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Irreducible Compact Operators

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The purpose of the present paper is to show that a positive compact (ideal-) irreducible operator T in a Banach lattice (of dimension greater than one) has spectral radius $r(T)>0$, i.e., T is not quasi-nilpotent. It was shown in [6] that there exist positive irreducible operators which are quasi-nilpotent and the question under what conditions a positive irreducible operator T satisfies $r(T)>0$ has been studied extensively (see e.g. [6] and [7], Sect. V.6). In particular we mention the Ando-Krieger theorem (see e.g. [9], Theorem 136.9), which states that a positive irreducible kernel operator T in a Banach function space has $r(T)>0$. Recently in [8], some special situations are discussed in which a compact positive irreducible operator has a strictly positive spectral radius.

Let L be a (real or complex) Banach lattice. By $\mathscr{L}(L)$ we denote the Banach space of all bounded linear operators in L. As usual, we write $S \leqq T$ in $\mathscr{L}(L)$ whenever $Su \leqq Tu$ for all $0 \leqq u \in L$. Given $0 \leqq T \in \mathscr{L}(L)$, the closed ideal J in L is called T-invariant if $T(J) \subseteq J$, and T is called (ideal-) irreducible if the only closed T-invariant ideals are $\{0\}$ and L (see [7], III.8). For the general theory of Banach lattices and positive operators we refer to the books [7] and [9].

In the proof of the main result of the paper we will use some properties of the center of a Banach lattice. We recall some elementary facts. Let L be a real Banach lattice. The center $Z(L)$ of L is the subspace of $\mathscr{L}(L)$ consisting of all operators π for which there exists $0 \leqq \lambda \in \mathbb{R}$ such that $|\pi f| \leqq \lambda |f|$ for all $f \in L$. The center has the structure of a vector lattice with $(\pi_1 \vee \pi_2) u = (\pi_1 u) \vee (\pi_2 u)$ and $(\pi_1 \wedge \pi_2) u = (\pi_1 u) \wedge (\pi_2 u)$ for all $0 \leqq u \in L$ and all $\pi_1, \pi_2 \in Z(L)$. Moreover, $Z(L)$ is a commutative algebra (with respect to composition as multiplication) and the identity operator I in L is a strong unit, i.e., for any $\pi \in Z(L)$ there exists $0 \leqq \lambda \in \mathbb{R}$ such that $|\pi| \leqq \lambda I$ (see [9], Chap. 20). Note that if $L = C(K)$ for some compact Hausdorff space K, then $Z(L)$ can be identified with $C(K)$ acting on itself by multiplication.

If L is a Dedekind complete Banach lattice and $0 \leqq u \leqq v$ in L, then it is an easy consequence of the Freudenthal spectral theorem ([4], Theorem 40.2) that there exists $\pi \in Z(L)$ such that $\pi v = u$ and $0 \leqq \pi \leqq I$. However, in an arbitrary Banach lattice such a π need not exist, as is shown by the Banach lattice $L = C([0,1])$. The next lemma shows that, under rather mild conditions, we can

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always find an approximation of such an operator π. Recall that the element $0 \leqq w \in L$ is called a quasi-interior point in L if the principal ideal I_w generated by w is norm dense in L.

Lemma 1. *Let L be a real Banach lattice with quasi-interior point. If $0 \leqq u \leqq |f|$ in L, then there exists a sequence $\{\pi_n: n=1,2,\dots\}$ in $Z(L)$ such that $\pi_n f \to u$ (norm) as $n \to \infty$ and $|\pi_n| \leqq I$ for all n.*

Proof. Taking the supremum of an arbitrary quasi-interior point and $|f|$, we may assume that w is a quasi-interior point in L such that $0 \leqq u \leqq |f| \leqq w$. By the Kakutani representation theorem, there exists an isomorphism $g \to \hat{g}$ from the principal ideal I_w onto the space $C(K)$ for some compact Hausdorff space K, such that $\hat{w} = \mathbb{1}$, the unit element in $C(K)$. Now define the functions $p_n \in C(K)$ by

$$p_n = \hat{u} \hat{f} (|\hat{f}| + n^{-1} \mathbb{1})^{-2}, \qquad n = 1, 2, \dots.$$

Then $|p_n| \leqq \mathbb{1}$ for all n and $p_n \hat{f} \to \hat{u}$ uniformly on K as $n \to \infty$. Let $\tilde{\pi}_n$ be the element in $Z(I_w)$ corresponding to multiplication by p_n in $C(K)$. Being norm continuous, $\tilde{\pi}_n$ extends uniquely to an element $\pi_n \in Z(L)$ with $|\pi_n| \leqq I$. Finally, $\pi_n(f) \to u$ (w-uniformly) as $n \to \infty$, which implies that $\pi_n(f) \to u$ (norm).

The proof of the following proposition uses the ideas of Hilden's proof of Lomonosov's invariant subspace theorem (see e.g. [5]). Although the last part of the present proof is almost similar to the proof of the invariant subspace theorem, for the sake of convenience we present the complete argument.

Proposition 2. *If L is a (complex) Banach lattice,* $\dim L > 1$, *and if T is a positive compact quasi-nilpotent linear operator from L into itself, then there exists a non-trivial closed T-invariant ideal in L.*

Proof. First observe that it is sufficient to consider a real Banach lattice L and a non-zero compact operator $0 \leqq T: L \to L$ with $\|T^n\|^{1/n} \to 0$ $(n \to \infty)$. Take $0 < u \in L$ and define

$$w = \sum_{k=0}^{\infty} 2^{-k} \|T\|^{-k} T^k u$$

(norm convergent series). The principal ideal I_w is T-invariant, and so the closure $\bar{I}_w$ is T-invariant. If $\bar{I}_w \neq L$, then $\bar{I}_w$ is a non-trivial closed T-invariant ideal and the proof is finished. Therefore we assume that $L = \bar{I}_w$, and so, from now on, we assume that L has a quasi-interior point.

We introduce some notation. Let

$$\mathscr{C}^+(T) = \{0 \leqq R \in \mathscr{L}(L): RT = TR\},$$
$$\mathscr{J}^+ = \{0 \leqq S \in \mathscr{L}(L): \exists R \in \mathscr{C}^+(T) \text{ with } 0 \leqq S \leqq R\},$$

and let $\mathscr{J} = \{S_1 - S_2 : S_1, S_2 \in \mathscr{J}^+\}$. Clearly, $\mathscr{J}$ is a linear subspace of $\mathscr{L}(L)$. Furthermore observe that $\pi S \in \mathscr{J}$ whenever $S \in \mathscr{J}$ and $\pi \in Z(L)$. Indeed, suppose that $|\pi| \leqq \lambda I$, $0 \leqq \lambda \in \mathbb{R}$ and $S = S_1 - S_2$ with $0 \leqq S_j \leqq R_j \in \mathscr{C}^+(T)$, $j = 1, 2$.

Then $\pi S = (\pi^+ S_1 + \pi^- S_2) - (\pi^+ S_2 + \pi^- S_1)$ and

$$0 \leqq \pi^+ S_1 + \pi^- S_2, \qquad \pi^+ S_2 + \pi^- S_1 \leqq \lambda(R_1 + R_2),$$

which shows that $\pi S \in \mathscr{J}$. Finally note that $TS \in \mathscr{J}$ for all $S \in \mathscr{J}$.

For $f \in L$ we now define

$$\mathscr{J}[f] = \{Sf \colon S \in \mathscr{J}\},$$

which is, by the above observations, a T-invariant subspace of L. Note that $f \in \mathscr{J}[f]$. We show next that the closure $\overline{\mathscr{J}[f]}$ is an ideal in L. To this end, first suppose that $0 \leqq u \leqq |Sf|$ in L for some $S \in \mathscr{J}$. By Lemma 1 there exists a sequence $\{\pi_n \colon n = 1, 2, \ldots\}$ in $Z(L)$ such that $|\pi_n| \leqq I$ and $\pi_n(Sf) \to u$ (norm) as $n \to \infty$. As observed above, $\pi_n S \in \mathscr{J}$ and so $\pi_n Sf \in \mathscr{J}[f]$ for all n, which shows that $u \in \overline{\mathscr{J}[f]}$. Let $\langle \mathscr{J}[f] \rangle$ denote the ideal generated by $\mathscr{J}[f]$ and take $0 \leqq v \in \langle \mathscr{J}[f] \rangle$. Then exist $S_1, \ldots, S_n \in \mathscr{J}$ such that $0 \leqq v \leqq \sum_{k=1}^{n} |S_k f|$, and so we can write $v = \sum_{k=1}^{n} v_k$ with $0 \leqq v_k \leqq |S_k f|$ $(k = 1, \ldots, n)$. By the above, $v_k \in \overline{\mathscr{J}[f]}$ for all k, and hence $v \in \overline{\mathscr{J}[f]}$. This shows that $\mathscr{J}[f] \subseteq \langle \mathscr{J}[f] \rangle \subseteq \overline{\mathscr{J}[f]}$, which implies that $\overline{\mathscr{J}[f]} = \overline{\langle \mathscr{J}[f] \rangle}$. Since the closure of an ideal is an ideal, we may conclude that $\overline{\mathscr{J}[f]}$ is and ideal.

We thus have shown that $\overline{\mathscr{J}[f]}$ is a closed ideal in L which is T-invariant. Therefore it is sufficient to show that there exists $f \neq 0$ in L such that $\overline{\mathscr{J}[f]} \neq L$. Suppose, on the contrary, that $\overline{\mathscr{J}[f]} = L$ for all $f \neq 0$, in L, and take $0 \leqq u \in L$ such that $Tu > 0$. Let $\mathscr{U}$ be an open ball with center u such that $0 \notin \overline{T(\mathscr{U})}$ and $0 \notin \overline{\mathscr{U}}$. For any $f \in \overline{T(\mathscr{U})}$ we have $f \neq 0$ and so, by hypothesis, $\overline{\mathscr{J}[f]} = L$. In particular $u \in \overline{\mathscr{J}[f]}$, hence there exists $S_f \in \mathscr{J}$ such that $S_f f \in \mathscr{U}$. Let $\mathscr{U}_f$ be an open neighborhood of f such that $S_f(\mathscr{U}_f) \subseteq \mathscr{U}$. Now $\{\mathscr{U}_f \colon f \in \overline{T(\mathscr{U})}\}$ is an open covering of the compact set $\overline{T(\mathscr{U})}$, hence there exist $f_1, \ldots, f_n$ in $\overline{T(\mathscr{U})}$, with corresponding $\mathscr{U}_j = \mathscr{U}_{f_j}$, $S_j = S_{f_j}$ $(j = 1, \ldots, n)$ such that $\overline{T(\mathscr{U})} \subseteq \mathscr{U}_1 \cup \ldots \cup \mathscr{U}_n$. Since $Tu \in \overline{T(\mathscr{U})}$, there exists $j_1 \in \{1, \ldots, n\}$ such that $Tu \in \mathscr{U}_{j_1}$, so $S_{j_1} Tu \in \mathscr{U}$ and hence $TS_{j_1} Tu \in \overline{T(\mathscr{U})}$. Repeating the argument we obtain a sequence $\{j_m \colon m = 1, 2, \ldots\}$ in $\{1, \ldots, n\}$ such that

$$g_m = S_{j_m} T S_{j_{m-1}} T \ldots S_{j_1} T u \in \mathscr{U}$$

for all $m = 1, 2, \ldots$. Now write $S_j = S_j^{(1)} - S_j^{(2)}$ with $0 \leqq S_j^{(i)} \leqq R_j^{(i)} \in \mathscr{C}^+(T)$ $(j = 1, \ldots, n;\ i = 1, 2)$ and let $C = \max(\|R_j^{(i)}\| \colon j = 1, \ldots, n;\ i = 1, 2)$. Then

$$\begin{aligned} |g_m| &= |S_{j_m} T \ldots S_{j_1} T u| \\ &\leqq (R_{j_m}^{(1)} + R_{j_m}^{(2)})\, T \ldots (R_{j_1}^{(1)} + R_{j_1}^{(2)})\, T u \\ &= (R_{j_m}^{(1)} + R_{j_m}^{(2)}) \ldots (R_{j_1}^{(1)} + R_{j_1}^{(2)})\, T^m u \end{aligned}$$

and hence $\|g_m\| \leqq (2C)^m \|T^m\| \cdot \|u\|$, i.e., $\|g_m\|^{1/m} \leqq 2C \|T^m\|^{1/m} \|u\|^{1/m}$ for all $m = 1, 2, \ldots$. Since T is quasi-nilpotent this implies that $\|g_m\| \to 0$ as $m \to \infty$. Now it follows from $g_m \in \mathscr{U}$ that $0 \in \overline{\mathscr{U}}$, which contradicts the choice of $\mathscr{U}$. By this the proof of the proposition is complete.

The following theorem is an immediate corollary of the above proposition.

Theorem 3. *If T is a positive irreducible compact operator in the (complex) Banach lattice L, $\dim L > 1$, then $r(T) > 0$.*

Note that in the above result the case $\dim L = 1$ has to be excluded, since in a one dimensional space the zero operator is irreducible.

Recently it was shown by V. Caselles [2] that the Ando-Krieger theorem concerning irreducible kernel operators can be extended to so-called Harris operators (positive order continuous operators for which some power majorizes a non-trivial kernel operator) in Banach function spaces. Next we will show that there is an analogous extension of the above theorem. Before stating the result we recall that, if L is a Banach lattice and if $0 \leqq S \leqq R$ in $\mathscr{L}(L)$ with R compact, then in general S need not be compact (if L and its dual L^* both have order continuous norm then by the Dodds-Fremlin theorem [3] we can conclude that S is compact; see also [9], Theorem 124.10). However, it follows from a theorem of Aliprantis and Burkinshaw [1] (see also [9], Theorem 124.5) that in the above situation S^3 is always compact. This last result will be used in the proof of the next proposition.

Proposition 4. *Let L be a Banach lattice and let T be a positive irreducible operator in L. Suppose that $S, Q \in \mathscr{L}(L)$ are such that $0 < S \leqq Q$ with Q compact and suppose that $S \leqq T^n$ for some natural number n. Then $r(T) > 0$.*

Proof. First consider the case that S itself is compact. Take $\lambda > r(T)$. The resolvent $R(\lambda; T)$ has the property that $R(\lambda; T)u$ is a quasi-interior point for all $0 < u \in L$ (see the beginning of Sect. V.6 in [7]). Put $A = R(\lambda; T)S$, which is a non-zero positive compact operator. Let N_S be the null ideal of S, i.e., $N_S = \{f \in L\colon S|f| = 0\}$. Clearly, N_S is an A-invariant closed ideal in L. Let A_0 be the operator induced by A in L/N_S, i.e., $A_0[f] = [Af]$ for all $[f] \in L/N_S$. Then A_0 is compact and $r(A_0) \leqq r(A)$ (this last inequality follows immediately from the fact that N_S is $R(\mu; A)$ - invariant for all $\mu > r(A)$). We claim that A_0 is irreducible. Indeed, take $0 \leqq u \in L$ such that $[u] > 0$ in L/N_S. Then $Su > 0$, so $Au = R(\lambda; T)Su$ is a quasi-interior point, which implies that $A_0[u] = [Au]$ is a quasi-interior point in L/N_S. Hence A_0 is irreducible. Now it follows from Theorem 3 that $r(A_0) > 0$, and therefore $r(A) > 0$. Furthermore, $0 \leqq S \leqq T^n$ implies that $0 \leqq A \leqq R(\lambda; T)T^n$, so $r(R(\lambda; T)T^n) > 0$. By the spectral mapping theorem we conclude that $r(T) > 0$.

Now consider the situation in which $0 < S \leqq Q$ with Q compact. Again take $\lambda > r(T)$, and note that $\{R(\lambda; T)S\}^3 > 0$ (whereas it is possible that $S^3 = 0$). Since $0 \leqq R(\lambda; T)S \leqq R(\lambda; T)Q$ and $R(\lambda; T)Q$ is compact, it follows that $\{R(\lambda; T)S\}^3$ is compact as well. Hence, since $R(\lambda, T)T^n$ is irreducible and $0 < \{R(\lambda; T)S\}^3 \leqq \{R(\lambda; T)T^n\}^3$, it follows from the first part of the proof that $r(R(\lambda; T)T^n) > 0$, and therefore $r(T) > 0$.

In a Dedekind complete Banach lattice L the above result can be reformulated as follows.

Corollary 5. *If T is a positive irreducible quasi-nilpotent operator in the Dedekind complete Banach lattice L, then all powers of T are disjoint to all positive compact operators in L, i.e., $T^n \wedge Q = 0$ for all $n = 1, 2, \ldots$ and all positive com-*

pact Q (where the infimum is taken in the Dedekind complete vector lattice of all order bounded operators in L).

As noted at the beginning of the paper, examples of positive irreducible quasi-nilpotent operators can be found in [6].

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