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Two-Parameter Maximal Functions in the Heisenberg Group

Fulvio Ricci [1] and Peter Sjögren [2]

[1] Department of Mathematics, Politecnico di Torino, Corso Duca degli Abruzzi 24, I-10129 Torino, Italy

[2] Department of Mathematics, Chalmers University of Technology and University of Göteborg, S-41296 Göteborg, Sweden

1. Introduction

Let H_1 be the three-dimensional Heisenberg group, consisting of the lower-triangular matrices

$$g=\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix}=(x, y, z),$$

where x, y, z are real numbers. The product in H_1 is the ordinary matrix product, so that

$$(x, y, z)(x', y', z')=(x+x', y+y', z+z'+yx'). \tag{1}$$

We are interested in studying some maximal operators on H_1 associated with the two-parameter family of (automorphic) dilations

$$D_{\delta,\varepsilon}(x, y, z)=(\delta x, \varepsilon y, \delta\varepsilon z), \tag{2}$$

and in determining their L^p-boundedness properties.

The Heisenberg group H_1 is just the simplest example of a nilpotent Lie group with a multiple-parameter family of dilations. The most interesting groups of this type are those which appear in Iwasawa decompositions of semisimple groups with real rank larger than one. In this way, H_1 appears in connection with $SL(3, \mathbb{R})$. The rank of the semisimple group is the number of parameters for the dilations on the nilpotent group.

The operators we consider in this paper have relevance in the analysis on $SL(3, \mathbb{R})$. Some of them actually arise in the study of the boundary behaviour of Poisson integrals on the symmetric space $SL(3, \mathbb{R})/SO(3)$, see Sjögren [8, Sect. 6].

The first maximal operator to be considered is obtained by fixing some nice bounded set Q in H_1, e.g.

$$Q = \{(x, y, z) \colon |x| < 1, |y| < 1, |z| < 1\}$$

and defining, for $g \in H_1$,

$$M_Q f(g) = \sup_{\delta, \varepsilon > 0} \int_Q |f(g(\delta x, \varepsilon y, \delta \varepsilon z)^{-1}|\, dx\, dy\, dz. \tag{3}$$

It is easily seen that M_Q is bounded on $L^p(H_1)$ for $1 < p \leqq \infty$. We sketch the proof, which is given, in a more general context, in Korányi [5, Sect. 3]. It follows from the multiplication law (1) that

$$(x, y, z) = (x, 0, 0)(0, y, 0)(0, 0, z),$$

so that

$$\begin{aligned} &M_Q f(g) \\ &\quad = \sup_{\delta, \varepsilon > 0} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} |f(g(0, 0, -\delta \varepsilon z)(0, -\varepsilon y, 0)(-\delta x, 0, 0))|\, dx\, dy\, dz \\ &\quad \leqq \sup_{\delta, \varepsilon, \rho > 0} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} |f(g(0, 0, -\rho z)(0, -\varepsilon y, 0)(-\delta x, 0, 0))|\, dx\, dy\, dz \\ &\quad = M_S f(g). \end{aligned}$$

If we define

$$M_x f(g) = \sup_{\delta > 0} \int_{-1}^{1} |\mathrm{f}(\mathrm{g}(-\delta \mathrm{x}, 0, 0))|\, \mathrm{dx} \tag{4}$$

and M_y, M_z similarly, we see that

$$M_Q f(g) \leqq M_S f(g) \leqq M_z M_y M_x f(g).$$

Since M_x, M_y, and M_z are Hardy-Littlewood maximal operators along one-parameter subgroups, each of them is bounded on $L^p(H_1)$, $1 < p \leqq \infty$, by transference methods, see Coifman and Weiss [3] and Ricci and Stein [7]. The L^p-boundedness of M_Q and the strong maximal operator M_S follows.

The next operators we consider are obtained by integrating along appropriate dilation-invariant embedded submanifolds of H_1.

The embedded curves that are invariant under the $D_{\delta, \varepsilon}$ are the three one-parameter groups $\{(x, 0, 0)\}$, $\{(0, y, 0)\}$, and $\{(0, 0, z)\}$, and their positive and negative halves. The associated maximal functions are M_x, M_y, M_z, respectively, which we have already discussed.

One finds that the invariant embedded two-dimensional manifolds are $\{x = 0\}$, $\{y = 0\}$, $\{z = \alpha x y\}$, $\alpha \in \mathbb{R}$, and unions of their "quadrants". Among these, $\{x = 0\}$, $\{y = 0\}$, $\{z = 0\}$, and $\{z = xy\}$ are products of two of the three one-parameter groups above. There are four such products because of the lack of commutativity. The two-parameter maximal operators connected with these special cases are dominated by products of two of the operators M_x, M_y, M_z, so that L^p boundedness in the range $1<p\leqq\infty$ is easy to establish.

What remains is to consider the surfaces $\{z=\alpha xy\}$ with $\alpha\neq 0, 1$. As a marginal remark, we observe that a coordinate change $x'=x$, $y'=y$, $z'=z+rxy$ (such as the passage to canonical coordinates of the first kind) preserves the formula for $D_{\delta,\varepsilon}$ and the family of surfaces $\{z=\alpha xy\}$. However, the values of α for which the surface "degenerates" into the product of two one-parameter groups depend on the choice of the coordinates. For simplicity, we restrict ourselves to the quadrant $\{x>0, y>0\}$ and define for $\alpha\in\mathbb{R}$

$$(M_\alpha f)(g)=\sup_{\delta>0,\varepsilon>0}\int_0^1\int_0^1 |f(g(\delta x,\varepsilon y,\delta\varepsilon\alpha xy)^{-1})|\,dx\,dy. \tag{5}$$

The aim of the present paper is to prove the following result.

Theorem 1. *For every* $\alpha\in\mathbb{R}$, *the operator* M_α *is bounded on* $L^p(H_1)$ *for* $1<p\leqq\infty$.

If convolution on the Heisenberg group is replaced by convolution in $\mathbb{R}^3$ in the definition of M_α, the resulting maximal operator is bounded on $L^p(\mathbb{R}^3)$ for $1<p\leqq\infty$, as shown by Carlsson, Sjögren and Strömberg [2].

After completing this work, we learned about a recent paper by M. Christ [10]. Our Theorem 1 can also be derived from his Theorem 2.3, proved by a method entirely different from ours.

We prove Theorem 1 by adapting to the present situation a rather standard complex interpolation argument (cf. Geller and Stein [4] and Stein and Wainger [9]). The basic measure defining M_α, i.e., the measure $dx\,dy$ concentrated on a compact portion of the surface $\{z=\alpha xy\}$, is embedded in an analytic family of distributions depending on a complex parameter s. For $\operatorname{Re} s>0$ one obtains more regular maximal operators, which can be proven to be bounded on $L^p(H_1)$ for $p>1$. The operators corresponding to $\operatorname{Re} s<0$ are more singular than M_α, but still they are bounded on $L_2(H_1)$ if $\operatorname{Re} s$ is close to zero.

This last result will follow from L^2 estimates for some singular integral operators which are of independent interest. The class of operators we consider contains the double Hilbert transform along the surface $z=\alpha xy$, as well as other operators given by kernels with a singularity on this surface and higher order singularities on the x and y axes (see Sect. 2). It is interesting to observe that certain of these operators are bounded on L^2 only in the nondegenerate case $\alpha\neq 0, 1$. In turn, the proofs of these estimates use representation theory and the analysis of some integral operators in $\mathbb{R}$.

2. Some Singular Integrals

We consider a class of convolution operators on H_1 which commute with the dilations $D_{\delta,\varepsilon}$. They are defined by means of a kernel concentrated on the surface $\{z=\alpha xy\}$, possibly composed with a fractional derivation in the z direction.

For $\operatorname{Re} s \geqq 0$, let D^s be the fractional derivative on the real line, having $\Gamma((1+s)/2)^{-1}|\xi|^s$ as its Fourier transform. We then define the fractional derivation in the z direction on H_1 as $(D_z^s f)(x, y, z) = (D^s f(x, y, \cdot))(z)$. Explicit examples of our operators are

$$(T_s f)(g) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |x|^{s-1} |y|^{s-1} [D_z^s f(g(x, y, z)^{-1})]_{z=\alpha xy} \, dx \, dy$$

for $0 \leqq \operatorname{Re} s \leqq 1$ and $s \neq 0$, and the double Hilbert transform

$$(Hf)(g) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{xy} f(g(x, y, \alpha xy)^{-1}) \, dx \, dy.$$

Hilbert transforms of this kind in $\mathbb{R}^n$ have been considered by Nagel and Wainger [6].

More generally, when $0 < \operatorname{Re} s < 1$, we consider two locally integrable functions $K_1^s(t)$, $K_2^s(t)$ on the line, such that for $t \neq 0$ and $j = 1, 2$.

$$|K_j^s(t)| \leqq C|t|^{\operatorname{Re} s - 1}, \quad |\hat{K}_j^s(t)| \leqq C|t|^{-\operatorname{Re} s}. \tag{6}$$

If $\operatorname{Re} s = 0$, we assume that K_1^s, K_2^s are tempered distributions which are locally integrable away from 0 and

$$|K_j^s(t)| \leqq C|t|^{-1}, \quad |\hat{K}_j^s(t)| \leqq C, \quad \left|\frac{d}{dt}\hat{K}_j^s(t)\right| \leqq C|t|^{-1}. \tag{7}$$

If $\operatorname{Re} s = 1$, we assume instead that K_1^s and K_2^s are bounded functions whose Fourier transforms are locally integrable away from the origin, and such that

$$|K_j^s(t)| \leqq C, \quad \left|\frac{d}{dt}K_j^s(t)\right| \leqq C|t|^{-1}, \quad |\hat{K}_j^s(t)| \leqq C|t|^{-1}. \tag{8}$$

Theorem 2. *Assume that* $\alpha \neq 0, 1$, $0 \leqq \operatorname{Re} s \leqq 1$, *and that* K_1^s, K_2^s *are as above. Then the operator*

$$(T_s f)(g) = \iint K_1^s(x) K_2^s(y) [D_z^s f(g(x, y, z)^{-1})]_{z=\alpha xy} \, dx \, dy = f * K_s(g), \tag{9}$$

initially defined on $\mathcal{S}(H_1)$, *is bounded on* $L^2(H_1)$.

In the proof of Theorem 2, we need the following generalization of Schur's boundedness criterion. It was proved by Brown, Halmos, and Shields [1].

Lemma 1. *Let* $K \geqq 0$ *be a measurable function on* $\mathbb{R}^2$. *Assume that there exists a function* $g \in L^1_{\text{loc}}(\mathbb{R})$ *with* $g > 0$ a.e. *such that*

$$(Tg)(x) = \int K(x, y) g(y) \, dy \leqq Cg(x) \text{ a.e.}$$

and

$$(T^* g)(x) = \int K(y, x) g(y) \, dy \leqq Cg(x) \text{ a.e.}$$

Then the operator T *defined by the kernel* K *is bounded on* $L^2(\mathbb{R})$.

Proof of Theorem 2. We verify the norm estimate, assuming that K_1^s and K_2^s are in $\mathscr{S}$. Since K_1^s and K_2^s can be approximated in the distribution sense by functions in $\mathscr{S}$ satisfying uniformly the same conditions (6), (7), or (8), respectively, we can assume that our kernels are in $\mathscr{S}$.

For $\lambda \neq 0$, let π_λ be the Schrödinger representation of H_1 in $L^2(\mathbb{R})$ given by

$$(\pi_\lambda(x, y, z) f)(\xi) = e^{i\lambda(z - \xi y - xy)} f(\xi + x). \tag{10}$$

By Plancherel's formula, we need only prove that the operators $\pi_\lambda(K_s)$ given by

$$\pi_\lambda(K_s) f = \iint K_1^s(x) K_2^s(y) [D_z^s(\pi_\lambda(x, y, z)^{-1} f)]_{z=\alpha xy}\, dx\, dy$$

are bounded on $L^2(\mathbb{R})$, uniformly in λ. We have

$$\begin{aligned}(\pi_\lambda(K_s) f)(\xi) &= \iint K_1^s(x) K_2^s(y) [D_z^s(e^{i\lambda(-z+\xi y)} f(\xi - x)]_{z=\alpha xy}\, dx\, dy \\ &= \Gamma\left(\frac{1+s}{2}\right)^{-1} |\lambda|^s \iint K_1^s(x) K_2^s(y) e^{-i\lambda(\alpha xy - \xi y)} f(\xi - x)\, dx\, dy \\ &= \Gamma\left(\frac{1+s}{2}\right)^{-1} |\lambda|^s \int K_1^s(x) \hat{K}_2^s(\lambda(\alpha x - \xi)) f(\xi - x)\, dx. \end{aligned} \tag{11}$$

Assume first that $0 < \operatorname{Re} s = \rho < 1$. By (6),

$$\begin{aligned}|(\pi_\lambda(K_s) f)(\xi)| &\leqq C_s \int |x|^{\rho-1} |\alpha x - \xi|^{-\rho} |f(\xi - x)|\, dx \\ &\leqq C_s \int |\xi - x|^{\rho-1} |(\alpha - 1)\xi - \alpha x|^{-\rho} |f(x)|\, dx.\end{aligned}$$

Applying Lemma 1 with $K(\xi, x) = |\xi - x|^{\rho-1} |(\alpha-1)\xi - \alpha x|^{-\rho}$ and $g(x) = |x|^{-1/2}$, we conclude that

$$\|\pi_\lambda(K_s) f\|_2 \leqq C \|f\|_2,$$

where C does not depend on λ. This settles the case $0 < \rho < 1$.

Assume now that $\operatorname{Re} s = 0$. By (11),

$$\begin{aligned}(\pi_\lambda(K_s) f)(\xi) &= \Gamma((1+s)/2)^{-1} |\lambda|^s \int K_1^s(x) \hat{K}_2^s(\lambda(\alpha x - \xi)) f(\xi - x)\, dx \\ &= \Gamma((1+s)/2)^{-1} |\lambda|^s \hat{K}_2^s(-\lambda\xi) \int K_1^s(x) f(\xi - x)\, dx \\ &\quad + \Gamma((1+s)/2)^{-1} |\lambda|^s \int K_1^s(x) (\hat{K}_2^s(\lambda(\alpha x - \xi)) - \hat{K}_2^s(-\lambda\xi)) f(\xi - x)\, dx.\end{aligned}$$

Since $|\lambda|^s$ has absolute value one and both $\hat{K}_1^s$ and $\hat{K}_2^s$ are bounded functions, the L^2 norm of the first term is dominated by the L^2 norm of f, independently of λ. As to the last term, we have

$$\int |K_1^s(x)| |\hat{K}_2^s(\lambda(\alpha x - \xi)) - \hat{K}_2^s(-\lambda\xi)| |f(\xi - x)|\, dx = \int_{|x| < 2|\xi|} + \int_{|x| > 2|\xi|}.$$

If $|x|<|\xi|/2|\alpha|$, we can use the smoothness of $\hat{K}_2^s$ away from the origin:

$$|\hat{K}_2^s(\lambda(\alpha x-\xi))-\hat{K}_2^s(-\lambda\xi)|\leqq C\frac{|\lambda\alpha x|}{|\lambda\xi|}=C\frac{|x|}{|\xi|}.$$

This estimate also holds if $|\xi|/2|\alpha|<|x|<2|\xi|$, because $\hat{K}_2^s$ is bounded. Therefore, since $|K_1^s(x)|\leqq C|x|^{-1}$,

$$\int_{|x|<2|\xi|}\leqq C|\xi|^{-1}\int_{|x|<2|\xi|}|f(\xi-x)|\,dx\leqq CMf(\xi),$$

where M is the Hardy-Littlewood maximal operator on the line.

For the remaining integral we have

$$\begin{aligned}\int_{|x|>2|\xi|} &\leqq \int_{|x|>2|\xi|}|x|^{-1}|f(\xi-x)|\,dx\\ &\leqq C\int_{|t|>|\xi|}|t|^{-1}|f(t)|\,dt.\end{aligned}$$

Now the L^2 maximal function estimate and Hardy's inequality give the desired L^2 estimate for $\pi_\lambda(K_s)$.

Let now Re $s=1$. Making the change of variable $t=x-\xi/\alpha$, we have

$$\begin{aligned}(\pi_\lambda(K_s)f)(\xi)&=\Gamma((1+s)/2)^{-1}|\lambda|^s\int K_1^s(x)\hat{K}_2^s(\lambda(\alpha x-\xi))f(\xi-x)\,dx\\ &=\Gamma\left(\frac{1+s}{2}\right)^{-1}|\lambda|^s\int K_1^s\left(t+\frac{\xi}{\alpha}\right)\hat{K}_2^s(\lambda\alpha t)f\left(\frac{\alpha-1}{\alpha}\xi-t\right)dt\\ &=\Gamma\left(\frac{1+s}{2}\right)^{-1}|\lambda|^sK_1^s\left(\frac{\xi}{\alpha}\right)\int\hat{K}_2^s(\lambda\alpha t)f\left(\frac{\alpha-1}{\alpha}\xi-t\right)dt\\ &\quad+\Gamma\left(\frac{1+s}{2}\right)^{-1}|\lambda|^s\int\left(K_1^s\left(t+\frac{\xi}{\alpha}\right)-K_1^s\left(\frac{\xi}{\alpha}\right)\right)\hat{K}_2^s(\lambda\alpha t)f\left(\frac{\alpha-1}{\alpha}\xi-t\right)dt.\end{aligned}$$

The first term is a convolution operator followed by multiplication by a bounded function. Since K_2^s is bounded, the convolution operators with kernels $|\lambda|\hat{K}_2^s(\lambda\alpha t)$ are uniformly bounded on $L^2(\mathbb{R})$ with respect to λ. It follows that the L^2 norm of the first term is dominated by the L^2 norm of $f((\alpha-1)\xi/\alpha)$, and thus by that of f, uniformly in λ. The last term is controlled in the same way as in the case Re $s=0$.

Theorem 2 is proved.

What will be needed in the sequel is already contained in Theorem 2. However, we also present a result regarding the degenerate cases $\alpha=0$ and $\alpha=1$.

Theorem 3. *Assume that* $\alpha=0$ *or* 1, $0\leqq\text{Re}\, s<1/2$, *and that* K_1^s, K_2^s *satisfy* (6) *or* (7) *according to the value of s. Then the operator* T_s *defined by* (9) *is bounded on* $L^2(H_1)$.

A simple example shows that Theorem 3 is false for $s=1/2$. Let $\alpha=0$, $K_1^{1/2}(t)=K_2^{1/2}(t)=|t|^{-1/2}$, and $f=\chi_Q$, where Q is the unit cube defined in the introduction. Then

$$(T_{1/2}f)(g)=c \int_{g(x,y,z)^{-1}\in Q} |x|^{-1/2}|y|^{-1/2}|z|^{-3/2}\,dx\,dy\,dz$$

for any point $g\in H_1$ such that the closure of $Q^{-1}g$ does not intersect the plane $\{z=0\}$. Let $g=(x_1,y_1,z_1)$ with $|x_1|<1$, $10<z_1<11$, $|y_1|$ large. Then $|T_{1/2}f(g)|\geqq \text{const.}\,|y_1|^{-1/2}$, so that $T_{1/2}f\notin L^2(H_1)$.

Proof of Theorem 3. Assume first that $\alpha=0$. Then by (11),

$$(\pi_\lambda(K_s)f)(\xi)=\Gamma((1+s)/2)^{-1}|\lambda|^s\hat{K}_2^s(-\lambda\xi)(K_1^s*f)(\xi).$$

If Re $s=0$, the assumptions (7) imply that $\pi_\lambda(K_s)$ is bounded on $L^2(\mathbb{R})$ uniformly in λ.

If $0<\operatorname{Re} s=\rho<1/2$, then by (6)

$$|(\pi_\lambda(K_s)f)(\xi)|\leqq C_s\int|\xi|^{-\rho}|x-\xi|^{\rho-1}|f(x)|\,dx.$$

We now apply Lemma 1 with $K(\xi,x)=|\xi|^{-\rho}|x-\xi|^{\rho-1}$ and $g(x)=|x|^{-1/2}$. This ends the case Re $s>0$.

If $\alpha=1$, we observe that

$$\pi_\lambda(K_s)f(\xi)=\Gamma((1+s)/2)^{-1}|\lambda|^s(K_1^s*(\hat{K}_2^s(-\lambda\cdot)f))(\xi),$$

and this operator is the transpose of one of those arising in the case $\alpha=0$.

3. Proof of Theorem 1

Instead of M_α, we consider a modified maximal operator. First of all, we can restrict ourselves to the non-degenerate case $\alpha\neq 0, 1$. Let $\psi\geqq 0$ be a C^∞ function on $\mathbb{R}$ supported on $[1/4, 2]$, never vanishing on $[1/2, 1]$. If $f\geqq 0$, as we can also assume, we have

$$\begin{aligned} M_\alpha f(g)&\leqq C\sup_{i,j\in\mathbb{Z}}\iint f(gD_{2^i,2^j}(x,y,\alpha xy)^{-1})\,\psi(x)\,\psi(y)\,dx\,dy\\ &=C\sup_{i,j\in\mathbb{Z}}\iint f(g(x,y,\alpha xy)^{-1})\,2^{-i-j}\psi(2^{-i}x)\,\psi(2^{-j}y)\,dx\,dy. \qquad (12)\end{aligned}$$

For a distribution T on H_1, we denote by T_{ij} the dilated distribution given by

$$\langle T_{ij},\varphi\rangle=\langle T,\varphi\circ D_{2^i,2^j}\rangle.$$

If μ is the measure on the surface $\{z=\alpha xy\}$ given by $\psi(x)\psi(y)\,dx\,dy$, then μ_{ij} is supported on the same manifold and given by $2^{-i-j}\psi(2^{-i}x)\psi(2^{-j}y)\,dx\,dy$. We are then led to consider the maximal operator

$$(M_\mu f)(g)=\sup_{i,j\in\mathbb{Z}}|(f*\mu_{ij})(g)|. \tag{13}$$

Let I^s be the fractional integration kernels $c_s|t|^{s-1}$ on the real line, with $\operatorname{Re} s>0$ and $c_s=2^{-s}\pi^{-1/2}\Gamma(s/2)^{-1}$. Its analytic extension in s to the complex plane is such that $I^0=\delta_0$ and $I^{-s}=D^s$ for $\operatorname{Re} s>0$.

For a distribution S on the line, we define S_z on H_1 as

$$\langle S_z, \varphi\rangle=\langle S, \varphi(0,0,\cdot)\rangle.$$

We want to estimate the operators that are obtained by replacing μ_{ij} in (13) by $(\mu*I_z^s)_{ij}$, with $\operatorname{Re} s$ both positive and negative. This is done in the following three lemmas, whose proofs are postponed to the end.

Lemma 2. *Let* $0<\operatorname{Re} s\leqq 1$. *The operator*

$$f\to\sup_{i,j\in\mathbb{Z}}|f*(\mu*D_z^s)_{ij}|$$

is bounded on $L^2(H_1)$, *and its norm increases at most exponentially in* $\operatorname{Im} s$ *for* $\operatorname{Re} s$ *fixed.*

Take $0\leqq\eta\in C_0^\infty(\mathbb{R})$ with $\eta=1$ near zero and $\operatorname{supp}\eta\subset[-1,1]$.

Lemma 3. *Let* $\operatorname{Re} s>0$. *The operator*

$$f\to\sup_{i,j\in\mathbb{Z}}|f*(\mu*(\eta I^s)_z)_{ij}|$$

is bounded on $L^p(H_1)$ *for* $1<p\leqq\infty$, *and its norm increases at most exponentially in* $\operatorname{Im} s$ *for* $\operatorname{Re} s$ *fixed.*

Lemma 4. *Let* $\operatorname{Re} s>0$. *The operator*

$$f\to\sup_{i,j\in\mathbb{Z}}|f*(\mu*((1-\eta)D^s)_z)_{ij}|$$

is bounded on $L^p(H_1)$ *for* $1<p\leqq\infty$, *and its norm increases at most exponentially in* $\operatorname{Im} s$ *for* $\operatorname{Re} s$ *fixed.*

We can now complete the proof of Theorem 1. From Lemmas 2 and 4, we see that

$$\sup_{i,j}|f*(\mu*(\eta I^s)_z)_{ij}|$$

defines a bounded operator on L^2 for $-1\leqq\operatorname{Re} s<0$. Lemma 3 says that the same operator is bounded on L^p, $p>1$, when $\operatorname{Re} s>0$. Since $\eta I^0=\delta_0$, complex interpolation at $s=0$ (worked out as in [2]) gives the L^p boundedness for M_μ and, therefore, for M_α.

Proof of Lemma 2. We use a square-function argument. Observe first that $T_{ij}f = f*(\mu * D_z^s)_{ij}$ is given by

$$T_{ij}f(g)=2^{(s-1)(i+j)}\int [D_z^s f(g(x,y,z)^{-1})]_{z=\alpha xy}\,\psi(2^{-i}x)\,\psi(2^{-j}y)\,dx\,dy.$$

Let κ_i and λ_j take values in $\{\pm 1\}$. Any finite sum $\sum_{|i|,|j|\leqq N} \kappa_i\lambda_j T_{ij}f$ is of the form $K_s * f$ with

$$K_1^s(x)=\sum_i \kappa_i 2^{(s-1)i}\psi(2^{-i}x),$$

$$K_2^s(x)=\sum_j \lambda_j 2^{(s-1)j}\psi(2^{-j}y).$$

Then $\hat{K}_1^s(x)=\sum_i \kappa_i 2^{si}\hat{\psi}(2^i t)$, $\hat{K}_2^s(t)=\sum_j \lambda_j 2^{sj}\hat{\psi}(2^j t)$, and (6) (or (8) if Re $s=1$) will be valid, with constants of at most exponential growth in Im s.

Theorem 2 and Khinchine's inequality now show that

$$\|(\sum_{i,j}|T_{ij}f|^2)^{1/2}\|_2 \leqq C_s\|f\|_2,$$

for finite sums in i, j and thus also for the sum over $i, j\in\mathbb{Z}$. This gives an estimate for $\sup_{i,j}|T_{ij}f|$ which ends the proof of Lemma 2.

Proof of Lemma 3. We use the method of Lemma 2 in [2]. If Re $s\geqq 1$, the distribution ηI^s is a bounded function and the statement is trivial. We therefore assume that $0<\text{Re}\, s<1$. The convolution $\mu*(\eta I^s)_z$ is supported in the set $\{0\leqq x, y\leqq 2, |z-\alpha xy|\leqq 1\}$. Its absolute value is bounded by $C_s|z-\alpha xy|^{\text{Re}\, s-1}$, where C_s grows at most exponentially in Im s. Hence,

$$|\mu*(\eta I^s)_z|\leqq C_s\sum_{m\geqq 0}2^{m(1-\text{Re}\, s)}\chi_m,$$

where χ_m is the characteristic function of the set $\{0<x, y<2, |z-\alpha xy|\leqq 2^{-m}\}$. This set is contained in a union of boxes $I_k^m\times I_l^m\times I_{kl}^m$, $0\leqq k$, $l<2^{m+1}$. Here $I_k^m=\{k2^{-m}, (k+1)2^{-m}\}$ and similarly for I_l^m, and $I_{kl}^m=\{t: |t-\alpha kl 2^{-2m}|\leqq C2^{-m}\}$ with a suitable C.

We obtain

$$\sup_{i,j}|f*(\mu*(\eta I^s)_z)_{ij}(g)|\leqq C_s\sum_{m\geqq 0}2^{m(1-\text{Re}\, s)}\sum_{k,l}\sup_{i,j}$$
$$\cdot\iiint_{I_k^m\times I_l^m\times I_{kl}^m}|f(gD_{2^i,2^j}(x,y,z)^{-1})|\,dx\,dy\,dz. \quad (14)$$

Define operators

$$M_{k,x}^m f(g)=\sup_{i\in\mathbb{Z}} 2^m\int_{I_k^m}|f(g(-2^i x,0,0))|\,dx,$$

$M^m_{l,y}$ similarly, and

$$M^m_{kl,z} f(g) = \sup_{q \in \mathbb{Z}} 2^m \int_{I^m_{kl}} |f(g(0,0,-2^q z))| \, dz.$$

They are one-dimensional operators transferred to H_1. The three corresponding operators in $\mathbb{R}$ are bounded on L^p with norm at most $C(1+m)^{1/p}$ for $p>1$, because of Lemma 4 in [2]. The transfer does not increase the norm, see e.g. [7, Prop. 5.1].

The integral in (14) equals

$$\int_{I^m_{kl}} dz \int_{I^m_l} dy \int_{I^m_k} dx \, |f(g(0,0,-2^{i+j}z)(0,-2^j y,0)(-2^i x,0,0))|$$
$$\leqq 2^{-3m} M^m_{kl,z} M^m_{l,y} M^m_{k,x} f(g).$$

The L^p norm of the operator in the lemma is then at most

$$C_s \sum_{m \geqq 0} 2^{-m(2+\operatorname{Re} s)} (1+m)^{3/p} 2^{2m} < \infty.$$

This proves Lemma 3.

Proof of Lemma 4. If $\operatorname{Re} s > 0$, the kernel of D^s decreases like $|t|^{-1-\operatorname{Re} s}$ at ∞ (or is supported at 0 when s an even integer). Therefore,

$$|(1-\eta(t)) D^s(t)| \leqq C \sum_{m>0} 2^{-m(1+\operatorname{Re} s)} \chi_{|t| \leqq 2^m}.$$

Hence

$$|\mu * ((1-\eta) D^s)_z (x,y,z)| \leqq C \sum_{m>0} 2^{-m(1+\operatorname{Re} s)} \chi_m,$$

where this time χ_m denotes the characteristic function of the box $\{0 \leqq x, y \leqq 2, |z| \leqq C 2^m\}$. This gives

$$\sup_{i,j} |f * (\mu * ((1-\eta) D^s)_z)_{ij}| \leqq C_s \sum_{m>0} 2^{-m(1+\operatorname{Re} s)} \sup_{i,j} |f * (\chi_m)_{ij}|.$$

The last supremum here is dominated by $2^m M_S f$, where M_S was defined in the introduction. The L^p estimate of the lemma now follows from the L^p boundedness of M_S.

References

1. Brown, A., Halmos, P.R., Shields, A.L.: Cesàro operators. Acta Szeged **26**, 125–137 (1965)
2. Carlsson, H., Sjögren, P., Strömberg, J.-O.: Multiparameter maximal functions along dilation-invariant hypersurfaces. Trans. Am. Math. Soc. **292**, 335–343 (1985)
3. Coifman, R.R., Weiss, G.: Transference methods in analysis. CBMS Reg. Conf. Series in Math. **31**, Am. Math. Soc., Providence 1977
4. Geller, D., Stein, E.M., Estimates for singular convolution operators on the Heisenberg group. Math. Ann. **267**, 1–15 (1984)

5. Korányi, A.: Harmonic functions on symmetric spaces, pp. 379–412 in Symmetric spaces, ed. Boothby and Weiss. New York: Marcel Dekker 1972
6. Nagel, A., Wainger, S.: L^2 boundedness of Hilbert transforms along surfaces and convolution operators homogeneous with respect to a multiple parameter group. Am. J. Math. **99**, 761–785 (1977)
7. Ricci, F., Stein, E.M.: Harmonic analysis on nilpotent groups and singular intergrals. II: Singular kernels supported on submanifolds. J. Funct. Anal. **78**, 56–84 (1988)
8. Sjögren, P.: Convergence for the square root of the Poisson kernel. Pacific J. Math. **131**, 361–391 (1988)
9. Stein, E.M., Wainger, S.: Problems in harmonic analysis related to curvature. Bull. A.M.S. **84**, 1239–1295 (1978)
10. Christ, M.: The strong maximal function on a nilpotent group. Research Report, Australian Nat. Univ., CMA-R 30-87

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