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Solvable Lie Algebras and Generalized Cartan Matrices Arising from Isolated Singularities

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§ 1. Introduction

In this paper we shall provide a general method of constructing solvable Lie algebras from isolated hypersurface singularities. The ideas come from our previous result [9] with Mather, which says that isolated hypersurface singularities determine and are determined by their moduli algebras. There are two natural problems raised by this result. The first one is the recognition problem: when is a commutative local algebra a moduli algebra of an isolated hypersurface singularity? The second question asks what kind of information does one need from the moduli algebra in order to determine the topological type of the singularity. Since moduli algebras are Artinian algebras, their associated algebras of derivations are finite dimensional Lie algebras. It is these Lie algebras that we are interested in. We conjectured in 1982 these Lie algebras to be solvable, which would give a necessary condition for the first problem. The second question has been studied by many authors including Lê and Ramanujam [7], Pham [13], Teissier [18, 19], and Zariski [23, 24]. Zariski shows that two irreducible plane curves are topological equivalent if and only if their associated numerical invariants so called Puiseux characteristic are the same (cf. also Pham [13]). Until now the higher dimension problem remains unsolved. Actually there is not even a conjecture of what the result should be. By [9], in order to determine the singularity $(V, 0)$ topologically, we need only to know partial information from $A(V)$. So we want to forget some information in $A(V)$. This leads us to consider $L(V)$. We conjecture that $L(V)$ is sufficient to determine the topological type of the singularity $(V, 0)$. The examples in [22] and the examples in the present article support our conjecture. They also show that $L(V)$ is not a topological invariant but only a "generic" topological invariant in some sense. Therefore $L(V)$ still contains too much information, so we want to forget some information in $L(V)$. This leads us to consider the generalized Cartan matrix $C(V)$. We suspect that $C(V)$ is actually a topological

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invariant. In particular, if $C(V)$ and $C(W)$ are of different types, then V is not topologically equivalent to W. Unlike the resolution matrix which is defined only for surface singularities, our definition for $C(V)$ should work for singularities of arbitrary dimension. There is a natural map from the algebra of derivations of the local ring of the singularity to this Lie algebra (cf. Lemma 2.1). In general this is not surjective. So the finite dimensional Lie algebras we consider here are quite different from those infinite dimensional Lie algebras which were considered before by K. Saito, Scheja-Wiebe, C.T.C. Wall and J. Wahl. We prove that if $(V,0)$ admits a $\mathbb{C}^*$-action, then the Lie algebra is abelian if and only if $(V,0)$ is either A_1 or A_2 singularity (cf. Proposition 2.4). In §3, we first write down an interesting one parameter family of inequivalent finite dimensional representations of a fixed Lie algebra. This family is non-trivial in the sense of Proposition 3.1. We shall restrict ourselves to two dimensional isolated hypersurface singularities in §4 and prove that the Lie algebras which we consider here are solvable. The higher dimension case will be discussed in a future paper. In §5, in view of the recent work of Santharoubane [26], we are able to attach a generalized Cartan matrix and hence a Kac-Moody Lie algebra to any isolated hyperface singularity. This generalized Cartan matrix is a new analytic invariant of isolated hypersurface singularities.

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§2. Isolated Singularities and Finite Dimensional Lie Algebras

In this section, we shall first establish a connection between the set of isolated hypersurface singularities and the set of finite dimensional Lie algebras. Let $(V,0)$ be an isolated hypersurface singularity in $(\mathbb{C}^{n+1},0)$ defined by the zero of a holomorphic function f. The moduli algebra $A(V)$ of V is

$$\mathbb{C}\{z_0, z_1, \ldots, z_n\} \Big/ \left(f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}\right).$$

Recall that in [18], Mather-Yau prove that the natural mapping

$$\begin{Bmatrix} \text{isolated hypersurface} \\ \text{singularities of dimension } n \end{Bmatrix} \to \begin{Bmatrix} \text{Commutative local Artinian} \\ \text{algebras} \end{Bmatrix}$$
$$(V,0) \to A(V) = \text{moduli algebra of } V$$

is one to one.

We define $L(V)$ to be the algebra of derivations of $A(V)$. Since $A(V)$ is finite dimensional as $\mathbb{C}$-vector space and $L(V)$ is contained in the endomorphism algebra of $A(V)$; consequently $L(V)$ is a finite dimensional Lie algebra. Thus we have the following natural mapping

$$\begin{Bmatrix} \text{isolated hypersurface} \\ \text{singularities} \end{Bmatrix} \to \begin{Bmatrix} \text{finite dimensional Lie} \\ \text{algebras} \end{Bmatrix}$$
$$(V,0) \to L(V).$$

Let $\mathcal{O}_{n+1}$ denote the ring of germs of the origin of holomorphic functions $(\mathbb{C}^{n+1},0)\to\mathbb{C}$. Let $\mathcal{O}_{V,0}=\mathcal{O}_{n+1}/(f)$ be the local ring of V at 0. Then we have the following lemma.

Lemma 2.1. *A derivation of $\mathcal{O}_{V,0}$ induces a derivation of $A(V)$. Hence there is a natural map from the algebra of derivations of $\mathcal{O}_{V,0}$ to $L(V)$.*

Proof. Let D be a derivation of $\mathcal{O}_{V,0}$. Then $D=\sum_{i=0}^{n} a_i\frac{\partial}{\partial z_i}$ where $a_i\in\mathcal{O}_{n+1}$ for all $0\leqq i\leqq n$ and $D(f)=bf$ for some $b\in\mathcal{O}_{n+1}$.

To prove that D induces a derivation of $A(V)$, we have to prove

$$D\left(\frac{\partial f}{\partial z_j}\right)\in\left(f,\frac{\partial f}{\partial z_0},\dots,\frac{\partial f}{\partial z_n}\right)\mathcal{O}_{n+1}\quad\text{for all } 0\leqq j\leqq n.$$

$$D\left(\frac{\partial f}{\partial z_j}\right)=\frac{\partial b}{\partial z_n}f+b\frac{\partial f}{\partial z_j}-\sum_{i=0}^{n}\frac{\partial a_i}{\partial z_j}\frac{\partial f}{\partial z_i}\in\left(f,\frac{\partial f}{\partial z_0},\dots,\frac{\partial f}{\partial z_n}\right)\mathcal{O}_{n+1}.\quad\text{Q.E.D.}$$

Remark 2.2. The above natural map is not surjective in general. This can be seen as follows:

Let us assume for a moment that f is a weighted homogeneous function, i.e., there exist $q_0,\dots,q_n$, $d\in\mathbb{N}$ (the set of positive integers) such that

$$f(t^{q_0}z_0,\dots,t^{q_n}z_n)=t^d f(z_0,\dots,z_n)\tag{2.1}$$

for all $(z_0,\dots,z_n)\in\mathbb{C}^{n+1}$ and $t\in\mathbb{C}^*=\mathbb{C}-\{0\}$. Then $E=\sum_{i=0}^{n} q_i z_i\frac{\partial}{\partial z_i}$ is a derivation of the local ring $\mathcal{O}_{V,0}$. This distinguished derivation is called Euler derivation. The following proposition is well known (cf. J. Wahl; Proc. Symp. Pure Math. AMS, Vol. 40, 2, p. 615).

Proposition 2.3. *Let $(V,0)$ be an isolated singularity with $\mathbb{C}^*$-action i.e., $\mathcal{O}_{V,0}=\mathcal{O}_{n+1}/(f)\,\mathcal{O}_{n+1}$ where f is a weighted homogeneous holomorphic function. Then the algebra of derivations of $\mathcal{O}_{V,0}$ is generated as an $\mathcal{O}_{V,0}$ module by the Euler derivation E and the following derivations:*

$$\frac{\partial f}{\partial z_j}\frac{\partial}{\partial z_i}-\frac{\partial f}{\partial z_i}\frac{\partial}{\partial z_j}.$$

Proof. Derivations of $\mathcal{O}_{V,0}$ are induced by derivations of $\mathcal{O}_{n+1}$ sending $(f)\mathcal{O}_{n+1}$ into $(f)\mathcal{O}_{n+1}$. Let D be any derivation of $\mathcal{O}_{V,0}$, then $Df=hf$ for some $h\in\mathcal{O}_{n+1}$. Since $Ef=df$, we have $D'f=0$ where $D'=D-\frac{h}{d}E$. Let $D'=a_i\frac{\partial}{\partial z_i}$, then $a_i\frac{\partial f}{\partial z_i}=0$. Because the singularity is isolated, $\left\{\frac{\partial f}{\partial z_0},\dots,\frac{\partial f}{\partial z_n}\right\}$ forms a regular sequence, whence the only relations are those generated by the obvious ones. Q.E.D.

Proposition 2.3 says that in case $(V,0)$ admits a $\mathbb{C}^*$-action then the image of the natural map defined in Lemma 2.1 is $A(V)\cdot E\subseteq\mathrm{Der}(A(V))=L(V)$.

Proposition 2.4. *Let $f(z_0,\dots,z_n)$ be a weighted homogeneous function. Suppose $V=\{(z_0,\dots,z_n)\in\mathbb{C}^{n+1}\colon f(z_0,\dots,z_n)=0\}$ has an isolated singularity at the origin.*

Then the Lie algebra $L(V)$ associated to the singularity $(V,0)$ is abelian if and only if $(V,0)$ is either an A_1 or A_2 singularity.

Proof. "if" the defining equation for A_1 singularity is $z_0^2+z_1^2+\ldots+z_n^2$. Its moduli algebra $A(V)$ is isomorphic to $\mathbb{C}$. Therefore the derivation algebra $L(V)$ of the moduli algebra is zero.

The defining equation for A_2 singularity is $z_0^3+z_1^2+\ldots+z_n^2$. Its moduli algebra $A(V)$ is a $\mathbb{C}$-vector space spanned by 1 and z_0 with multiplication rule $z_0^2=0$. Therefore the derivation algebra $L(V)$ of the moduli algebra is a 1-dimensional $\mathbb{C}$-vector space spanned by $z_0\frac{\partial}{\partial z_0}$, in particular $L(V)$ is abelian.

"only if". By [14], after analytic change of coordinates, we may write f in the following form.

$$f=h(z_0,\ldots,z_r)+z_{r+1}^2+\ldots+z_n^2$$

where h is a weighted homogeneous polynomial with multiplicity at least three. Notice that the moduli algebra of f is isomorphic to the moduli algebra of h.

Suppose the Lie algebra $L(V)$ is abelian. Let $\sum_{i=0}^{r} a_i\frac{\partial}{\partial z_i}$ be any derivation of the moduli algebra. Then

$$\left[\sum_{i=0}^{r} q_i z_i\frac{\partial}{\partial z_i}, \sum_{i=0}^{r} a_i\frac{\partial}{\partial z_i}\right]=\sum_{i=0}^{r}\left(\sum_{j=0}^{r} q_j z_j\frac{\partial a_i}{\partial z_j}\right)\frac{\partial}{\partial z_i}-\sum_{i=0}^{r} q_i a_i\frac{\partial}{\partial z_i}.$$

The fact that the algebra of derivations is abelian implies

$$\sum_{j=0}^{r} q_j z_j\frac{\partial a_i}{\partial z_j}\equiv q_i a_i \bmod \left(\frac{\partial h}{\partial z_0},\frac{\partial h}{\partial z_1},\ldots,\frac{\partial h}{\partial z_r}\right) \quad \text{for } 0\leqq i\leqq r. \tag{2.2}$$

We observe that since $(V,0)$ has isolated singularity at the origin, the ideal

$$\left(\frac{\partial h}{\partial z_0},\frac{\partial h}{\partial z_1},\ldots,\frac{\partial h}{\partial z_r}\right)\mathcal{O}_{r+1}$$

contains the maximal ideal $m_{r+1}\subseteq\mathcal{O}_{r+1}$ to certain power. Let k be the least positive integer such that

$$\left(\frac{\partial h}{\partial z_0},\frac{\partial h}{\partial z_1},\ldots,\frac{\partial h}{\partial z_r}\right)\mathcal{O}_{r+1}\supseteq m_{r+1}^k. \tag{2.3}$$

since multiplicity of h is at least 3, it follows that multiplicity of $\frac{\partial h}{\partial z_i}$ is at least 2 for all $0\leqq i\leqq r$. Therefore $m_{r+1}^2\subseteq\left(\frac{\partial h}{\partial z_0},\frac{\partial h}{\partial z_1},\ldots,\frac{\partial h}{\partial z_r}\right)\mathcal{O}_{r+1}$ and $k\geqq 2$.

If $k\geqq 3$, then there exists a monomial b in $m_{r+1}^{k-1}\subseteq m_{r+1}^2$ such that

$$b\notin\left(\frac{\partial h}{\partial z_0},\ldots,\frac{\partial h}{\partial z_r}\right)\mathcal{O}_{r+1}.$$

Since multiplicity of h is at least 3, it follows that $\frac{\partial^2 h}{\partial z_j \partial z_i} \in m_{r+1}$ for all $0 \leqq i$, $j \leqq r$. Therefore

$$b \frac{\partial}{\partial z_j}\left(\frac{\partial h}{z_i}\right) \in m_{r+1}^{k-1} \cdot m_{r+1} = m_{r+1}^k \subseteq \left(\frac{\partial h}{\partial z_0}, \frac{\partial h}{\partial z_1}, \ldots, \frac{\partial h}{\partial z_r}\right) \mathcal{O}_{r+1} \quad \text{for all } 0 \leqq i,\ j \leqq r$$

by (2.3). This simply means that $b \frac{\partial}{\partial z_j}$ is an element in $L(V)$ for all $0 \leqq j \leqq r$. Eq. (2.2) implies that

$$\sum_{j=0} q_j z_j \frac{\partial b}{\partial z_j} \equiv q_i b \bmod \left(\frac{\partial h}{\partial z_0}, \ldots, \frac{\partial h}{\partial z_r}\right) \mathcal{O}_{r+1} \quad \text{for } 0 \leqq i \leqq r \tag{2.4}$$

since b is a monomial of degree $\geqq 2$, $b = z_0^{n_0} z_1^{n_1} \ldots z_r^{n_r}$ with $n_0 + n_1 + \ldots + n_r \geqq 2$. The left hand side of (2.4) is $\left(\sum_{j=0}^{r} n_j q_j\right) b$. As $n_0 + \ldots + n_r \geqq 2$, there exists an i such that $\sum_{j=0}^{r} n_j q_j > q_i$. This will imply that $b \in \left(\frac{\partial h}{\partial z_0}, \ldots, \frac{\partial h}{\partial z_r}\right) \mathcal{O}_{r+1}$ by (2.4), which is a contradiction to our choice of b.

By the argument above, we conclude that $k=2$. In this case, we have

$$m_{r+1}^2 = \left(\frac{\partial h}{\partial z_0}, \ldots, \frac{\partial h}{\partial z_r}\right) \mathcal{O}_{r+1}. \tag{2.5}$$

Since the minimal number of generators for m_{r+1}^2 is $\frac{(2+r)!}{2!r!}$ and the minimal number of generators for $\left(\frac{\partial h}{\partial z_0}, \ldots, \frac{\partial h}{\partial z_r}\right) \mathcal{O}_{r+1}$ is $r+1$, (2.5) holds only if $\frac{(2+r)!}{2!r!} = r+1$, i.e., $r=0$. Clearly we may assume without loss of generality that $f = z_0^{\ell+1} + z_1^2 + \ldots + z_n^2$. The moduli algebra is isomorphic to $\mathbb{C}\{z_0\}/(z_0^{\ell})\mathbb{C}\{z_0\}$. The algebra of derivations $L(V)$ of the moduli algebra is spanned by $\langle z_0 \frac{\partial}{\partial z_0}, z_0^2 \frac{\partial}{\partial z_0}, \ldots, z_0^{\ell-1} \frac{\partial}{\partial z_0}$. This Lie algebra is abelian if and only if $\ell = 1$ or 2. Q.E.D.

§ 3. A Continuous Family of Finite Dimensional Representations of a Lie Algebra

Let us consider a family of simple elliptic singularities in $\mathbb{C}^3$ defined by

$$x^3 + y^3 + z^3 + txyz = 0$$

where $t^3 + 27 \neq 0$. For each fixed t with $t^3 + 27 \neq 0$, the moduli algebra is given by

$$A(V_t) = \langle 1, x, y, z, xy, yz, zx, zyx \rangle$$

with multiplication rules: $x^2=-\frac{t}{3}yz,\ y^2=-\frac{t}{3}zx,\ z^2=-\frac{t}{3}xy$

$$x^2y=xy^2=y^2z=yz^2=x^2z=xz^2=0.$$

We shall assume $t\neq 0$ and $\frac{t^6}{27}-7t^3-216\neq 0$. Under these assumptions

$$\begin{aligned}L(V_t)=\Big\langle & xy\frac{\partial}{\partial x}-\frac{t}{6}zx\frac{\partial}{\partial y},\ zx\frac{\partial}{\partial x}-\frac{t}{6}xy\frac{\partial}{\partial z},\ xyz\frac{\partial}{\partial x}\\ & -\frac{t}{6}yz\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y},\ yz\frac{\partial}{\partial y}-\frac{t}{6}xy\frac{\partial}{\partial z},\ xyz\frac{\partial}{\partial y}\\ & -\frac{t}{6}zx\frac{\partial}{\partial y}+yz\frac{\partial}{\partial z},\ -\frac{t}{6}yz\frac{\partial}{\partial x}+zx\frac{\partial}{\partial z},\ xyz\frac{\partial}{\partial z},\ x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\Big\rangle\end{aligned}$$

$L(V_t)$ is independent of t and will be denoted by $L(\tilde{E}_6)$.

The natural representations ρ_t of $L(V_t)$ on $A(V_t)$ can be extended to $\mathbb{C}$. Then we have the following result, which can be checked by computation.

Proposition 3.1. *For $j(t)\neq j(t')$, the representations ρ_t^A and $\rho_{t'}$ are not equivalent for any automorphism A of the Lie algebra $L(\tilde{E}_6)$.*

§4. Solvability of $L(V)$

Theorem 4.1. *Suppose that $V=\{(x,y,z)\}\in\mathbb{C}^3\colon f(x,y,z)=0\}$ has an isolated singularity at $(0,0,0)$. Then the finite dimensional Lie algebra $L(V)$ associated to the singularity is solvable.*

We first begin with two observations.

Lemma 4.2. *Let $D=a(x,y,z)\frac{\partial}{\partial x}+b(x,y,z)\frac{\partial}{\partial y}+c(x,y,z)\frac{\partial}{\partial z}$ be an element in $L(V)$ where $a(x,y,z)$, $b(x,y,z)$ and $c(x,y,z)$ are in $\mathcal{O}_3=\mathbb{C}\{x,y,z\}$. Then $a(0,0,0)=0=b(0,0,0)=c(0,0,0)$. In particular $L(V)$ acts on m/m^2 where m is the unique maximal ideal of $\mathcal{O}_3$.*

Proof. Suppose on the contrary that $a(0,0,0)\neq 0$. Then $a(x,y,z)$ would be an unit in $\mathcal{O}_3$. Since V has only isolated singularity at the origin,

$$\Delta(f)=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)\mathcal{O}_3$$

contains the maximal ideal m raised to certain power. Hence we can choose a smallest positive integer k such that

$$x^k\in(f)+\Delta(f).$$

As D is an element in $L(V)$, D leaves the ideal $(f)+\Delta(f)$ invariant; in particular

$$D(x^k)\in(f)+\Delta(f).$$

On the other hand $D(x^k)=k\,a(x,y,z)\,x^{k-1}$. Since $a(x,y,z)$ is an unit, this implies $x^{k-1}\in(f)+\Delta(f)$, which contradicts to our original choice of k.

Similary, we can prove $b(0,0,0)=0=c(0,0,0)$. Q.E.D.

Lemma 4.3. *Let* $L_1=\{D\in L(V)\colon D(m)\subseteq m^2\}$. *Then* L_1 *is a nilpotent ideal.*

Proof. By Lemma 4.2, it is clear that L_1 is an ideal in $L(V)$. Let $L_1^1=[L_1,L_1]$, $L_1^2=[L_1,L_1^1],\ldots,L_1^r=[L_1,L_1^{r-1}]$. We claim that for any $D\in L_1^r$, $D(m)\subseteq m^{r+2}$. We shall prove this by induction. Without loss of generality, we shall assume that $D=[D_1,D_2]$ where $D_1\in L_1$ and $D_2\in L_1^{r-1}$

$$\begin{aligned} D(m)&=D_1(D_2(m))-D_2(D_1(m))\\ &\subseteq D_1(m^{r+1})-D_2(m^2)\\ &\subseteq m^{r+2}-m^{r+2}\\ &=m^{r+2}. \end{aligned}$$

Let k be a positive integer such that $m^k\subseteq(f)+\Delta(f)$. Then $L_1^{k-2}(m)\subseteq m^k\subseteq(f)+\Delta(f)$. This means that $L_1^{k-2}=0$, i.e. L_1 is a nilpotent Lie algebra. Q.E.D.

Let us now recall the well-known $s\ell(2,C)$ representation theory.

Theorem 4.4 (Weyl). *Let* $\phi\colon L\to g\ell(V)$ *be a finite dimensional representation of a semisimple Lie algebra,* $V\neq 0$. *Then* ϕ *is completely reducible i.e.* V *is a direct sum of irreducible L-submodules.*

Recall that $A_1=(s\ell(2,C))$ is the complex Lie algebra with basis

$$\tau=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\quad X_+=\begin{pmatrix}0&1\\0&0\end{pmatrix}\quad X_-=\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

and relations $[\tau,X_+]=2X_+$, $[\tau,X_-]=-2X_-$, $[X_+,X_-]=\tau$. Let V be an arbitrary A_1-module. Since h is semisimple, V can be written as direct sum of eigenspaces $V_\lambda=\{v\in V\colon \tau v=\lambda v\}$, $\lambda\in\mathbb{C}$. λ is called the weight of τ in V and V_λ is called a weight space.

Theorem 4.5. *Let* V *be an irreducible module for* $A_1=s\ell(2,\mathbb{C})$.

(a) *Relative to* τ, V *is the direct sum of weight spaces* V_μ, $\mu=m$, $m-2,\ldots,-(m-2),-m$; *where* $m+1=\dim V$ *and* $\dim V_\mu=1$ *for each* μ.

(b) V *has (up to nonzero scalar multiples) a unique maximal vector* v_0 *whose weight (called the highest weight of* V*) is* m.

(c) *The action of* A_1 *on* V *is given explicitly by the following formulas, if the basis is chosen to be* $\{v_0,v_1,\ldots,v_m\}$ *where* $v_i=X_-^i\,v_0$.

In fact, the matrices representation with respect to this basis are given as follows

$$\tau \to \begin{pmatrix} m & & & & 0 \\ & m-2 & & & \\ & & \ddots & & \\ & & & -(m-2) & \\ 0 & & & & -m \end{pmatrix} \quad X_+ \to \begin{pmatrix} 0 & \mu_1 & & & 0 \\ & 0 & \mu_2 & & \\ & & & \ddots & \\ & & & & \mu_m \\ 0 & & & & 0 \end{pmatrix}$$

$$X_- \to \begin{pmatrix} 0 & & & & 0 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ 0 & & \ddots & & \\ & & & 1 & 0 \end{pmatrix}$$

where $\mu_i = i(m-i+1)$.

We now apply the above theory to some concrete cases.

Lemma 4.6. *Let M_2^k be the space of homogeneous polynomials of degree k in x and y variables. Let A_1 act on M_2^k via the following actions.*

$$\tau = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y},$$

$$X_+ = x\frac{\partial}{\partial y},$$

$$X_- = y\frac{\partial}{\partial y}.$$

Then M_2^k is an irreducible A_1-module.

Lemma 4.7. *Let M_3^k be the space of homogeneous polynomials of degree k in x, y and z variables. Let A_1 act on M_3^k via the following actions*

$$\tau = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y},$$

$$X_+ = x\frac{\partial}{\partial y},$$

$$X_- = y\frac{\partial}{\partial x}.$$

Then M_3^k decomposes as follows

$$M_3^k = (k+1) \oplus (k) \oplus \ldots \oplus (1)$$

where $(i) := \langle x^{i-1} z^{k-i+1}, x^{i-2} y z^{k-i+1}, \ldots, x^{i-j} y^{j-1} z^{k-i+1}, \ldots, y^{i-1} z^{k-i+1} \rangle$ is an i-dimensional irreducible representation of A_1.

Proof. Obvious. Q.E.D.

The following lemma is a special case of the theorem of Cayley-Silvester to be found in (3.4.2), Chap. 3 of [17].

Lemma 4.8. *Let A_1 act on M_3^k, the space of homogeneous polynomials of degree k in x, y and z variables, via the following actions*

$$\tau = 2x\frac{\partial}{\partial x} - 2z\frac{\partial}{\partial z},$$

$$X_+ = 2x\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial z},$$

$$X_- = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}.$$

Then M_3^k decomposes as follows.

(a) *If $k=2\ell$ is an even integer, then $M_3^k = (1)\oplus(5)\oplus\ldots\oplus(4i+1)\oplus\ldots\oplus(4\ell+1)$, where $(4i+1)$ is a $4i+1$ dimensional irreducible representation of A_1. Moreover (1) is spanned by (y^2-2xz).*

(b) *If $k=2\ell+1$ is an odd integer, then $M_3^k=(3)\oplus(7)\oplus\ldots\oplus(4i+3)\oplus\ldots\oplus(4\ell+3)$, where $(4i+3)$ is a $4i+3$ dimensional irreducible representation of A_1. Moreover (3) is spanned by*

$$\langle x(y^2-2xz)^\ell, y(y^2-2xz)^\ell, z(y^2-2xz)^\ell\rangle.$$

Proof. It follows immediately from the weight decomposition of the k^{th}-symmetric power of $\mathbb{C}^3$. This decomposition is trivially obtained if coordinates on $\mathbb{C}^3$ of weights $2, 0, -2$ are chosen.

Lemma 4.9. *Let f be a homogenous polynomial of degree k and q an irreducible polynomial of degree n, where $k>n$. Suppose that $\frac{\partial f}{\partial x}=q^\ell\cdot a$, $\frac{\partial f}{\partial y}=q^\ell\cdot b$, $\frac{\partial f}{\partial z}=q^\ell\cdot c$ for $\ell\geqq 1$ and suitable a, b, c homogenous polynomials of degree $k-n\ell-1$. Then f is divisible by $q^{\ell+1}$. In particular, if $k=n\ell+1$, i.e. a, b, $c\in\mathbb{C}$ and $n\geqq 2$, then $f=0$.*

Proof. $kf = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = q^\ell(xa+yb+zc)$

$\Rightarrow f=q^\ell\cdot p$ where p is a homogenous polynomial of degree $k-n\ell$

$$\Rightarrow \frac{\partial f}{\partial x} = q^\ell\frac{\partial p}{\partial x} + \ell q^{\ell-1}\frac{\partial q}{\partial x}\cdot p.$$

Since q^ℓ divides $\frac{\partial f}{\partial x}$, q must divide $\frac{\partial q}{\partial x}\cdot p$. As q is irreducible, q divides p. Hence we get f divisible by $q^{\ell+1}$.

Proof of Theorem 4.1. By the Levi decomposition, if the Lie algebra is not solvable, then the Lie algebra $L(V)$ contains A_1 as subalgebra. By Lemma 4.3, we shall assume that A_1 acts on m/m^2 nontrivially.

Suppose that the multiplicity of f is two. After a biholomorphic change of coordinates, we can assume that $f=z^2-g(x,y)$. In this case $L(V)=L(W)$ where $W=\{(x,y)\in\mathbb{C}^2: g(x,y)=0\}$. If multiplicity of g is equal to 2, then again by a biholomorphic change of coordinate, we can assume that $g(x,y)=x^2+y^{n+1}$ where $n\geqq 1$. The associated moduli algebra $A(W)$ is spanned by $1, y, \dots, y^{n-1}$ with multiplication rule $y^n=0$. The Lie algebra $L(W)$ is spanned by $y\frac{\partial}{\partial y}$, $y^2\frac{\partial}{\partial y}, \dots, y^{n-1}\frac{\partial}{\partial y}$ and hence is solvable.

Suppose that multiplicity of g is $k+1$ with $k\geqq 2$. Since A_1 acts on m/m^2 nontrivially, by Theorem 4.5, we know that the representation of A_1 on m/m^2 has the following forms

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \quad \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.$$

Hence there are three elements in $L(W)$ of the following forms.

$$D_1=x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}+\text{higher order operator},$$

$$D_2=x\frac{\partial}{\partial y}+\text{higher order operator},$$

$$D_3=y\frac{\partial}{\partial x}+\text{higher order operator}$$

where higher order operator means operator of the form $p(x,y)\frac{\partial}{\partial x}+q(x,y)\frac{\partial}{\partial y}$ with $p(x,y)$, $q(x,y)\in m^2$. Write

$$g=\sum_{i=k+1}^{\infty} g_i$$

where g_i is a homogenous polynomial of degree i in x and y variable. Clearly D_1, D_2, D_3 act on m^k/m^{k+1}. We can replace D_1 by $\tau=x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}$, D_2 by $X_+ =x\frac{\partial}{\partial y}$ and D_3 by $X_-=y\frac{\partial}{\partial x}$ without changing the actions on $m^k/m^{k+1}\cong M_2^k$, the space of homogenous polynomials of degree k in x and y variables. By Lemma 4.6, M_2^k is an irreducible A_1-module of dimension $k+1$. On the other hand, $L(W)$ leaves $(g)+\Delta(g)$ invariant. In particular, it leaves the initial ideal in $[(g)+\Delta(g)]$ of $(g)+\Delta(g)$ invariant. The space of initial forms of degree k is spanned by $\frac{\partial g_{k+1}}{\partial x}$ and $\frac{\partial g_{k+1}}{\partial y}$. It is a nontrivial invariant subspace of dimension at most two in M_2^k. Since $k+1>2$, this contradicts the irreducibility of M_2^k.

From now on, we shall assume that multiplicity of $f=k+1\geqq 3$. By Theorem 4.5, we know that the representation of A_1 on m/m^2 has one of the following forms.

Case 1. A_1 has the following form

$$\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix}\quad\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}\quad\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}.$$

Then there are three elements in $L(V)$ of the following forms

$$D_1 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + \text{higher order operator},$$

$$D_2 = x\frac{\partial}{\partial y} \qquad + \text{higher order operator},$$

$$D_3 = y\frac{\partial}{\partial x} \qquad + \text{higher order operator}$$

where higher order operator means operator of the form $p(x,y,z)\frac{\partial}{\partial x} + q(x,y,z)\frac{\partial}{\partial y} + r(x,y,z)\frac{\partial}{\partial z}$ with $p(x,y,z)$, $q(x,y,z)$ and $r(x,y,z) \in m^2$. Write

$$f(x,y,z) = \sum_{j=k+1}^{\infty} f_j(x,y,z)$$

where $f_j(x,y,z)$ is a homogenous polynomial of degree j. D_1, D_2, D_3 act on m^k/m^{k+1}. We can replace D_1 by $\tau = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$, D_2 by $X_+ = x\frac{\partial}{\partial y}$ and D_3 by $X_- = y\frac{\partial}{\partial x}$ without changing the actions on $m^k/m^{k+1} \cong M_3^k$, the space of homogenous polynomials of degree k in x, y and z variables. $L(V)$ leaves the initial ideal in $[(f)+\Delta(f)]$ of $(f)+\Delta(f)$ invariant. We now prove by induction on the degree j that all homogenous components f_j of f are divisible by z^2. The space J_k of initial forms of degree k is spanned by $\frac{\partial f_{k+1}}{\partial x}$, $\frac{\partial f_{k+1}}{\partial y}$ and $\frac{\partial f_{k+1}}{\partial z}$. It is an invariant subspace of dimension at most three in M_3^k. If $\dim J_m = 1$ or $\dim J_m = 2$, then $J_m = \langle z^k \rangle$ or $J_m = \langle x z^{k-1}, y z^{k-1} \rangle$ respectively by Lemma 4.7. In both cases, $\frac{\partial f_{k+1}}{\partial x}$, $\frac{\partial f_{k+1}}{\partial y}$ and $\frac{\partial f_{k+1}}{\partial z}$ are divisible by z^{k-1}. Hence by Lemma 4.9, f_{k+1} is divisible by z^k. If $\dim J_m = 3$ and J_m is reducible, then $J_m = \langle z^k \rangle \oplus \langle x z^{k-1}, y z^{k-1} \rangle$ by Lemma 4.7. Again f_{k+1} is divisible by z^k in view of Lemma 4.8. If $\dim J_m = 3$ and J_m is irreducible, then $J_m = \langle x^2 z^{k-2}, xyz^{k-2}, y^2 z^{k-2} \rangle$ by Lemma 4.7. In this case, f_{k+1} cannot be a polynomial in x, y variables alone otherwise $\dim J_m < 3$. Hence z appear either in $\frac{\partial f_{k+1}}{\partial x}$ or $\frac{\partial f_{k+1}}{\partial y}$. This implies $k \geqq 3$. Lemma 4.7 shows that f_{k+1} is divisible by z^{k-1}, in particular, divisible by z^2.

Assume all f_j divisible by z^2 for $j \leqq n$, i.e.

$$f = z^2 \cdot p + f_{n+1} + f_{n+2} + \cdots$$

where p is a polynomial of degree $n-2$. D_1, D_2, D_3 act on $\mathcal{O}/\langle m^{n+1}+(z)\rangle$, the space of polynomials of degree at most n in x and y variables. Denote the image of $(f)+\Delta(f)$ in $\mathcal{O}/\langle m^{n+1}+(z)\rangle$ by $\overline{J(f)}$. It is easy to see that we can replace D_1 by $\tau = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$, D_2 by $X_+ = x\frac{\partial}{\partial y}$ and D_3 by $X_- = y\frac{\partial}{\partial x}$ without changing the actions on $\langle m^n+(z)\rangle/\langle m^{n+1}+(z)\rangle$. $\overline{J(f)}$ is an invariant subspace of dimension at most 3 in $\mathcal{O}/\langle m^{n+1}+(z)\rangle$. $\overline{J(f)}$ is spanned by $\left\langle \frac{\partial f_{n+1}}{\partial x}, \frac{\partial f_{n+1}}{\partial y}, \frac{\partial f_{n+1}}{\partial z} \right\rangle$ and hence may be identified with a subspace of M_2^n. By Lemma 4.6 M_2^n is an irreducible A_1-module of dimension $n+1 \geq k+2 > 4$. Therefore $\overline{J(f)} = 0$. This means that $\frac{\partial f_{n+1}}{\partial x}$, $\frac{\partial f_{n+1}}{\partial y}$ and $\frac{\partial f_{n+1}}{\partial z}$ are divisible by z. By Lemma 4.9, we obtain f_{n+1} divisible by z^2. We have proved that f is divisible by z^2. In particular $(V, 0)$ is not an isolated singularity, a contradiction to our assumption.

Case II. A_1 has the following form

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Hence after change of coordinates there are three elements in $L(V)$ of the following forms.

$$D_1 = 2x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z},$$

$$D_2 = 2x\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial z},$$

$$D_3 = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}.$$

Write

$$f(x, y, z) = \sum_{j=k+1}^{\infty} f_j(x, y, z)$$

where $f_j(x, y, z) \in M_3^j$. D_1, D_2, D_3 act on m^k/m^{k+1}. Recall that $L(V)$ leaves the initial ideal $\mathrm{In}[(f)+\Delta(f)]$ of $(f)+\Delta(f)$ invariant. We first prove that $k+1 \geq 4$. The space J_k of initial form of degree k is spanned by $\frac{\partial f_{k+1}}{\partial x}$, $\frac{\partial f_{k+1}}{\partial y}$ and $\frac{\partial f_{k+1}}{\partial z}$. It is an invariant subspace of dimension at most three in M_3^k. From Corollary 4.8, we get

$k = 2\ell$ even integer $\Rightarrow J_k = \langle (y^2 - 2xz)^\ell \rangle$,

$k = 2\ell + 1$ odd integer $\Rightarrow J_k = \langle x(y^2 - 2xz)^\ell, y(y^2 - 2xz)^\ell, z(y^2 - 2xz)^\ell \rangle$.

By Lemma 4.9, the first case is excluded since $f_{k+1}\neq 0$. Thus k is odd and $k\geqq 3$. Again by Lemma 4.9

$$f_{k+1}=(y^2-2xz)^{\ell+1} \quad \text{where } \ell\geqq 1.$$

We now prove by induction on the degree j that all the homogeneous components f_j of f are divisible by $(y^2-2xz)^{\ell+1}$. Assume all f_j divisible by $(y^2-2xz)^{\ell+1}$ for $j\leqq n$ i.e.,

$$f=(y^2-2xz)^{\ell+1}\cdot p+f_{n+1}+f_{n+2}+\dots$$

where p is a polynomial of degree at most $n-2\ell-2=n-k-1$. D_1, D_2, D_3 acts on $\mathcal{O}/\langle m^{n+1}+(y^2-2xz)^{\ell}\rangle$. Denote the image of $(f)+\Delta(f)$ in $\mathcal{O}/\langle m^{n+1}+(y^2-2xz)^{\ell}\rangle$ by $\overline{J(f)}$. $\overline{J(f)}$ is an invariant subspace of dimension at most 3 in $\langle m^n+(y^2-2xz)^{\ell}\rangle/\langle m^{n+1}+(y^2-2xz)^{\ell}\rangle$. $\overline{J(f)}$ is spanned by $\frac{\partial f_{n+1}}{\partial x}, \frac{\partial f_{n+1}}{\partial y}, \frac{\partial f_{n+1}}{\partial z}$ and hence may be identified with a subspace of M_3^n. By Corollary 4.8, we get

$$n=2\ell_1 \quad \text{even integer} \Rightarrow \overline{J(f)}=\langle (y^2-2xz)^{\ell_1}\rangle,$$
$$n=2\ell_1+1 \quad \text{odd integer} \Rightarrow \overline{J(f)}=\langle x(y^2-2xz)^{\ell_1}, y(y^2-2xz)^{\ell_1}, z(y^2-2xz)^{\ell_1}$$

where in both cases $\ell_1\geqq\ell$. By Lemma 4.9, f_{n+1} is divisible by $(y^2-2xz)^{\ell+1}$. We have proved that f is divisible by $(y^2-2xz)^{\ell+1}$ with $\ell\geqq 1$. In particular $(V,0)$ is not an isolated singularity, a contradiction to our original assumption.

§ 5. Kac-Moody Lie Algebras and Isolated Hypersurface Singularities

In this section we shall attach a Kac-Moody Lie algebra to every isolated hypersurface singularity. Let $(V,0)$ be an isolated hypersurface singularity. Let $\underline{g}(V)$ be the maximal ideal of $L(V)$ consisting of nilpotent elements. We recall from [16] how to construct a generalized Cartan matrix $C(V)$ from $\underline{g}(V)$, which is a new invariant of $(V,0)$.

Definition 5.1 ([16], 2.1). An $\ell\times\ell$ matrix with entries in Z, $C=(c_{ij})$ is a generalized Cartan matrix if

a) $c_{ii}=2 \qquad \forall i=1,\dots,\ell,$

b) $c_{ij}\leqq 0 \qquad \forall i,j=1,\dots,\ell,\ i\neq j,$

c) $c_{ij}=0$ if and only if $c_{ji}=0 \quad \forall i,j=1,\dots,\ell,\ i\neq j.$

Let $\underline{g}(V)$ be the set of all nilpotent elements in $L(V)$. Then $\underline{g}(V)$ is the maximal nilpotent Lie subalgebra of $L(V)$. Let $\operatorname{Der}\underline{g}(V)$ be the derivation algebra of the Lie algebra $\underline{g}(V)$.

Definition 5.2 ([16], 1.2). A torus on $\underline{g}(V)$ is a commutative subalgebra of $\operatorname{Der}\underline{g}(V)$ whose elements are semi-simple endomorphisms. A maximal torus is a torus not contained in any other torus. The dimensional of maximal torus is called Mostow number.

Mostow number is an invariant of isolated singularity $(V,0)$.

Theorem 5.3 (Mostow 4.1 of [22]). *If T_1 and T_2 are maximal tori of $\underline{g}(V)$, then there exists $\theta \in \operatorname{Aut} \underline{g}(V)$ (automorphism group of $\underline{g}(V)$) such that $\theta T_1 \theta^{-1} = T_2$.*

Let T be a maximal torus and consider the root space decomposition of $\underline{g}(V)$ relatively to T, (cf. [16], 1.4)

$$\underline{g}(V) = \sum_{\beta \in R(T)} \underline{g}(V)^{\beta}$$

where $\underline{g}(V)^{\beta} = \{x \in \underline{g}(V) \colon tx = \beta(t)x \, \forall t \in T\}$ and $R(T) = \{\beta \in T^* \colon \underline{g}(V)^{\beta} \neq (0)$. We denote: $m = \dim T$

$$\begin{aligned} R^1(T) &= \{\beta \in R(T) \colon \underline{g}(V)^{\beta} \nsubseteq [\underline{g}(V), \underline{g}(V)]\}, \\ \ell_{\beta} &= \dim(\underline{g}(V)^{\beta}/[\underline{g}(V), \underline{g}(V)] \cap \underline{g}(V)^{\beta}) \, \forall \beta \in R^1(T), \\ d_{\beta} &= \dim \underline{g}(V)^{\beta} \qquad \beta \in R^1(T). \end{aligned}$$

The map: $\beta \to d_{\beta}$, $R^1(T) \to \mathbb{N}^*$ gives the partition:

$$R^1(T) = R^1(T)_{p_1} \cup \ldots \cup R^1(T)_{p_q}$$

where $p_1 < \ldots < p_q$, $R^1(T)_{p_i} \neq \phi$ and $R^1(T)_p = \{\beta \in R^1(T) \colon d_{\beta} = p\}$.

Let $s_i = \# R^1(T)_{p_i}$ and $s = s_1 + \ldots + s_q$; we number the elements of $R^1(T) = \{\beta_1, \ldots, \beta_s\}$ in such a way that:

$$R^1(T)_{p_1} = \{\beta_1, \ldots, \beta_s\}, \qquad R^1(T)_{p_2} = \{\beta_{s_1+1}, \ldots, \beta_{s_1+s_2}\}, \ldots.$$

Let $d_i = d_{\beta_i}$, $\ell_i = \ell_{\beta_i}$ and $\ell = \ell_1 + \ldots + \ell_s$ (one checks that $\ell = \dim \underline{g}(V)/[\underline{g}(V), \underline{g}(V)]$). Let $P_s^{s_1 \ldots s_q}$ be the group of permutations of $\{1, \ldots, s\}$ which leaves $\{1, \ldots, s_1\}$, $\{s_1+1, \ldots, s_1+s_2\}, \ldots$ invariant.

Lemma 5.4 ([16], 1.5). *The integers $m, q, p_1, \ldots, p_q$, $s_1, \ldots, s_q$, $d_1, \ldots, d_s$, $\ell_1, \ldots, \ell_s$, defined above are invariants of isolated hypersurface singularity $(V,0)$.*

The map θ induces a bijection between: $R(T)$ and $R(T')$, $R^1(T)$ and $R^1(T')$, $R^1(T)_{p_i}$ and $R^1(T')_{p_i}$ $1 \leqq i \leqq q$; thus there exists$\to \in P_s^{s_1 \ldots s_q}$ such that

$$\check{\theta} \beta_a = \beta'_{\tau a} \qquad 1 \leqq a \leqq s.$$

Therefore, if T, T' are two maximal torus on $\underline{g}(V)$, then there exists $\theta \in \operatorname{Aut} \underline{g}(V)$ and $\tau \in P_s^{s_1 \ldots s_1}$ such that $\theta \, \underline{g}(V)^{\beta_a} = \underline{g}(V)^{\beta'_{\tau a}}$ $1 \leqq a \leqq s$. Let $f \colon \{1, \ldots, \ell\} \to \{1, \ldots, s\}$ be defined by

$$f(i) = \begin{cases} 1 & \text{if } 1 \leqq i \leqq \ell_1 \\ 2 & \text{if } \ell_1 \leqq i \leqq \ell_1 + \ell_2 \\ \vdots & \\ s & \text{if } \ell_1 + \ldots + \ell_{s-1} < i \leqq \ell. \end{cases}$$

For $\sigma \in P_s^{s_1 \ldots s_q}$, we lift σ to $\hat{\sigma} \in P_\ell$-(Permutation group of ℓ elements) such that $f \circ \hat{\sigma} = \sigma \circ f$. Define an action of $P_s^{s_1 \ldots s_q}$ on the set of $\ell \times \ell$ matrices by setting

$$\sigma(c_{ij})_{1 \leqq i, j \leqq \ell} = (c_{\hat{\sigma} i \hat{\sigma} j})_{1 \leqq i, j \leqq \ell}.$$

Theorem 5.5 ([16], 1.7). *For* $i, j \{1, \ldots, \ell\}$, $i \neq j$; *let*

$$-c_{ij}(T) = \mathrm{Min}\left\{ -n \in \mathbb{N} \cup \{0\} \colon (\mathrm{ad} v)^{-n+1} w = 0 \begin{array}{l} \forall v \in \underline{g}(V)^{\beta_{f(i)}} \\ \forall w \in \underline{g}(V)^{\beta_{f(j)}} \end{array} \right\}$$

with $(\mathrm{ad} 0)^0 = 0$ *and let* $c_{ii}(T) = 2$ *for* $i = 1, \ldots, \ell$. *Then*

$$C(T) = (c_{ij}(T))_{1 \leqq i, j \leqq \ell} \text{ is a generalized Cartan matrix.}$$

For any $\sigma \in P_s^{s_1 \ldots s_q}$, *the action of* σ *on* $C(T)$ *is independent of the lifting* $\hat{\sigma}$ *of* σ. *Furthermore the* $P_s^{s_1 \ldots s_1}$ *orbit of* $C(T)$ *is an invariant of* $(V, 0)$.

Definition 5.6 ([16], 1.8). We choose arbitrarily A in $P_s^{s_1 \ldots s_q}$-orbit of $C(T)$ (which has at most $s!/s_1! \ldots s_q!$ elements) and we say by an abuse of language: "$\underline{g}(V)$ is of type C" or "C is the Cartan matrix of $\underline{g}(V)$". We denote:

$$\mathscr{J}_V(C) = \{T \colon T \text{ is a maximal torus on } \underline{g}(V), C(T) = C\},$$
$$P_s^{s_1 \ldots s_q}(C) = \{\sigma \in P_s^{s_1 \ldots s_q} \colon \sigma C = C\}.$$

Lemma 5.7 ([16], 1.9). *If* T, $T' \in \mathscr{J}_V(C)$, *then there exist* $\theta \in \mathrm{Aut}\, \underline{g}(V)$ *and* $\tau \in P_s^{s_1 \ldots s_q}(C)$ *such that:*

$$\theta \in \underline{g}(V)^{\beta_a} = \underline{g}(V)^{\beta'_{\tau a}} \; \forall a = 1, \ldots, s.$$

We denote by $msg(\underline{g}(V))$ the set of minimal systems of generators of $\underline{g}(V)$; by [(Bourbaki; Lie Algebra, Chap. I), Sect. 4, p. 119]: $(x_1, x_2 \ldots) \in msg(\underline{g}(V))$ if and only if $(x_1 + [\underline{g}(V), \underline{g}(V)],\ x_2 + [\underline{g}(V), \underline{g}(V)], \ldots)$ is a basis of $\underline{g}(V)/[\underline{g}(V), \underline{g}(V)]$. Therefore each element of $msg(\underline{g}(V))$ is an ℓ-triple $(x_1, \ldots, x_\ell)$ where $\ell = \dim \underline{g}(V)/[\underline{g}(V), \underline{g}(V)]$.

Let $T \in \mathscr{J}_V(C)$ and denote:

$$msg(T) = msg(\underline{g}(V)) \in ((\underline{g}(V)^{\beta_1})^{\ell_1} \times \ldots \times (\underline{g}(V)^{\beta_s})^{\ell_s}).$$

For all $(x_1, \ldots, x_\ell) \in msg(T)$ one has:

$$(\mathrm{ad}\, x_i)^{-c_{ij}+1} x_j = 0 \qquad 1 \leqq i \neq j \leqq \ell.$$

We shall now apply the above theory to study Lie algebras of rational double points. We list the results in a table in the appendix.

We include the following proposition as an example:

Proposition 5.8. *Let* $V = \{(x, y, z) \in \mathbb{C}^3 \colon z^4 + y^3 + x^2 = 0\}$ *be the* E_6 *singularity. Then* $A(V) = \langle 1, z, z^2, y, yz, z^2 y \rangle$ *with multiplication rules:* $z^3 = 0$, $y^2 = 0$

$$L(V)=\left\langle z\frac{\partial}{\partial z}, z^2\frac{\partial}{\partial z}, yz\frac{\partial}{\partial z}, yz^2\frac{\partial}{\partial z}, y\frac{\partial}{\partial y}, yz\frac{\partial}{\partial y}, yz^2\frac{\partial}{\partial y}\right\rangle,$$

$$\underset{=}{g}(V)=\left\langle z^2\frac{\partial}{\partial z}, yz\frac{\partial}{\partial z}, yz\frac{\partial}{\partial y}, yz^2\frac{\partial}{\partial z}, yz^2\frac{\partial}{\partial y}\right\rangle \quad \textit{with multiplication rules}$$

$$\begin{array}{ccccc} \| & \| & \| & \| & \| \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$

$$[x_1,x_2]=-x_4, [x_1,x_3]=x_5, [x_1,x_4]=0, [x_1,x_5]=0, [x_2,x_3]=-x_4,$$
$$[x_2,x_4]=0, \quad [x_2,x_5]=0, \quad [x_3,x_4]=0, [x_3,x_5]=0, [x_4,x_5]=0.$$

The type of the E_6 *singularity*$:=\dim \underset{=}{g}/[g,g]=3$.
The nilpotency of the E_6 *singularity*$:=\min\{p\in\mathbb{N}\cup\{0\}: \underset{=}{g}^{p+1}=0\}=1$.

Let δ: $\underset{=}{g}\to\underset{=}{g}$ be a derivation such that $\delta(x_i)=a_{i1}x_1+a_{i2}x_2+a_{i3}x_3+a_{i4}x_4+a_{i5}x_5$ for $1\leqq i\leqq 5$. Then the matrix A representing δ is of the following form

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{21} & a_{24} & a_{25} \\ a_{31} & a_{32} & (a_{11}+a_{31}-a_{13}) & a_{34} & a_{35} \\ 0 & 0 & 0 & (a_{11}+a_{22}-a_{13}) & -a_{21} \\ 0 & 0 & 0 & -(a_{12}+a_{32}) & (2a_{11}+a_{31}-a_{13}) \end{bmatrix}.$$

Clearly δ is semisimple if and only if $AA^*=A^*A$. Let t_1,t_2,t_3 be the three derivations of $\underset{=}{g}(V)$ defined by

$$t_1(x_1)=x_1,\ t_1(x_2)=0,\quad t_1(x_3)=x_3,\ t_1(x_4)=x_4,\quad t_1(x_5)=2x_5,$$
$$t_2(x_1)=0,\quad t_2(x_2)=x_2,\ t_2(x_3)=0,\quad t_2(x_4)=x_4,\quad t_2(x_5)=0,$$
$$t_3(x_1)=x_3,\ t_3(x_2)=0,\quad t_3(x_3)=x_1,\ t_3(x_4)=-x_4,\ t_3(x_5)=0.$$

Then $T=\mathbb{C}t_1\oplus\mathbb{C}t_2\oplus\mathbb{C}t_3$ is a torus of $\underset{=}{g}(V)$. Since $\dim T=3=$the type of E_6, T is a maximal torus of $\underset{=}{g}(V)$. Let β_i: $T\to C$ be a linear map with $\beta_i(t_j)=\delta_{ij}$ for $i,j=1,2,3$. Then

$$\underset{=}{g}(V)=\underset{=}{g}^{\beta_2}\oplus\underset{=}{g}^{2\beta_1}\oplus\underset{=}{g}^{\beta_1+\beta_3}\oplus\underset{=}{g}^{\beta_1-\beta_3}\oplus\underset{=}{g}^{\beta_1+\beta_2-\beta_3}$$
$$=\mathbb{C}x_2\oplus\mathbb{C}x_5\oplus\mathbb{C}(x_1+x_3)\oplus\mathbb{C}(x_1-x_3)\oplus\mathbb{C}x_4.$$

$(x_1+x_3,\ x_1-x_3,\ x_2)$ is a T-minimal system of generators. The generalized Cartan matrix associated to E_6 singularity is

$$C(E_6)=\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Appendix

	dim $A(V)$	dim $L(V)$	dim $g(V)$	Type of $g(V)$	Nil-potency of $g(V)$	Mostow number	Generalized Cartan matrix
A_4	4	3	2	2	0	2	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
A_5	5	4	3	2	1	2	$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$
A_6	6	5	4	2	2	2	$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$
A_k $k \geqq 7$	k	$k-1$	$k-2$	2	$k-4$	1	k odd $\begin{pmatrix} 2 & -(k-4) \\ \frac{k-3}{2} & 2 \end{pmatrix}$ k even $\begin{pmatrix} 2 & -(k-4) \\ \frac{(k-4)}{2} & 2 \end{pmatrix}$
D_4	4	4	2	2	0	2	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
D_5	5	5	4	2	1	2	$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$
D_6	6	6	5	3	2	2	$\begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$
D_7	7	7	6	3	2	2	$\begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$
D_k $k \geqq 8$	k	k	$k-1$	3	$k-5$	1	$\begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -(k-5) \\ 0 & -(k-6) & 2 \end{pmatrix}$
E_6	6	7	5	3	1	3	$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$
E_7	7	8	7	3	4	1	$\begin{pmatrix} 2 & -4 & -3 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$
E_8	8	10	8	4	2	2	$\begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 \\ -2 & -1 & 2 & -2 \\ -2 & -1 & -2 & 2 \end{pmatrix}$

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Added in Proof. Theorem 4.1 is now being generalized to any isolated hypersurface singularities in $\mathbb{C}^n$ for $n \leqq 5$. This can be found in our new article "singularities defined by $s\ell(2, \mathbb{C})$ invariant polynomials and solvability of Lie algebras arising from isolated singularities".