so that we may assume it is harmless, by induction on the degree (= maximum number of points in a geometric fiber). As in (1.15) we find a finite $A$ algebra $C$, free as an $A$ module, and $f \in C$ with $[f]\,\varphi_*(\alpha)=0$, where $\varphi: A \to C$ is such that $A \to C_f$ is étale and faithfully flat. To proceed as in (1.18) we need transfer maps $\varphi^*: H^j(C) \to H^j(A)$. Now $NK_n(C) \overline{\underset{W(C)}{\otimes}} W(B \underset{A}{\otimes} C) \simeq NK_n(C) \overline{\underset{W(A)}{\otimes}} W(B)$ by part (iii) of Theorem 2.4. Further the projection formula shows that $\varphi^*: NK_n(C) \to NK_n(A)$ is $W(A)$ linear, so that it induces

$$\varphi^*: NK_n(C) \underset{W(A)}{\otimes} W(B) \to NK_n(A) \underset{W(A)}{\otimes} W(B)$$

and one easily sees from (3.1) that this factors through

$$\varphi^*: NK_n(C) \overline{\underset{W(A)}{\otimes}} W(B) \to NK_n(A) \overline{\underset{W(A)}{\otimes}} W(B).$$

From here the way is clear. □

(3.5) *Exercise.* Rederive Theorem 1.2 from Theorem 2.4(v) and Theorem 3.2.

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