Math. Z. 194, 173–182 (1987)

Mathematische Zeitschrift

Asymptotic Behavior of Solutions of $-\Delta v + q v = \lambda v$ and the Distance of λ to the Essential Spectrum

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0. Introduction

In $L_2(\mathbb{R}^n)$, $n \geqq 2$, we study the operator $-\Delta + q$ defined on $C_0^\infty(\mathbb{R}^n)$ with a real-valued $q \in L_{2,\mathrm{loc}}(\mathbb{R}^n)$. Under further assumptions on q, this operator will be essentially self-adjoint.

There are many connexions between the asymptotic behavior at infinity of non-trivial solutions $v \in L_{2,\mathrm{loc}}(\mathbb{R}^n)$ of the equation $-\Delta v + q v = \lambda v$ (i.e. $\forall \phi \in C_0^\infty(\mathbb{R}^n)$: $\int v(-\Delta + q - \lambda)\phi = 0$) and the location of $\lambda \in \mathbb{R}$ with respect to the different parts of the spectrum of the operator $\overline{-\Delta + q}$ (for the notation of the spectrum, see [9]). A trivial prototype of this kind of statements is

$$v \text{ decays very rapidly} \Rightarrow \lambda \in \sigma_p.$$

Many results concern the other direction: if $\lambda \in \sigma_p$, then every eigenfunction decays, depending on the behavior of q near infinity, more or less rapidly. See [5] for some examples. In that paper, however, it was pointed out that in general eigenfunctions need not go to zero pointwise if q is not bounded from below (Example 1 in [5]). On the other hand, if $\lambda \in \sigma_d$, then the pointwise decay of eigenfunctions is usually exponential (i.e. faster than any negative power of $|x|$). For potentials q which are bounded from below, this has been proved by Shnol' (Theorem 2 in [3], p. 179; cp. Satz 0 in [4]). In Corollary 2 we will show that the same is true, if $q(x) \geqq -o(|x|^2)$. An application lies in the manifestation of the existence of embedded eigenvalues for a special kind of operators with a potential unbounded from below (Example 1). To arrive at these results, we establish, in Theorem 1, the connexion between the decay of eigenfunctions and the distance of λ to σ_e.

The same method will be used to yield links between $\mathrm{dist}(\lambda, \sigma_e)$ and the growth of solutions which are not in L_2 (Theorem 2). It is generally believed that the following is true:

$$\exists v \in L_\infty(\mathbb{R}^n) \setminus \{0\},\ -\Delta v + q v = \lambda v \ \Rightarrow \lambda \in \sigma. \qquad (*)$$

Again Shnol' (Theorem 5 in [3], p. 182) provided a proof for the case of potentials which are bounded from below. This was generalized by Simon [7] to potentials the negative parts of which lie in K_n. By Theorem 2, we can establish (*) if $q(x) \geqq -o(|x|^2)$ (Corollary 3) (The latter result has also been obtained by Shnol', but in a non-quantitative way; see Shnol''s original works as cited in [3].). We are, however, led to believe that (*) in general fails if $q(x) \sim -O(|x|^2)$ (Conjecture).

Another consequence of Theorem 2 are lower bounds for non-L_2-solutions of the equation $-\Delta v + qv = \lambda v$ if $\lambda \notin \sigma_e$ (Proposition), which fit perfectly with earlier results. This fact provides some hope to find lower bounds for eigenfunctions by a similar method.

1. Some Tools

In Lemmas 1 to 3 we will give some abstract tools necessary in the proofs of Theorems 1 and 2. Because of the structure of our problems here, we can assume, throughout this paper, every appearing function to be real-valued.

Lemma 1. *Let T be a self-adjoint operator on a Hilbert space, $\lambda \in \mathbb{R}$.*

If there is a sequence $(u_k)_{k\in\mathbb{N}} \subset D(T)$ with $\|u_k\| = 1$ $\forall k \in \mathbb{N}$, $u_k \xrightarrow{w} 0$ as $k \to \infty$, and $a := \liminf_{k\to\infty} \|(T-\lambda)u_k\| < \infty$, then $\operatorname{dist}(\lambda, \sigma_e(T)) \leqq a$.

Proof. Assume that $\sigma_e(T) \cap [\lambda - a, \lambda + a] = \varnothing$. Then by Theorem 7.24 in [9] and compactness of $[\lambda - a, \lambda + a]$ we have (E being the spectral family of T):

$$\exists \varepsilon > 0\colon \ \dim \mathrm{R}(E(\lambda + a + \varepsilon) - E(\lambda - a - \varepsilon)) < \infty,$$

so that, using Theorem 6.3 in [9],

$$(E(\lambda + a + \varepsilon) - E(\lambda - a - \varepsilon))\, u_k \to 0 \qquad \text{as } k \to \infty. \tag{1}$$

On the other hand

$$\begin{aligned} \|(T-\lambda)u_k\|^2 &= \int |t-\lambda|^2 \,\mathrm{d}\,\|E(t)\,u_k\|^2 \\ &\geqq (a+\varepsilon)^2 \Big[\int \mathrm{d}\,\|E(t)\,u_k\|^2 - \int \chi_{]\lambda-a-\varepsilon,\,\lambda+a+\varepsilon]}(t) \quad \mathrm{d}\,\|E(t)\,u_k\|^2\Big] \\ &= (a+\varepsilon)^2 \big[\|u_k\|^2 - \|(E(\lambda+a+\varepsilon) - E(\lambda-a-\varepsilon))\,u_k\|^2\big]. \end{aligned}$$

By (1), the right-hand side tends to $(a+\varepsilon)^2$ as $k \to \infty$, so $\liminf_{k\to\infty} \|(T-\lambda)u_k\|^2 \geqq (a+\varepsilon)^2$, which is in contradiction to the assumptions of the lemma.

Lemma 2. *Let v, $\partial_i v \in L_{2,\mathrm{loc}}(\mathbb{R}^n)$, $\psi \in C^\infty(\mathbb{R}^n)$. Then $\partial_i(v\psi) = \partial_i v\,\psi + v\,\partial_i\psi$.*

Proof. $\forall \phi \in C_0^\infty(\mathbb{R}^n)$: $-\int v\psi\,\partial_i\phi = -\int v\,\partial_i(\psi\,\phi) + \int v\,\partial_i\psi\,\phi$.

Lemma 3. *Let $2 \leqq \rho \leqq \infty$. If $v \in L_{\rho,\mathrm{loc}}(\mathbb{R}^n)$, $\Delta v \in L_{\frac{\rho}{\rho-1},\mathrm{loc}}(\mathbb{R}^n)$, then*

a) $\forall i \in \{1, \ldots, n\}$: $\partial_i v \in L_{2,\mathrm{loc}}(\mathbb{R}^n)$,

b) $\Delta(v^2) = 2\Delta v\, v + 2|\nabla v|^2$.

Proof. a) Let $R>0$ and define

$$\zeta(x)\begin{cases} :=1, & |x|\leqq R\\ \in[0,1], & R\leqq|x|\leqq 2R\\ :=0, & |x|\geqq 2R\end{cases}\Bigg\}\in C_0^\infty(\mathbb{R}^n).$$

Then (v_ε being the mollified v, see [1])

$$-\int v_\varepsilon\,\Delta(v_\varepsilon)\,\zeta=\int|\nabla(v_\varepsilon)|^2\,\zeta+\tfrac{1}{2}\int\nabla(v_\varepsilon^2)\cdot\nabla\zeta$$
$$=\int|\nabla(v_\varepsilon)|^2\,\zeta-\tfrac{1}{2}\int v_\varepsilon^2\,\Delta\zeta,$$

so that

$$\int_{B(0,R)}|\nabla(v_\varepsilon)|^2\leqq\int|\nabla(v_\varepsilon)|^2\,\zeta=\tfrac{1}{2}\int v_\varepsilon^2\,\Delta\zeta-\int v_\varepsilon\,\Delta(v_\varepsilon)\,\zeta$$
$$\leqq\tfrac{1}{2}\|v_\varepsilon\|^2_{L_2(B(0,R))}\,\|\Delta\zeta\|_{L_\infty(\mathbb{R}^n)}$$
$$+\|v_\varepsilon\|_{L_p(B(0,R))}\,\|(\Delta v)_\varepsilon\|_{L_{\frac{p}{p-1}}(B(0,R))}\,\|\zeta\|_{L_\infty(\mathbb{R}^n)}\leqq const(R).$$

Theorem 4.25 in [9] guarantees the existence of a subsequence $(\partial_i(v_{\varepsilon_k}))_{k\in\mathbb{N}}$ and a $v_i\in L_2(B(0,R))$ with $\partial_i(v_{\varepsilon_k})\xrightarrow{w}v_i$ as $k\to\infty$.

Thus

$$\forall\phi\in C_0^\infty(B(0,R)):\ -\int v\,\partial_i\phi=-\lim_{k\to\infty}\int v_{\varepsilon_k}\,\partial_i\phi=\lim_{k\to\infty}\int\partial_i(v_{\varepsilon_k})\,\phi=\int v_i\,\phi.$$

Therefore $\partial_i v=v_i$ on $B(0,R)$.

b) $\forall\phi\in C_0^\infty(\mathbb{R}^n)$: $\int\Delta v\,v_\varepsilon\,\phi=\int v\,\Delta(v_\varepsilon\phi)$

$$=\int v\,\Delta(v_\varepsilon)\,\phi+2\int v\,\nabla(v_\varepsilon)\cdot\nabla\phi+\int v\,v_\varepsilon\,\Delta\phi$$
$$=-\int v\,\Delta(v_\varepsilon)\,\phi-2\int\nabla v\cdot\nabla(v_\varepsilon)\,\phi+\int v\,v_\varepsilon\,\Delta\phi$$

with the aid of Lemma 2.

For $\varepsilon\to 0$ on a subsequence as in the proof of a), we arrive at the conclusion of b).

2. Decay of Eigenfunctions and the Essential Spectrum

We start with the principal result of this chapter. (For the definition of $M_{2,\mathrm{loc}}(\mathbb{R}^n)$, see [5].)

Theorem 1. *Let $q\in M_{2,\mathrm{loc}}(\mathbb{R}^n)$, fulfilling $q(x)\geqq-\beta|x|^{2\gamma}$ outside a ball $B(0,R_0)$ for a $\beta>0$ and a $\gamma\in[0,1]$. Suppose $\lambda\in\sigma_p(\overline{-\Delta+q})$ with an eigenfunction v having the property: there is a $\mu>0$ such that*

$$e^{2\mu r^{1-\gamma}}\int_{|y|\geqq r}|v(y)|^2\,dy\quad\textit{is unbounded in } [R_0,\infty[,\ \textit{if } 0\leqq\gamma<1;$$
$$r^{2\mu}\int_{|y|\geqq r}|v(y)|^2\,dy\quad\textit{is unbounded in } [R_0,\infty[,\ \textit{if } \gamma=1.$$

Then

$$\operatorname{dist}(\lambda,\sigma_e(\overline{-\Delta+q}))\leqq\begin{cases} d_1\mu^2+d_2\sqrt{(\beta+\lambda)_+}\,\mu, & \gamma=0\\ d(1-\gamma)\sqrt{\beta}\,\mu, & 0<\gamma<1\\ \sqrt{\beta}(d_1\mu+d_2\sqrt{\mu}), & \gamma=1.\end{cases}$$

($(\beta+\lambda)_+ := \max\{0, \beta+\lambda\}$; *the constants* d, d_1, d_2 *can be given explicitely and do not depend on any of the quantities appearing in the assumptions.*)

Remark. For $\gamma=0$, β is allowed to be non-positive.

Before the proof of Theorem 1, we will present two important consequences.

Corollary 1. *Assumptions on* q *as in Theorem 1. Let* $\lambda \in \sigma_d(\overline{-\Delta+q})$ *with eigenfunction* v. *Then there is a* $\mu>0$ *and a* $c>0$ *such that for all* $x \in \mathbb{R}^n$:

$$|v(x)| \leqq \begin{cases} c\, \mathrm{e}^{-\mu|x|^{1-\gamma}}, & 0 \leqq \gamma < 1; \\ c(1+|x|)^{(n-\mu)/2}, & \gamma = 1. \end{cases}$$

Proof. As in the proof of Theorem 1 in [5], we have for $|x| \geqq 4R_0+4$:

$$\begin{aligned} |v(x)|^2 &\leqq c_1 |x|^{n\gamma} \int_{B\left(x, \frac{1}{2|x|^\gamma}\right)} |v(y)|^2 \,\mathrm{d}y \\ &\leqq c_1 |x|^{n\gamma} \int_{|y| \geqq |x| - \frac{1}{2|x|^\gamma}} |v(y)|^2 \,\mathrm{d}y \\ &\leqq c_2 |x|^{n\gamma} \cdot \begin{cases} \mathrm{e}^{-2\mu\left(|x| - \frac{1}{2|x|^\gamma}\right)^{1-\gamma}}, & 0 \leqq \gamma < 1 \\ \left(|x| - \frac{1}{2|x|}\right)^{-\mu}, & \gamma = 1, \end{cases} \end{aligned}$$

the last inequality is true for a small μ, depending on the distance of λ to $\sigma_e(\overline{-\Delta+q})$, by Theorem 1.

This, together with the continuity of v (Lemma 1 in [5]), proves the corollary.

Corollary 2. *Let* $q \in M_{2,\mathrm{loc}}(\mathbb{R}^n)$ *with* $q(x) \geqq -\mathrm{o}(|x|^2)$. *Let* $\lambda \in \sigma_d(\overline{-\Delta+q})$ *with eigenfunction* v. *Then* v *decays exponentially.*

Proof. As $\forall \beta>0\ \exists B(0,R_0)$: $q(x) \geqq -\beta|x|^2$ outside $B(0,R_0)$, we see that we can find μ as large as we wish in Corollary 1, case $\gamma=1$.

Proof of Theorem 1. A. By Satz 7 in [6], $-\Delta+q$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$. Let $T := \overline{-\Delta+q}$.

B. Define

$$\Theta(r) \begin{Bmatrix} :=0, & r \leqq 0 \\ \in [0,1], & 0 \leqq r \leqq 1 \\ :=1, & r \geqq 1 \end{Bmatrix} \in C^\infty(\mathbb{R}).$$

Let $S>0$ be fixed (will be specified later). For all $R>0$ we define

$$\forall x \in \mathbb{R}^n\colon\ \Theta_R(x) := \Theta\left(\frac{|x|-R}{SR^\gamma}\right) \quad \text{and} \quad v_R := \Theta_R\, v.$$

As v is an eigenfunction, $v_R\in L_2(\mathbb{R}^n)$ and $\forall\phi\in C_0^\infty(\mathbb{R}^n)$:

$$\begin{aligned}\int v_R(-\Delta+q)\phi&=\int\lambda v_R\phi+\int(q-\lambda)v\Theta_R\phi-\int v\Delta(\Theta_R\phi)+\int v\Delta\Theta_R\phi+2\int v\nabla\Theta_R\cdot\nabla\phi\\&=\int\lambda v_R\phi-\int v\Delta\Theta_R\phi-2\int\nabla v\cdot\nabla\Theta_R\phi,\end{aligned}$$

where we have made use of Lemmas 2 and 3a) (by Lemma 1 in [5], $v\in C^0(\mathbb{R}^n)$).

So we have $v_R\in D((-\Delta+q)^*)=D(T)$ and $(T-\lambda)v_R=-v\Delta\Theta_R-2\nabla v\cdot\nabla\Theta_R$. Therefore

$$\|(T-\lambda)v_R\|^2\leqq 2\|v\Delta\Theta_R\|^2+8\|\nabla v\cdot\nabla\Theta_R\|^2. \tag{2}$$

Lemma 3b) says that $\forall\psi\in C_0^\infty(\mathbb{R}^n)$:

$$\int v^2\Delta\psi=2\int(q-\lambda)v^2\psi+2\int|\nabla v|^2\psi.$$

So, putting $\psi:=|\nabla\Theta_R|^2$, we get

$$\|\nabla v\cdot\nabla\Theta_R\|^2=\tfrac{1}{2}\int v^2\Delta(|\nabla\Theta_R|^2)-\int v^2(q-\lambda)|\nabla\Theta_R|^2,$$

and with (2) we arrive at

$$\begin{aligned}&\|(T-\lambda)v_R\|^2\leqq\int v^2\{2(\Delta\Theta_R)^2+4\Delta(|\nabla\Theta_R|^2)-8(q-\lambda)|\nabla\Theta_R|^2\}\\&\leqq\int v^2\{2(\Delta\Theta_R)^2+4\Delta(|\nabla\Theta_R|^2)+8(\beta|\cdot|^{2\gamma}+\lambda)|\nabla\Theta_R|^2\}\\&\leqq\left\{\frac{10c_2^2+8c_1c_3}{S^4R^{4\gamma}}+\frac{12(n-1)c_1c_2}{S^3R^{3\gamma+1}}+\frac{2(n-1)^2c_1^2}{S^2R^{2\gamma+2}}+\frac{8(\beta(R+SR^\gamma)^{2\gamma}+\lambda)_+c_1^2}{S^2R^{2\gamma}}\right\}\\&\quad\int_{R\leqq|y|\leqq R+SR^\gamma}v^2(y)\,dy,\end{aligned}\tag{3}$$

where $c_i:=\max_{r\in\mathbb{R}}\{\Theta^{(i)}(r)\}$, $R\geqq R_0$.

Observing that $\int_{r\leqq|y|}v^2(y)\,dy\neq 0$ by the assumptions of the theorem, we get

$$\|(T-\lambda)v_R\|^2\leqq\{\dots\}(R)\frac{\int_{R\leqq|y|\leqq R+SR^\gamma}v^2(y)\,dy}{\int_{R+SR^\gamma\leqq|y|}v^2(y)\,dy}\|v_R\|^2.$$

Let $u_R:=\frac{v_R}{\|v_R\|}$. Then $(u_R)_{R\geqq R_0}\subset D(T)$, $\|u_R\|=1$, and $u_R\xrightarrow{w}0$ as $R\to\infty$, because $\operatorname{supp}u_R\subset\complement B(0,R)$.

So in order to complete the proof of the theorem with the aid of Lemma 1, we just have to find estimates for

$$\liminf_{R\to\infty}\{\dots\}(R)\frac{\int\dots}{\int\dots}.$$

C. Observing that

$$\lim_{R\to\infty}\{\dots\}(R)=\begin{cases}\dfrac{10c_2^2+8c_1c_3}{S^4}+\dfrac{8(\beta+\lambda)_+c_1^2}{S^2}, & \gamma=0\\ \dfrac{8\beta c_1^2}{S^2}, & 0<\gamma<1\\ \dfrac{8\beta(1+S)^2c_1^2}{S^2}, & \gamma=1,\end{cases}$$

we only have to investigate $\liminf\limits_{R\to\infty}\dfrac{\int\dots}{\int\dots}$.

Let $R\geqq R_0$ be fixed and define

$$b:=\inf_{r\geqq R}\frac{\int\limits_{r\leqq|y|\leqq r+Sr^\gamma} v^2(y)\,dy}{\int\limits_{r+Sr^\gamma\leqq|y|} v^2(y)\,dy}+1.$$

Then $b\geqq 1$ and $\forall r\geqq R$: $\int\limits_{r\leqq|y|} v^2(y)\,dy\geqq b\int\limits_{r+Sr^\gamma\leqq|y|} v^2(y)\,dy$.

Let $r_0:=R$ and $r_{k+1}:=r_k+Sr_k^\gamma$. Then by induction,

$$\forall k\in\mathbb{N}_0:\ \int\limits_{r_k\leqq|y|} v^2(y)\,dy\leqq b^{-k}\int\limits_{R\leqq|y|} v^2(y)\,dy.$$

Also by induction we see that

$$k\geqq\begin{cases}\dfrac{\ln r_k-\ln R}{\ln(S+1)}, & \gamma=1\\ \dfrac{r_k^{1-\gamma}-R^{1-\gamma}}{S(1-\gamma)}, & 0\leqq\gamma<1.\end{cases}\tag{4}$$

(For $\gamma=1$ this is immediate; for $\gamma\neq 1$, we observe that $r_{k+1}^{1-\gamma}-r_k^{1-\gamma}=Sr_k^\gamma(1-\gamma)\,\xi^{-\gamma}$ with a $\xi\in[r_k,r_{k+1}]$, so that $r_{k+1}^{1-\gamma}\leqq r_k^{1-\gamma}+S(1-\gamma)$.)

So we find for $\gamma=1$ that

$$\forall k\in\mathbb{N}_0:\ \int\limits_{r_k\leqq|y|} v^2(y)\,dy\leqq b^{-\ln r_k/\ln(S+1)}\,b^{\ln R/\ln(S+1)}\int\limits_{R\leqq|y|} v^2(y)\,dy,$$

i.e.

$\forall k\in\mathbb{N}_0\ \forall r_k\leqq r\leqq r_{k+1}$:

$$r^{2\mu}\int\limits_{r\leqq|y|} v^2(y)\,dy\leqq r^{2\mu}\int\limits_{r_k\leqq|y|} v^2(y)\,dy\leqq(1+S)^{2\mu}r_k^{2\mu-\ln b/\ln(S+1)}\,b^{\ln R/\ln(S+1)}\int\limits_{R\leqq|y|} v^2(y)\,dy,$$

which for $b\geqq(S+1)^{2\mu}$ would lead to a contradiction, because $r^{2\mu}\int\limits_{r\leqq|y|} v^2(y)\,dy$ was supposed to be unbounded. So $b<(S+1)^{2\mu}$. A similar procedure for $0\leqq\gamma<1$ yields $b<e^{2\mu S(1-\gamma)}$. As these bounds for b do not depend on R, we arrive at

$$\liminf_{R\to\infty}\frac{\int\limits_{R\leqq|y|\leqq R+SR^\gamma}|v(y)|^2\,dy}{\int\limits_{R+SR^\gamma\leqq|y|}|v(y)|^2\,dy}\leqq\begin{cases}e^{2\mu S(1-\gamma)}-1, & 0\leqq\gamma<1\\ (S+1)^{2\mu}-1, & \gamma=1.\end{cases}$$

D. Combining the results of B and C and putting

$$S=\begin{cases}\dfrac{1}{2\mu(1-\gamma)}, & \gamma\neq 1\\ e^{1/\mu}-1, & \gamma=1 \text{ and } \mu\geqq 1\\ 1, & \gamma=1 \text{ and } \mu<1,\end{cases}$$

we have

$$\operatorname{dist}(\lambda,\sigma_e(\overline{-\Delta+q})\leqq\begin{cases}d_1\mu^2+d_2\sqrt{(\beta+\lambda)_+}\,\mu, & \gamma=0\\ d(1-\gamma)\sqrt{\beta}\,\mu, & 0<\gamma<1\\ \sqrt{\beta}(d_1\mu+d_2\sqrt{\mu}), & \gamma=1.\end{cases}$$

As an application of the results of this chapter, we prove the existence of an embedded eigenvalue for an operator with the potential not bounded from below.

Example 1. Let $q:=p$ with the p as in Example 1 of [5]. Then $0\in\sigma_p(\overline{-\Delta+q})\cap\sigma_e(\overline{-\Delta+q})$.

Proof. In [5] it was shown that $0\in\sigma_p$. For the eigenfunction v constructed there, we found a sequence $x_i\to\infty$ with

$$v(x_i)>|x_i|^{1-\varepsilon}\qquad(\varepsilon>0 \text{ small}).$$

Now assuming $0\in\sigma_d(\overline{-\Delta+q})$, Corollary 1 tells us (as $q(x)\geqq-\beta|x|$):

$$\exists\mu>0,\ c>0\ \forall x\in\mathbb{R}^n\colon\ |v(x)|\leqq c\,e^{-\mu\sqrt{|x|}},$$

in obvious contradiction with what we said before.

In a similar way, a related theorem for one dimension could be used to prove the statements in [2], p. 91f., that their eigenvalue μ is embedded in the essential spectrum if $k<2$. For $k=2$, however, it seems possible that $\mu\in\sigma_d$!

3. Growth of Solutions and the Essential Spectrum

The analogue of Theorem 1 for non-L_2-solutions is

Theorem 2. *q as in Theorem 1. Suppose for a $\lambda\in\mathbb{R}$ we have a solution*

$$v\in L_{2,\mathrm{loc}}(\mathbb{R}^n)\setminus L_2(\mathbb{R}^n)\qquad \text{of}\ \ -\Delta v+qv=\lambda v,$$

having the property: there is a $\mu>0$ such that

$$e^{-\mu|\cdot|^{1-\gamma}}v\in L_2(\mathbb{R}^n),\qquad \text{if } 0\leqq\gamma<1;$$
$$(1+|\cdot|)^{-\mu}v\in L_2(\mathbb{R}^n),\qquad \text{if } \gamma=1.$$

Then the same estimates for $\operatorname{dist}(\lambda,\sigma_e(\overline{-\Delta+q}))$ *as in Theorem 1 hold.*

Remark. As after Theorem 1.

Again we continue with some interesting consequences.

Corollary 3. *Let $q \in M_{2,\mathrm{loc}}(\mathbb{R}^n)$ with $q(x) \geqq -o(|x|^2)$; $\lambda \in \mathbb{R}$. If there is a bounded solution $v \not\equiv 0$ of $-\Delta v + qv = \lambda v$, then $\lambda \in \sigma(\overline{-\Delta + q})$.*

Proof. Without loss, $v \in L_\infty \setminus L_2(\mathbb{R}^n)$. We apply Theorem 2 with $\gamma = 1$, $\mu > \frac{n}{2}$, $\beta > 0$, and find $\operatorname{dist}(\lambda, \sigma_e) = 0$ for $\beta \to 0$.

More precisely we have

Corollary 4. *Let q be as in Theorem 1. If for a $\lambda \in \mathbb{R}$ we have a solution $v \in L_{2,\mathrm{loc}}(\mathbb{R}^n) \setminus L_2(\mathbb{R}^n)$ with the property*

$$\forall \mu > 0: \quad \mathrm{e}^{-\mu|\cdot|^{1-\gamma}} v \in L_2(\mathbb{R}^n), \quad \text{if } 0 \leqq \gamma < 1;$$
$$(1 + |\cdot|)^{-\mu} v \in L_2(\mathbb{R}^n), \quad \text{if } \gamma = 1,$$

then $\lambda \in \sigma_e(\overline{-\Delta + q})$.

Proof. This follows immediately from Theorem 2, letting $\mu \to 0$.

The fact that our method fails to yield the conclusion of Corollary 3 in the case of $q(x) \sim -O(|x|^2)$, leads us to the following

Conjecture. *There is a $q \in M_{2,\mathrm{loc}}(\mathbb{R}^n)$ with $q(x) \geqq -O(|x|^2)$ and a $\lambda \in \mathbb{R} \setminus \sigma(\overline{-\Delta + q})$ such that a bounded solution $v \not\equiv 0$ of $-\Delta v + qv = \lambda v$ exists.*

An example of Halvorsen in [8], pp. 373–382, for one dimension supports this idea.

We now come to the

Proof of Theorem 2. A. We can follow nearly completely the proof of Theorem 1.

B. Instead of v_R we will use $\tilde{v}_R := v - v_R$. We observe that $\tilde{v}_R \in L_2(\mathbb{R}^n)$ because $v \in L_{2,\mathrm{loc}}(\mathbb{R}^n)$, and as before we see that $\tilde{v}_R \in D(T)$ with $(T - \lambda)\tilde{v}_R = -v\Delta\Theta_R - 2\nabla v \cdot \nabla\Theta_R$, and following the steps up to (3), we arrive at ($\int_{|y| \leqq r} v^2(y)\,\mathrm{d}y \neq 0$ for r large enough):

$$\|(T-\lambda)\tilde{v}_R\|^2 \leqq \{\ldots\}(R) \frac{\int_{R \leqq |y| \leqq R + SR^\gamma} v^2(y)\,\mathrm{d}y}{\int_{|y| \leqq R} v^2(y)\,\mathrm{d}y} \|\tilde{v}_R\|^2.$$

We define $\tilde{u}_R := \frac{\tilde{v}_R}{\|\tilde{v}_R\|}$. Then $(\tilde{u}_R)_{R \geqq R_0} \subset D(T)$, $\|\tilde{u}_R\| = 1$, and $\tilde{u}_R \xrightarrow{w} 0$ as $R \to \infty$, because $\forall \phi \in C_0^\infty(\mathbb{R}^n)$: $(\tilde{u}_R, \phi) = \frac{(v, \phi)}{\|\tilde{v}_R\|}$ for R large enough and $\|\tilde{v}_R\| \to \infty$ as $R \to \infty$ by the assumption $v \notin L_2(\mathbb{R}^n)$. So this time we have to estimate

$$\liminf_{R \to \infty} \frac{\int_{R \leqq |y| \leqq R + SR^\gamma} v^2(y)\,\mathrm{d}y}{\int_{|y| \leqq R} v^2(y)\,\mathrm{d}y}.$$

C. Let R (large enough) be fixed and define

$$b := \inf_{r\geqq R} \frac{\int\limits_{r\leqq|y|\leqq r+Sr^\gamma} v^2(y)\,dy}{\int\limits_{|y|\leqq r} v^2(y)\,dy}+1.$$

Then $b\geqq 1$ and $\forall r\geqq R$: $\int\limits_{|y|\leqq r+Sr^\gamma} v^2(y)\,dy \geqq b \int\limits_{|y|\leqq r} v^2(y)\,dy$.

Defining $r_0 := R$, $r_{k+1} := r_k + S r_k^\gamma$ we see by induction

$$\forall k\in\mathbb{N}_0:\ \int\limits_{|y|\leqq r_k} v^2(y)\,dy \geqq b^k \int\limits_{|y|\leqq R} v^2(y)\,dy.$$

Using (4), we have for $\gamma=1$:

$$\begin{aligned}\forall k\in\mathbb{N}_0:\ &\int\limits_{|y|\leqq r_k} (1+|y|)^{-2\mu} v^2(y)\,dy\\ &\geqq (1+r_k)^{-2\mu} b^{\ln r_k/\ln(S+1)} b^{-\ln R/\ln(S+1)} \int\limits_{|y|\leqq R} v^2(y)\,dy,\end{aligned}$$

from which we conclude $b\leqq (S+1)^{2\mu}$, because otherwise the left-hand side would be unbounded for $k\to\infty$.

A similar argument for $0\leqq\gamma<1$ yields $b\leqq e^{2\mu S(1-\gamma)}$ in that case.

Again, this does not depend on R, so that we arrive at

$$\liminf_{R\to\infty} \frac{\int\limits_{R\leqq|y|\leqq R+SR^\gamma} v^2(y)\,dy}{\int\limits_{|y|\leqq R} v^2(y)\,dy} \leqq \begin{cases} e^{2\mu S(1-\gamma)}-1, & 0\leqq\gamma<1\\ (S+1)^{2\mu}-1, & \gamma=1.\end{cases}$$

D. The rest of the proof is the same as for Theorem 1.

4. Lower Bounds for Generalized Solutions

As a modest approach to lower bounds of solutions of the equation $-\Delta v+qv=\lambda v$, we finally give as a further application of Theorem 2 the following

Proposition. *Let $q\in M_{2,\mathrm{loc}}(\mathbb{R}^n)$, fulfilling outside a ball $B(0,R_0)$ with a $\beta>0$:*

(i) $q(x)\geqq -\beta|x|^{2\gamma}$ *for a* $\gamma\in[0,1]$

or

(ii) $q(x)\geqq \beta|x|^{2\gamma}$ *for a* $\gamma>0$.

Let $\lambda\in\mathbb{R}\setminus\overline{\sigma_e(-\Delta+q)}$.

Then there is a $\mu>0$ such that for any solution $v\in L_{2,\mathrm{loc}}(\mathbb{R}^n)\setminus L_2(\mathbb{R}^n)$ of $-\Delta v+qv=\lambda v$:

(i) $(1+|\cdot|)^{-\mu} v\notin L_2(\mathbb{R}^n)$, *if* $\gamma=1$

$e^{-\mu|\cdot|^{1-\gamma}} v\notin L_2(\mathbb{R}^n)$, *if* $0\leqq\gamma<1$

or

(ii) $e^{-\mu|\cdot|^{1+\gamma}} v\notin L_2(\mathbb{R}^n)$, *respectively.*

Proof. A. Case (i) is a consequence of Corollary 4.

B. Case (ii) follows from Theorem 2 in [5]. In fact, if we put $\alpha(r) := \frac{\sqrt{\beta/2}}{\gamma+1} r^{\gamma+1}$ and assume

$$\forall \mu > 0 \colon \mathrm{e}^{-\mu|\cdot|^{1+\gamma}} v \in L_2(\mathbb{R}^n),$$

then the assumptions of that theorem are fulfilled and we get from it: $v \in L_2(\mathbb{R}^n)$.

Acknowledgements. This work has been supported by "Deutsche Forschungsgemeinschaft" under research project "Eigenfunktionen". I want to express my thanks to E. Wienholtz, the leader of that project, for valuable discussions. H. Kalf called my attention to Halvorsen's example and C.G. Simader sketched the very elegant proof of Lemma 3a).

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Received February 6, 1986; in final form June 29, 1986