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Variational Problems with Non-Convex Obstacles and an Integral-Constraint for Vector-Valued Functions

Frank Duzaar and Martin Fuchs

Mathematisches Institut der Universität, Universitätsstr. 1, D-4000 Düsseldorf, Federal Republic of Germany

0. Introduction

During the last years interesting regularity theorems have been proved for obstacle problems (with integral constraints) in the vector-valued case, we mention the papers [6–10, 12]. Unfortunately complete results for general non-convex obstacles were only obtained in two dimensions, in higher dimensions the obstacle had to be convex. Some progress has been made by Hildebrandt, Meier ([8]): If $u: \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$ minimizes $\int_\Omega |Dv|^2 dx$ under the condition $v(\Omega) \subset \bar{M}$ and if we assume M to be diffeomorphic to some convex set K then by transformation we get a minimum-problem with convex obstacle K and associated functional of the form $\int_\Omega b^{ij}(v) Dv^i \cdot Dv^j dx$ and according to [8], Theorem 3 regularity would follow if we impose some smallness condition relating the coefficients b^{ij} and the diameter of K which is in general not satisfied.

Very recently the second author ([4]) could extend some results in [12] to arbitrary dimensions by showing partial regularity in the special case of a graph-obstacle $u^N \geqq f(u^1, \ldots, u^{N-1})$ where $f: \mathbb{R}^{N-1} \to \mathbb{R}$ is a general smooth function satisfying some growth conditions. The argument rests on an appropriate linearisation of the minimum-property giving an elliptic system with a vector-measure on the right-hand side which does not behave too bad. In the sequel the first author ([1]) could apply this method to more complicated obstacles.

The plan and purpose of this paper can now be summarized as follows: We try to extend the above mentioned result of Hildebrandt, Meier ([8], Theorem 3) by considering the problem

$$(0.1)\qquad F_\Omega(u) = \tfrac{1}{2} \int_\Omega |Du|^2 dx \to \min$$

in the class

$$(0.2)\qquad \mathscr{C} := \{v \in H^{1,2}(\Omega)^N : v = u_0 \text{ on } \partial\Omega, G_\Omega(v) := \int_\Omega g(\cdot, v) dx = K, v(\Omega) \subset \bar{M}\}$$

where $\bar{M} \subset \mathbb{R}^N$ can be mapped on a closed convex set K in a Bi-Lipschitz way. In two steps we prove that the minimizer u is regular up to a set of $\mathbb{H}^{n-2}$-measure zero:

Step 1 consists in a suitable linearisation which transforms (0.1) into an elliptic system with right-hand side of quadratic growth in the gradient of u. Here we use ideas of [1, 4] combined with techniques due to Hildebrandt and others ([8, 9]) which enable us to handle the integral constraint. In the second step we prove a reverse Hölder-inequality for the minimum, well-known arguments of Giaquinta, Giusti (see [5]) give the desired result. We wish to remark that by different techniques replacing step 2 it is possible to handle more general side-conditions, but these arguments are much more technical than the approach described here so that they will be described in a forthcoming paper.

1. Notations and the Result

We fix the following general assumptions.

(A1) $\Omega \subset \mathbb{R}^n$, $n \geqq 3$, is a bounded Lipschitz-domain (see [11]), $N \geqq 2$ denotes a given integer.

(A2) $M \subset \mathbb{R}^N$ is bounded and open with smooth boundary ∂M of class C^2; $P: \bar{M} \to K := \{z \in \mathbb{R}^N : |z| \leqq 1\}$ is a given Bi-Lipschitz map.

(A3) $g: \bar{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ is continuous with continuous partial derivative $D_2 g$ (which means differentiation with respect to the $\mathbb{R}^N$-variable). For domains $D \subset \Omega$ and functions $u: D \to \mathbb{R}^N$ let

$$F_D(u) := \frac{1}{2} \int_D |Du|^2 \, dx, \qquad G_D(u) := \int_D g(\cdot, u) \, dx.$$

(A4) The boundary values u_0 belong to the space $H^{1,2}(\Omega)^N$ and satisfy $u_0(\Omega) \subset \bar{M}$ and the non-degeneracy condition

$$\mathbb{H}^{n-1}(\{x \in \partial\Omega : D_2 g(x, \Phi_0(x)) \neq 0\} \cap \Phi_0^{-1}(\mathbb{R}^N \sim \partial M)) > 0, \; \Phi_0 := u_0|_{\partial\Omega}.$$

(A5) For a given real number κ the class $\mathscr{C}$ defined in (0.2) is non-empty.

Then we have the following

Theorem. *Let* (A1)–(A5) *hold and let* u *be the solution of* (0.1). *Then there is an open subset* Ω_0 *of* Ω *such that* $\mathbb{H}^{n-2}(\Omega \sim \Omega_0) = 0$ *and* $u \in C^{1,\alpha}(\Omega_0)^N$ *for all* $0 < \alpha < 1$.

Remarks. 1) It is possible to replace F_Ω by more general functionals $\int_\Omega a_{\alpha\beta} D_\alpha u \cdot D_\beta u \, dx$ with smooth coefficients $a_{\alpha\beta}(x)$.

2) The non-degeneracy condition (A4) and the assumptions concerning g are essentially due to Hildebrandt, Meier ([8]), we use a simplified version which can be found (for the scalar case) in the papers of Eisen ([2, 3]). In contrast to [2, 3, 8] we do not have to impose growth conditions on g and $D_2 g$ with respect to the second argument since we assume M to be bounded.

2. Linearisation

We start with an auxiliary result which enables us to handle the integral contraint. The proof of Lemma 1 is essentially due to Hildebrandt, Wente ([9]).

Lemma 1. *Under the assumptions of the theorem there exist two balls B^1, B^2 and two functions f^1, f^2 having the following properties:*

(i) $B^i \Subset \Omega$, $\operatorname{dist}(B^1, B^2) > 0$, $f^i \in \mathring{H}^{1,2}(B^i)^N \cap L^\infty$.

(ii) $(u + tf^i)(B^i) \subset \bar{M}$ for $|t| \leqq t_0$.

(iii) $\delta G_\Omega(u, f^i) := \frac{d}{dt}\Big|_0 G_\Omega(u + tf^i) = 1$.

Proof. Since $\partial\Omega$ is Lipschitz we infer from [11], Thm. 3.6.1 and (A4)

$$\mathbb{L}^n(\{x \in \Omega : D_2 g(x, u(x)) \neq 0\} \cap u^{-1}(\mathbb{R}^N \sim \partial M)) > 0$$

and by a subdivision argument there are balls B^1, B^2 as above such that

$$\mathbb{L}^n(\{x \in B^i : D_2 g(x, u(x)) \neq 0\} \cap u^{-1}(M)) > 0, \qquad i = 1, 2. \tag{2.1}$$

Denote by M_δ the inner parallel set of M at distance $\delta > 0$. Obviously (2.1) holds with M replaced by M_δ if we take δ small enough. For $A := M_\delta$, $B := M_{\delta/2}$ let $\rho \in C_0^1(B, [0, 1])$, $\rho = 1$ on A. If

$$\frac{d}{dt}\Big|_0 \int_{B^i} g(\cdot, u + t\eta\rho(u))\, dx = 0$$

for all $\eta \in C_0^1(B^i)^N$ we would have a contradiction to (2.1) and our choice of δ. Thus we find $\eta^i \in C_0^1(B^i)^N$ such that $\delta G_{B^i}(u, \eta^i \rho(u)) \neq 0$ and we may assume that the first variation is $= 1$. Now define $f^i := \eta^i \rho(u) \in \mathring{H}^{1,2}(B^i) \cap L^\infty$; for t_0 sufficiently small (ii) is obviously satisfied. QED

To prove our theorem we can consider subdomains D of Ω having diameter d as small as we want; for d small enough either $D \cap B^1 = \varnothing$ or $D \cap B^2 = \varnothing$. So we may restrict ourselves to work on $\Omega_1 := \Omega \sim \bar{B}^1$.

Lemma 2. *On $\Omega_1 u$ satisfies the system of Euler-equations*

$$-\Delta u - \alpha D_2 g(\cdot, u) = \Lambda \cdot \lambda \tag{2.2}$$

with a positive Radon-measure λ and $\Lambda \in L^\infty(\Omega_1, \lambda)^N$, $\alpha := \delta F_\Omega(u, f^1)$. For λ we have the estimate

$$\lambda \leqq C(|Du|^2 + |D_2 g(\cdot, u)|) \cdot \mathbb{L}^n \tag{2.3}$$

where C denotes a positive constant.

Proof. For $r > 0$ fixed we take $h: [0, \infty) \to \mathbb{R}$ of class C^1 satisfying $h = 1$ on $[0, r]$, $h' \leqq 0$, $h = 0$ on $[2r, \infty)$. We let $d(y) := \operatorname{dist}(y, \partial M)$, $V(y) := Dd(y)$, $y \in \bar{M}$, and define for $\varepsilon > 0$, $\eta \in C_0^\infty(\Omega_1, [0, \infty))$ $v_\varepsilon := u + \varepsilon \cdot \eta \cdot V(u) \cdot h(d(u))$. For ε small enough

$v_\varepsilon(\Omega) \subset \bar{M}$ and if we introduce the functions

$$\varphi(\varepsilon, t) := \varphi_0 + \varphi_1(\varepsilon) + \varphi_2(t), \qquad \psi(\varepsilon, t) := \kappa + \psi_1(\varepsilon) + \psi_2(t),$$
$$\varphi_0 := F_\Omega(u), \qquad \varphi_1(\varepsilon) := F_\Omega(v_\varepsilon) - F_\Omega(u), \qquad \varphi_2(t) := F_\Omega(u + tf^1) - F_\Omega(u),$$
$$\psi_1(\varepsilon) := G_\Omega(v_\varepsilon) - G_\Omega(u), \qquad \psi_2(t) := G_\Omega(u + tf^1) - G_\Omega(u)$$

we have according to [8], Lemma 3:

$$\varphi_1'(0) - \alpha \cdot \psi_1'(0) \geqq 0.$$

By the Riesz-representation-theorem this inequality is equivalent to

$$\text{(2.4)} \qquad \int_{\Omega_1} Du \cdot D(\eta \cdot V(u) \cdot h(d(u)))\,dx - \alpha \int_{\Omega_1} D_2 g(\cdot, u) \cdot V(u) \cdot h(d(u)) \cdot \eta\,dx = \int_{\Omega_1} \eta \cdot d\lambda$$

for all functions $\eta \in C_0^\infty(\Omega_1)$.

Now we extend V to a neighborhood of ∂M. For $T: \mathbb{R}^N \to \mathbb{R}^N$ with support in a small ball $K_R(y_0)$ centered at $y_0 \in \partial M$ and the property $T \cdot V = 0$ we introduce the flow $\Phi(s, y)$ which satisfies

$$\Phi(0, y) = y, \frac{\partial}{\partial s|_0} \Phi(s, y) = T(y), \qquad \Phi(s, y) \in \bar{M} \quad \text{for } y \in \bar{M}.$$

Thus the function $u_\varepsilon := \Phi(\varepsilon \cdot \eta \cdot h(d(u)), u)$ respects the obstacle provided $|\varepsilon| \ll 1$ and $\eta \in C_0^\infty(\Omega_1)$. The argument from above gives

$$\text{(2.5)} \qquad \int_{\Omega_1} Du \cdot D(\eta \cdot h(d(u)) \cdot T(u))\,dx - \int_{\Omega_1} \alpha \cdot D_2 g(\cdot, u) \eta \cdot h(d(u)) T(u)\,dx = 0.$$

We cover a neighborhood of ∂M with balls $K_R(y_l)$, $l = 1, \ldots, L$, $y_l \in \partial M$ on which we can find vector-fields T_l^i, $i = 1, \ldots, N-1$, $l = 1, \ldots, L$ such that $T_l^i \cdot T_l^j = \delta_{ij}$, $T_l^i \cdot V = 0$ on $K_R(y_l)$. Let $\{\varphi^l\}$ be a partition of the unity subordinate to $\{K_R(y_l)\}$, $\varphi^1 + \ldots + \varphi^L = 1$ in a neighborhood of ∂M. For i and l fixed (2.5) holds for $T(z) := \varphi^l(z) \cdot T_l^i(z)$, that is

$$\text{(2.6)} \qquad \int_{\Omega_1} Du \cdot D(\eta \cdot \varphi^l(u) \cdot h(d(u)) \cdot T_l^i(u))\,dx - \int_{\Omega_1} \alpha \cdot D_2 g(\cdot, u) \cdot T_l^i(u) \cdot h(d(u)) \cdot \eta \cdot \varphi^l(u)\,dx = 0.$$

Now using (2.4) (with $\eta = \psi \cdot V(u) \cdot h(d(u)) \cdot \varphi^l(u)$), a smoothing argument and Hahn-Banach we infer from (2.4)

$$\text{(2.7)} \qquad \int_{\Omega_1} Du \cdot D\{(\psi \cdot V) \cdot h^2(d(u)) \cdot \varphi^l(u) \cdot V(u)\}\,dx - \alpha \cdot \int_{\Omega_1} D_2 g(\cdot, u) \cdot \{\ldots\}\,dx = \int_{\Omega_1} \Lambda^l \cdot \psi\,d\lambda \qquad \text{for all } \psi \in C_0^\infty(\Omega_1)^N,$$

where Λ^l is a function in the space $L^\infty(\Omega_1, \lambda)^N$, $|\Lambda^l| \leqq 1$. Next we take $\eta = T_l^i(u) \cdot h(d(u)) \cdot \psi$ in (2.6); adding the result to (2.7), summing over $i = 1, \ldots, N-1$

and then over $l=1,\dots,L$ we get

$$(2.8)\qquad \int_{\Omega_1} Du\cdot D\{h^2(d(u))\psi\}\,dx-\alpha\int_{\Omega_1} D_2 g(\cdot,u)\cdot\{\dots\}\,dx=\int_{\Omega_1} \Lambda\cdot\psi\,d\lambda,$$

$$\Lambda=\Lambda^1+\dots+\Lambda^L.$$

For all $|\varepsilon|\ll 1$ $w_\varepsilon:=u+\varepsilon(1-h^2(d(u)))\cdot\psi$ satisfies $w_\varepsilon(\Omega_1)\subset\bar{M}$, thus we get the equation

$$(2.9)\qquad \int_{\Omega_1} Du\cdot D\{(1-h^2(d(u)))\cdot\psi\}\,dx-\alpha\int_{\Omega_1} D_2 g(\cdot,u)\cdot\{\dots\}\,dx=0,$$

adding (2.8), (2.9) we arrive at (2.2). Finally we observe that the measure λ is independent of the parameter r (use the variation $v_\varepsilon:=u+\varepsilon\eta\{h_1(d(u))-h_2(d(u))\}V(u)$, $|\varepsilon|\ll 1$). The estimate (2.3) follows by passing to the limit $r\to 0$ in (2.4) observing the properties of h and h'. QED

3. Proof of the Theorem

According to [5], Theorem 1.3, Remark 1.3, Theorem 1.5 in Chapter VI and our Lemma 2 the statement of the theorem is an immediate consequence of

Lemma 3. *Under the assumptions of the theorem there exist constants R_0, $C>0$ and $p>2$ such that $Du\in L^p_{\mathrm{loc}}(\Omega)^{nN}$ and*

$$(3.1)\qquad \Big(\fint_{B_R(x)} |Du|^p\,dz\Big)^{1/p}\leq C\Big(\fint_{B_{2R}(x)} (1+|Du|)^2\,dz\Big)^{1/2}$$

for all $x\in\Omega$ and all $R\leq\min(R_0,\operatorname{dist}(x,\partial\Omega))$.

Proof. For R small enough every ball $B_R(x)\subset\Omega$ is contained either in Ω_1 or Ω_2. Let us consider the first case and assume for simplicity $x=0$. For $0<\rho<r<R$ fixed we choose a cut-off function $\eta=1$ on B_ρ, $\eta=0$ on $\Omega\sim B_r$ such that $|D\eta|\leq c/(r-\rho)$ and $0\leq\eta\leq 1$. We define

$$w:=P^{-1}(Pu+\eta\cdot(a-Pu)),\qquad a:=\fint_{B_R} Pu\,dx.$$

Since P is Bi-Lipschitz w belongs to $H^{1,2}(\Omega)^N$ and satisfies $w=u$ on $\partial\Omega$, $w(\Omega)\subset\bar{M}$. Moreover we have

$$(3.2)\qquad F_{B_r}(w)\leq F_{B_r}(Pu+\eta(a-Pu))\cdot\operatorname{Lip}(P^{-1})^2$$

and the properties of g imply

$$(3.3)\qquad |G_{B_r}(w)-G_{B_r}(u)|\leq CR^n.$$

Here and in the sequel we denote all constants independent of $B_R(x)$ by the same symbol C. By Lemma 1 there is a continuous function $\xi\colon[-t_0,t_0]\to\mathbb{R}$, $\xi(0)=0$, such that

$$\chi(t):=G_{B^1}(u+tf^1)-G_{B^1}(u)=t\cdot(1+\xi(t)).$$

We may assume $1+\xi(t)\geqq 1/2$ for $|t|\leqq t_0$, thus $[-t/2,t/2]\subset\chi([-t,t])$, $0<t\leqq t_0$. Combining this with (3.3) we see that

$$G_{B_r}(u)-G_{B_r}(w)=\chi(t) \tag{3.4}$$

for a suitable $|t|\leqq t_0$, provided we take $R\leqq R_0$. According to (3.4) the function

$$v:=w \quad \text{on } B_r, :=u+tf^1 \quad \text{on } B^1, :=u \text{ elsewhere}$$

belongs to the class $\mathscr{C}$, therefore by (3.2)

$$\begin{aligned} F_{B_r}(u) &\leqq CF_{B_r}(Pu+\eta(a-Pu))+F_{B^1}(u+tf^1)-F_{B^1}(u)\\ &\leqq C\int_{B_r}|(1-\eta)\cdot D(Pu)+D\eta\cdot(a-Pu)|^2\,dx+C|t|,\\ |t| &\leqq 2|G_{B_r}(u)-G_{B_r}(w)|\leqq CR^n. \end{aligned}$$

Applying Young's inequality we get the estimate

$$\int_{B_\rho}|Du|^2\,dx\leqq C\{\int_{B_r\sim B_\rho}|Du|^2\,dx+(r-\rho)^{-2}\int_{B_r}|Pu-a|^2\,dx+R^n\}.$$

We now fill the hole and use an iteration-lemma due to Giusti ([5], *V*, Lemma 3.1) to conclude

$$\int_{B_{R/2}}|Du|^2\,dx\leqq C\{R^{-2}\int_{B_R}|Pu-a|^2\,dx+R^n\}.$$

The desired result follows by applying the Sobolev-Poincaré inequality and [5], *V*, Proposition 1.1. QED

4. Some Remarks and Extensions

1) It is well-known (see [5]) that x is a regular point (i.e. $x\in\Omega_0$) if and only if $\otimes \lim_{R\to 0} R^{2-n}\int_{B_R(x)}|Du|^2\,dx=0$. According to Lemma 3 we thus can improve the estimate for the singular set to $\mathbb{H}^{n-p}(\Omega\sim\Omega_0)=0$.

2) It is easy to see by using $v(x):=u(Rx/|x|)$, $B_R(0)\subset\Omega$, combined with a suitable correcting variation as comparison map that $\psi(r):=r^{2-n}\int_{B_r}|Du|^2\,dx$ is more or less increasing. Despite of this fact we did not succeed in proving that $\otimes$ from above is satisfied everywhere. The approach described in [5], VII Section 3, fails in our situation since it essentially rests on a smallness condition which must not be satisfied.

3) In the absence of the integral constraint we get a further improvement of our result, we have

Theorem. *If u is a local minimum of the Dirichlet-integral under the side-condition $u(\Omega)\subset\bar{M}$, M as in* (A2), *then for $n=3$ the singularities of u are isolated in Ω, $\mathbb{H}$-$\dim(\Omega\sim\Omega_0)\leqq n-3$ for $n\geqq 3$.*

Here IH-dim denotes the Hausdorff-dimension of the set. For the proof one only minor modification occurs in the proof of [5a], Lemma 1, where it is changes, the boundedness of u required in [5a] is implied by the side-condition and all comparison-maps used by Giaquinta-Giusti respect the obstacle. The only minor dodification occurs in the proof of [5a], Lemma 1, where it is shown that certain blow-ups converge to a minimizer. The map $v^\nu := w + \eta(u^\nu - v)$ introduced there has to be replaced by $v^\nu := (P^{-1} \circ Q)\ (Pw + \eta(Pu^\nu - Pv))$ where P is defined in (A2) and $Q: K' \to K$, $K' := \{z \in \mathbb{R}^N : |z| \leqq 3\}$ denotes the standard retraction

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