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Majorizations for Generalized s-Numbers in Semifinite von Neumann Algebras

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Introduction

In the theory of matrices, several majorizations are known for the eigenvalues and the singular values of matrices, which are useful in deriving various norm inequalities for matrices. For each $n \times n$ matrix (resp. Hermitian matrix) A, let $\mu(A) = (\mu_1(A), \ldots, \mu_n(A))$ (resp. $\lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A))$) be the vectors in $\mathbb{R}^n$ whose coordinates are the singular values (resp. the eigenvalues) of A arranged in decreasing order. For two vectors $x = (\alpha_1, \ldots, \alpha_n)$ and $y = (\beta_1, \ldots, \beta_n)$ in $\mathbb{R}^n$, the submajorization $x \prec_w y$ means that $\sum_{j=1}^{k} \alpha_j^* \leqq \sum_{j=1}^{k} \beta_j^*$ for $k = 1, \ldots, n$, where $(\alpha_1^*, \ldots, \alpha_n^*)$ and $(\beta_1^*, \ldots, \beta_n^*)$ are the decreasing rearrangements of x and y. The majorization $x \prec y$ means that $x \prec_w y$ and $\sum_{j=1}^{n} \alpha_j = \sum_{j=1}^{n} \beta_j$. Then the following majorizations are the most important in various respects:

(1°) If A and B are $n \times n$ Hermitian matrices, then

$$\lambda(A) - \lambda(B) \prec \lambda(A - B) \prec \lambda(A) - \check{\lambda}(B)$$

where $\check{\lambda}(B)$ is the increasing rearrangement of $\lambda(B)$. This is the famous Lidskii-Wielandt theorem.

(2°) If A and B are any $n \times n$ matrices, then

$$|\mu(A) - \mu(B)| \prec_w \mu(A - B).$$

The submajorization (2°) for singular values (and to some extent also (1°) for eigenvalues) can be extended to the case of compact operators on a Hilbert space. See [1, 11, 12, 19] for detailed expositions on majorizations for matrices and compact operators. Furthermore analogous majorizations are known for the decreasing rearrangements of measurable functions on a measure space (see [4, 10]).

Recently Fack and Kosaki [6] (also [5]) introduced the notion of generalized

s-numbers for τ-measurable operators affiliated with a semifinite von Neumann algebra. This extends both the notion of singular values of compact operators and that of rearrangements of measurable functions. Similarly the notion of eigenvalues of matrices is extended to the case of selfadjoint τ-measurable operators (see [7, 16]). The purpose of this paper is to establish the majorizations as stated above in the general framework of semifinite von Neumann algebras with a surprisingly simple proof.

In Sect. 1 of this paper, after giving the definitions of the generalized s-numbers and the spectral scales of τ-measurable operators, we mention the concepts of majorization and submajorization together with their several characterizations. In Sect. 2, the L^1-norm inequalities for the generalized s-numbers and the spectral scales are established as preliminary lemmas. Finally in Sect. 3, we prove the main results concerning majorizations for the generalized s-numbers and the spectral scales by the real interpolation method (the K-method). Some norm inequalities are derived from the main results.

1. Preliminaries

Throughout this paper, let $\mathcal{M}$ be a semifinite von Neumann algebra on a Hilbert space $\mathcal{H}$ with a faithful normal semifinite trace τ. A densely-defined closed operator x affiliated with $\mathcal{M}$ is said to be *τ-measurable* if, for each $\delta>0$, there exists a projection e in $\mathcal{M}$ such that $e\mathcal{H}\subset\mathcal{D}(x)$ and $\tau(1-e)<\delta$. Let $\tilde{\mathcal{M}}$ denote the set of all τ-measurable operators affiliated with $\mathcal{M}$, which becomes a complete Hausdorff topological $*$-algebra equipped with the measure topology (see [14, 22]). For $1\leqq p\leqq\infty$, $L^p(\mathcal{M})=L^p(\mathcal{M};\tau)$ is the noncommutative L^p-space on $(\mathcal{M},\tau)$, that is, $L^\infty(\mathcal{M})=\mathcal{M}$ and, for $1\leqq p<\infty$, $L^p(\mathcal{M})$ is the Banach space consisting of all $x\in\tilde{\mathcal{M}}$ with $\|x\|_p=\tau(|x|^p)^{1/p}<\infty$ (see [14, 18]). Let $\tilde{\mathcal{M}}_{sa}$ (resp. $\tilde{\mathcal{M}}_+$) denote the set of all selfadjoint (resp. positive selfadjoint) elements in $\tilde{\mathcal{M}}$. Moreover let $\mathfrak{S}$ be the closure of $L^1(\mathcal{M})$ in $\tilde{\mathcal{M}}$ in the measure topology.

For each selfadjoint operator x affiliated with $\mathcal{M}$, we denote by $e_I(x)$ the spectral projection of x corresponding to an interval I in $\mathbb{R}$. Note that if x is a densely-defined closed operator affiliated with $\mathcal{M}$, then x belongs to $\tilde{\mathcal{M}}$ if and only if $\lim_{s\to\infty}\tau(e_{(s,\infty)}(|x|))=0$. According to [5, 6], the *generalized s-number* (singular value) $\mu_t(x)$, $t>0$, of $x\in\tilde{\mathcal{M}}$ is defined by

$$\mu_t(x)=\inf\{s\geqq 0:\tau(e_{(s,\infty)}(|x|))\leqq t\},$$

which is expressed also by

$$\mu_t(x)=\inf\{\|xe\|: e \text{ is a projection in } \mathcal{M} \text{ with } \tau(1-e)\leqq t\}.$$

See [6] for detailed properties of generalized s-numbers $\mu_t(x)$. We denote simply by $\mu(x)$ the function $t\mapsto\mu_t(x)$ on $(0,\infty)$ into $[0,\infty)$, which is non-increasing and right-continuous.

When τ is finite (i.e. $\tau(1)<\infty$) and $x\in\tilde{\mathcal{M}}_{sa}$, we define

$$\lambda_t(x)=\inf\{s\in\mathbb{R}:\tau(e_{(s,\infty)}(x)\leqq t\},\qquad t\in(0,\tau(1)),$$

and call it the *spectral scale* of x following Petz [16]. For each $t \in (0, \tau(1))$, $\lambda_t(x)$ admits the "min-max" expression:

$$\lambda_t(x) = \inf\{ \sup_{\substack{\xi \in e\mathcal{H} \\ \|\xi\|=1}} \langle x\xi, \xi \rangle : e \text{ is a projection in } \mathcal{M} \text{ with } \tau(1-e) \leqq t \}.$$

The properties of $\lambda_t(x)$ are analogous to those of $\mu_t(x)$ (cf. [16], [7, Sect. 6]). Obviously $\lambda_t(x) = \mu_t(x)$, $t \in (0, \tau(1))$, when $x \in \tilde{\mathcal{M}}_+$. Also let $\check{\lambda}_t(x) = -\lambda_t(-x)$, $t \in (0, \tau(1))$, for $x \in \tilde{\mathcal{M}}_{sa}$. It is easy to check that $\check{\lambda}_t(x) = \lambda_{\tau(1)-t-0}(x)$ for all $t \in (0, \tau(1))$. The function $t \mapsto \lambda_t(x)$ (resp. $t \mapsto \check{\lambda}_t(x)$) on $(0, \tau(1))$ into $\mathbb{R}$ is denoted by $\lambda(x)$ (resp. $\check{\lambda}(x)$), which is regarded as the decreasing (resp. increasing) rearrangement of "generalized eigenvalues" of x. The latter was first used by Murray and von Neumann [13].

For later convenience, we here mention the following facts:

(1°) For each $x \in \tilde{\mathcal{M}}$ and $1 \leqq p \leqq \infty$, $x \in L^p(\mathcal{M})$ if and only if $\mu(x) \in L^p(0, \infty)$, and then $\|x\|_p = \|\mu(x)\|_p$ (see [6, Lemma 2.5(i) and Corollary 2.8]).

(2°) For each $x \in \tilde{\mathcal{M}}_{sa}$ (when $\tau(1) < \infty$) and $1 \leqq p \leqq \infty$, $x \in L^p(\mathcal{M})$ if and only if $\lambda(x) \in L^p(0, \tau(1))$, and then $\|x\|_p = \|\lambda(x)\|_p$.

In the above, $L^p(0, \infty)$ and $L^p(0, \tau(1))$ are the L^p-spaces with respect to the Lebesgue measure. As for (2°), the case $p = \infty$ is easy and the case $1 \leqq p < \infty$ follows from the next proposition which is seen as [16, Proposition 1].

Proposition 1.1. *Assume* $\tau(1) < \infty$. *If* $x \in \tilde{\mathcal{M}}_{sa}$ *and* f *is a real Borel function on* $\mathbb{R}$, *then*

$$\tau(f(x)) = \int_0^{\tau(1)} f(\lambda_t(x))\, dt$$

in the sense that if either side of the equality exists permitting $\pm\infty$, *then so does the other and the two are equal.*

Furthermore, according to [6, Proposition 3.2 and Remark 3.3] (see also [7, Proposition 1.3]), an $x \in \tilde{\mathcal{M}}$ belongs to $\tilde{\mathfrak{S}}$ if and only if $\lim_{t \to \infty} \mu_t(x) = 0$, and $L^p(\mathcal{M})$ is contained in $\tilde{\mathfrak{S}}$ when $1 \leqq p < \infty$. If $\tau(1) < \infty$, then $\tilde{\mathfrak{S}} = \tilde{\mathcal{M}}$ is the set of all densely-defined closed operators affiliated with $\mathcal{M}$.

When $\mathcal{M} = \mathbf{B}(\mathcal{H})$ the algebra of all bounded operators on $\mathcal{H}$ with the canonical trace, $\tilde{\mathcal{M}} = \mathbf{B}(\mathcal{H})$ and $\tilde{\mathfrak{S}}$ is the algebra of all compact operators on $\mathcal{H}$. If x is a compact operator, then $\mu_t(x) = \mu_n$ for all $t \in [n-1, n)$, $n = 1, 2, \ldots$, where $\mu_1 \geqq \mu_2 \geqq \ldots$ are the usual singular values of x.

When $\mathcal{M}$ is commutative, that is, $\mathcal{M} = L^\infty(\Omega, m)$ and $\tau(f) = \int_\Omega f\, dm$ where (Ω, m) is a semifinite (=localizable) measure space, $\tilde{\mathcal{M}}$ consists of measurable functions f on Ω such that f is bounded except on a set of finite measure. Then $\mu(f)$ is nothing but the decreasing rearrangement $|f|^*$ of $|f|$:

$$\mu_t(f) = |f|^*(t) = \inf\{ s \geqq 0 : m(\{\omega \in \Omega : |f(\omega)| > s\}) \leqq t \}$$

for all $t \in (0, \infty)$. If $m(\Omega) < \infty$ and f is a real measurable function on Ω, then

$$\lambda_t(f)=f^*(t)=\inf\{s\in\mathbb{R}: m(\{\omega\in\Omega: f(\omega)>s\})\leqq t\}$$

for all $t\in(0, m(\Omega))$. For the rearrangements of measurable functions, see [3] for example.

Although this paper is concerned with (sub)majorizations of the functions $\mu(x)$ on $(0,\infty)$ and $\lambda(x)$ on $(0,\tau(1))$, we now introduce the concepts of majorization and submajorization in general setup. For $x, y\in\tilde{\mathcal{M}}_+$, x is said to be *submajorized* by y (we write $x\prec\!\!\!\!< y$) if $\int_0^t \mu_s(x)\,ds\leqq\int_0^t \mu_s(y)\,ds$ for all $t>0$, and x is said to be *majorized* by y (we write $x\prec y$) if $x\prec\!\!\!\!< y$ and $\int_0^\infty \mu_s(x)\,ds=\int_0^\infty \mu_s(y)\,ds$, i.e. $\tau(x)=\tau(y)$ (permitting the value ∞). When $\tau(1)<\infty$, these are extended to $x, y\in\tilde{\mathcal{M}}_{sa}$: $x\prec\!\!\!\!< y$ if $\int_0^t \lambda_s(x)\,ds\leqq\int_0^t \lambda_s(y)\,ds$ for all $t\in(0,\tau(1))$, and $x\prec y$ if $x\prec\!\!\!\!< y$ and $\int_0^{\tau(1)} \lambda_s(x)\,ds=\int_0^{\tau(1)} \lambda_s(y)\,ds$, i.e. $\tau(x)=\tau(y)$ (permitting $\pm\infty$). In the commutative case, the (sub)majorization is sometimes called the (weak) spectral order of Hardy, Littlewood and Pólya (see [3, 17]).

To clarify the meaning of majorization and submajorization, we present their characterizations in the next two propositions (almost given in [7, Propositions 2.3 and 2.4]; see also [8]).

Proposition 1.2. *For every $x, y\in\tilde{\mathcal{M}}_+$, the following conditions are equivalent:*

(i) $x\prec\!\!\!\!< y$;

(ii) $\tau((x-r)_+)\leqq\tau((y-r)_+)$ *for all* $r>0$;

(iii) $\tau(f(x))\leqq\tau(f(y))$ *for all non-decreasing continuous convex function f on* $[0,\infty)$ *with* $f(0)\geqq 0$;

(iv) $f(x)\prec\!\!\!\!< f(y)$ *for all f as in* (iii).

The above proposition is readily seen from [6, Lemma 2.5(iv)] and [17, Theorems 2.2 and 3.1].

Proposition 1.3. *Assume $\tau(1)<\infty$. For every $x, y\in L^1(\mathcal{M})_{sa}$ (the selfadjoint part of $L^1(\mathcal{M})$), the following conditions are equivalent:*

(i) $x\prec y$;

(ii) $\tau(x)=\tau(y)$ *and* $\tau((x-r)_+)\leqq t((y-r)_+)$ *for all* $r\in\mathbb{R}$;

(iii) $\tau(|x-r|)\leqq\tau(|y-r|)$ *for all* $r\in\mathbb{R}$;

(iv) $\tau(f(x))\leqq\tau(f(y))$ *for all convex function f on* $\mathbb{R}$;

(v) $f(x)\prec\!\!\!\!< f(y)$ *for all f as in* (iv).

Proof. The equivalence of (i), (ii) and (iv) follows from Proposition 1.1 and [3, Theorems 1.6 and 2.5]. It is obvious that (v)⇒(iv)⇒(iii). (iv)⇒(v) follows from the fact that if f is a convex function on $\mathbb{R}$, then so is $(f-r)_+$ for any $r\in\mathbb{R}$. Finally, since $2z_+=|z|+z$ and

$$\pm\tau(z)=\lim_{r\to\infty}\tau(r-|z\mp r|)$$

for every $z\in L^1(\mathscr{M})_{sa}$, (iii)⇒(ii) is obtained. □

2. Lemmas

In this section, we establish some lemmas on the generalized s-numbers $\mu(x)$ and the spectral scales $\lambda(x)$ which will be used in the next section.

When $\tau(1)<\infty$, we define a faithful normal finite trace $\hat{\tau}$ on $\mathscr{M}\otimes\mathbf{M}_2$ ($\mathbf{M}_2$ denotes the 2×2 matrix algebra) by

$$\hat{\tau}\left(\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{bmatrix}\right)=\tau(x_{11}+x_{22}), \quad \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{bmatrix}\in\mathscr{M}\otimes\mathbf{M}_2.$$

It is straightforward to see that $(\mathscr{M}\otimes\mathbf{M}_2)^{\sim}=\tilde{\mathscr{M}}\otimes\mathbf{M}_2$. The following lemma is a very useful device which is well known in the case of matrices.

Lemma 2.1. *Assume* $\tau(1)<\infty$. *If* $x\in\tilde{\mathscr{M}}$ *and* $\hat{x}=\begin{bmatrix} 0 & x^* \\ x & 0\end{bmatrix}$, *then* $\hat{x}\in(\mathscr{M}\otimes\mathbf{M}_2)^{\sim}_{sa}$ *and*

$$\lambda_t(\hat{x})=\begin{cases} \mu_t(x), & 0<t<\tau(1), \\ -\mu_{2\tau(1)-t-0}(x), & \tau(1)\leqq t<2\tau(1).\end{cases}$$

Proof. The first assertion is obvious. Since $|\hat{x}|=\begin{bmatrix} |x| & 0 \\ 0 & |x^*|\end{bmatrix}$, we get

$$\hat{x}_+=\frac{1}{2}(|\hat{x}|+\hat{x})=\frac{1}{2}\begin{bmatrix} |x| & x^* \\ x & |x^*|\end{bmatrix},$$

$$\hat{x}_-=\frac{1}{2}(|\hat{x}|-\hat{x})=\frac{1}{2}\begin{bmatrix} |x| & -x^* \\ -x & |x^*|\end{bmatrix},$$

so that

$$\begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix}\hat{x}_+\begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix}=\hat{x}_-.$$

This shows that

$$\hat{\tau}(e_{(s,\infty)}(|\hat{x}|))=\hat{\tau}(e_{(s,\infty)}(\hat{x}_+)+e_{(s,\infty)}(\hat{x}_-))=2\hat{\tau}(e_{(s,\infty)}(\hat{x}_+))$$

for all $s\geqq 0$. Furthermore, since we have $\tau(e_{(s,\infty)}(|x|))=\tau(e_{(s,\infty)}(|x^*|))$ by taking the polar decomposition of x, it follows that

$$\hat{\tau}(e_{(s,\infty)}(|\hat{x}|))=\hat{\tau}\left(\begin{bmatrix} e_{(s,\infty)}(|x|) & 0 \\ 0 & e_{(s,\infty)}(|x^*|)\end{bmatrix}\right)=2\tau(e_{(s,\infty)}(|x|)).$$

Hence $\hat{\tau}(e_{(s,\infty)}(\hat{x}_+))=\tau(e_{(s,\infty)}(|x|))$ for all $s\geqq 0$. Therefore, if $0<t<\tau(1)$, then

$$\lambda_t(\hat{x})=\lambda_t(\hat{x}_+)=\lambda_t(\hat{x}_-)=\mu_t(x),$$

and if $\tau(1)<t<2\tau(1)$, then

$$\begin{aligned}\lambda_t(\hat{x})&=\inf\{s\leqq 0\colon \hat{\tau}(e_{(s,\infty)}(\hat{x}))\leqq t\}\\&=\inf\{s\leqq 0\colon \hat{\tau}(1-e_{[-s,\infty)}(\hat{x}_-))\leqq t\}\\&=-\sup\{s\geqq 0\colon \hat{\tau}(e_{[s,\infty)}(\hat{x}_-))\geqq 2\tau(1)-t\}\\&=-\lambda_{2\tau(1)-t-0}(\hat{x}_-)\\&=-\mu_{2\tau(1)-t-0}(x).\quad\square\end{aligned}$$

Lemma 2.2. *If $\tau(1)<\infty$ and $x,y\in L^1(\mathcal{M})_{sa}$, then*

$$\|\lambda(x)-\lambda(y)\|_1\leqq\|x-y\|_1.$$

The above L^1-norm inequality was given in [16, Proposition 3] for $x, y\in\mathcal{M}_{sa}$. But the proof remains valid for $x, y\in L^1(\mathcal{M})_{sa}$ in view of Proposition 1.1.

Though the following lemma is an immediate consequence of our main theorem (see Corollary 3.3 in the next section), the theorem will be shown to follow from this special case.

Lemma 2.3. (1) *If $x, y\in\tilde{\mathcal{M}}$ and $x-y\in\mathcal{M}$, then*

$$\sup_{t>0}|\mu_t(x)-\mu_t(y)|\leqq\|x-y\|.$$

(2) *If $x, y\in L^1(\mathcal{M})$, then*

$$\|\mu(x)-\mu(y)\|_1\leqq\|x-y\|_1.$$

Proof. (1) is readily verified from [6, Lemma 2.5(i), (v)] as pointed out in the proof of [6, Proposition 2.7].

(2) Let $x, y\in L^1(\mathcal{M})$. When $\tau(1)<\infty$, we take $(\mathcal{M}\otimes\mathbf{M}_2, \hat{\tau})$ as in Lemma 2.1. let $\hat{x}=\begin{bmatrix}0 & x^*\\ x & 0\end{bmatrix}$ and $\hat{y}=\begin{bmatrix}0 & y^*\\ y & 0\end{bmatrix}$. Then $\hat{x}, \hat{y}\in L^1(\mathcal{M}\otimes\mathbf{M}_2)_{sa}$ and

$$\|\hat{x}-\hat{y}\|_1=\hat{\tau}\left(\begin{bmatrix}|x-y| & 0\\ 0 & |(x-y)^*|\end{bmatrix}\right)=2\|x-y\|_1.$$

By Lemmas 2.2 and 2.1, we have

$$\begin{aligned}\|\hat{x}-\hat{y}\|_1&\geqq\int_0^{2\tau(1)}|\lambda_t(\hat{x})-\lambda_t(\hat{y})|\,dt\\&=2\int_0^{\tau(1)}|\mu_t(x)-\mu_t(y)|\,dt,\end{aligned}$$

implying the desired inequality.

We next show the case $\tau(1)=\infty$. Take the polar decomposition $x=w|x|$ and define

$$x_n=e_{(1/n,\infty)}(|x^*|)\,x\,e_{(1/n,\infty)}(|x|),$$
$$y_n=e_{(1/n,\infty)}(|y^*|)\,y\,e_{(1/n,\infty)}(|y|), \qquad n\geqq 1.$$

Since

$$\|x_n-x\|_1\leqq\|e_{(1/n,\infty)}(|x^*|)\,w|x|\,e_{[0,1/n]}(|x|)\|_1+\|e_{[0,1/n]}(|x^*|)\,|x^*|\,w\|_1$$
$$\leqq\|\,|x|\,e_{[0,1/n]}(|x|)\|_1+\|e_{[0,1/n]}(|x^*|)\,|x^*|\,\|_1,$$

we get $\|x_n-x\|_1\to 0$ and analogously $\|y_n-y\|_1\to 0$. Moreover it follows from [6, Lemma 3.4] that $\mu_t(x_n)\to\mu_t(x)$ and $\mu_t(y_n)\to\mu_t(y)$ for all $t>0$. Now let

$$e_n=e_{(1/n,\infty)}(|x|)\vee e_{(1/n,\infty)}(|x^*|)\vee e_{(1/n,\infty)}(|y|)\vee e_{(1/n,\infty)}(|y^*|).$$

Then $\tau(e_n)<\infty$ and $x_n,\,y_n\in L^1(e_n\mathcal{M}e_n)$. Therefore we have

$$\int_0^\infty|\mu_t(x)-\mu_t(y)|\,dt\leqq\liminf_{n\to\infty}\int_0^\infty|\mu_t(x_n)-\mu_t(y_n)|\,dt$$
$$\leqq\lim_{n\to\infty}\|x_n-y_n\|_1$$
$$=\|x-y\|_1$$

by Fatou's lemma and the assertion already shown for the case $\tau(1)<\infty$. □

3. Majorizations

Concerning majorizations for the generalized s-numbers and the spectral scales of τ-measurable operators, our main results are stated as follows:

Theorem 3.1. *If $x\in\tilde{\mathcal{M}}$ and $y\in\tilde{\mathfrak{S}}$, then*

$$|\mu(x)-\mu(y)|\prec\mu(x-y).$$

Theorem 3.2. *If $\tau(1)<\infty$ and $x,\,y\in L^1(\mathcal{M})_{sa}$, then*

$$\lambda(x)-\lambda(y)\prec\lambda(x-y)\prec\lambda(x)-\check{\lambda}(y).$$

We give some comments before proving the theorems. First the submajorization in Theorem 3.1 means that

$$\int_0^t|\mu(x)-\mu(y)|^*(s)\,ds\leqq\int_0^t\mu_s(x-y)\,ds, \qquad t>0,$$

where $|\mu(x)-\mu(y)|^*$ is the decreasing rearrangement of $|\mu(x)-\mu(y)|$ on the Lebesgue measure space $(0,\infty)$ (note $(\mu(x-y))^*=\mu(x-y)$). The majorizations in Theorem 3.2 are analogous. These majorizations for τ-measurable operators are the general extensions of those for matrices stated in Introduction.

Secondly it should be pointed out that even in the case of matrices the derivation of majorizations in question is not at all easy; the known methods are essentially based on either the induction on matrix order or the smooth dependence of eigenvalues on matrix entries.

Thirdly the crucial point in the proof of Theorem 3.1 is the following formula given in [6, the remark after Theorem 4.4]:

$$\int_0^t \mu_s(x)\,ds=\inf\{\|x_1\|_1+t\|x_2\|: x=x_1+x_2,\ x_1\in L^1(\mathcal{M}),\ x_2\in\mathcal{M}\}$$

for every $x\in\tilde{\mathcal{M}}$ and $t>0$. This implies in particular that $x\in L^1(\mathcal{M})+\mathcal{M}$ if and only if $\int_0^t \mu_s(x)\,ds<\infty$ for some (hence all) $t>0$. Indeed the above formula is well known in the commutative case. Its right-hand side is familiar as the K-functional in the real interpolation theory and is denoted by $K(t,x)=K(t,x;L^1(\mathcal{M}),\mathcal{M})$ (see [2] for example). The following simple proof is based on the real interpolation method (the K-method) which is useful in the interpolation of Lipschitz continuous (non-linear) maps (cf. [15, 21]).

Proof of Theorem 3.1. First let $x\in\tilde{\mathcal{M}}$ and $y\in L^1(\mathcal{M})$. As remarked above, for every $t>0$ we have

$$\int_0^t \mu_s(x-y)\,ds=\inf\{\|x_1\|_1+t\|x_2\|: x-y=x_1+x_2,\ x_1\in L^1(\mathcal{M}),\ x_1\in\mathcal{M}\}$$

and

$$\int_0^t |\mu(x)-\mu(y)|^*(s)\,ds$$
$$=\inf\{\|f_1\|_1+t\|f_2\|_\infty: \mu(x)-\mu(y)=f_1+f_2,\ f_1\in L^1(0,\infty),\ f_2\in L^\infty(0,\infty)\}.$$

For each $x_1\in L^1(\mathcal{M})$ and $x_2\in\mathcal{M}$ with $x-y=x_1+x_2$, define

$$f_1=\mu(x_1+y)-\mu(y),$$
$$f_2=\mu(x)-\mu(x_1+y).$$

Then $\mu(x)-\mu(y)=f_1+f_2$. Since $x_1+y,\ y\in L^1(\mathcal{M})$ and $x-(x_1+y)=x_2\in\mathcal{M}$, it follows from Lemma 2.3 that $\|f_1\|_1\leqq\|x_1\|_1$ and $\|f_2\|_\infty\leqq\|x_2\|$. Hence

$$\int_0^t |\mu(x)-\mu(y)|^*(s)\,ds\leqq\|f_1\|_1+t\|f_2\|_\infty\leqq\|x_1\|_1+t\|x_2\|,$$

so that

$$\int_0^t |\mu(x)-\mu(y)|^*(s)\,ds\leqq\int_0^t \mu_s(x-y)\,ds.$$

Next let $x \in \tilde{\mathcal{M}}$ and $y \in \tilde{\mathfrak{S}}$. Choose a sequence $\{y_n\}$ in $L^1(\mathcal{M})$ such that $y_n \to y$ in the measure topology, and let $x_n = x + y_n - y$. By [6, Lemma 3.4], we get $\mu_t(x_n) \to \mu_t(x)$ and $\mu_t(y_n) \to \mu_t(y)$ for almost every $t > 0$. We now notice (cf. [20, p. 202]) that

$$\int_0^t |\mu(x) - \mu(y)|^*(s)\, ds$$

$$= \sup\{\int_E |\mu_s(x) - \mu_s(y)|\, ds \colon E \text{ is a Borel set in } (0, \infty) \text{ with } |E| = t\}$$

where $|E|$ denotes the Lebesgue measure of E. For every $t > 0$ and E as in the above expression, we have

$$\begin{aligned} \int_E |\mu_s(x) - \mu_s(y)|\, ds &\leqq \liminf_{n\to\infty} \int_E |\mu_s(x_n) - \mu_s(y_n)|\, ds \\ &\leqq \liminf_{n\to\infty} \int_0^t |\mu(x_n) - \mu(y_n)|^*(s)\, ds \\ &\leqq \int_0^t \mu_s(x-y)\, ds \end{aligned}$$

using Fatou's lemma, the case already shown and $x_n - y_n = x - y$. Thus the theorem is proved. □

Replacing x by $x + y$ in Theorem 3.1, we have

$$\begin{aligned} \int_0^t \{\mu_s(x+y) - \mu_s(y)\}\, ds &\leqq \int_0^t |\mu(x+y) - \mu(y)|^*(s)\, ds \\ &\leqq \int_0^t \mu_s(x)\, ds, \qquad t > 0, \end{aligned}$$

and hence $\mu(x+y) \prec \mu(x) + \mu(y)$. This weakened submajorization was shown in [6, Theorem 4.4] for all $x, y \in \tilde{\mathcal{M}}$ in connection with Minkowski's inequality in $L^p(\mathcal{M})$.

Proof of Theorem 3.2. To show the first majorization, since

$$\int_0^{\tau(1)} \{\lambda_s(x) - \lambda_s(y)\}\, ds = \tau(x-y) = \int_0^{\tau(1)} \lambda_s(x-y)\, ds$$

by Proposition 1.1, it suffices to prove that $\lambda(x) - \lambda(y) \prec \lambda(x-y)$. When $x, y \in \mathcal{M}_{sa}$, if we take $\alpha, \beta \in \mathbb{R}$ with $x + \alpha \geqq y + \beta \geqq 0$, then Theorem 3.1 gives

$$\lambda(x+\alpha) - \lambda(y+\beta) \prec \lambda(x - y + (\alpha - \beta)),$$

so that $\lambda(x) - \lambda(y) \prec \lambda(x-y)$ follows from $\lambda(x+\alpha) = \lambda(x) + \alpha$ and analogous equalities for $\lambda(y+\beta)$ and $\lambda(x - y + (\alpha-\beta))$.When $x, y \in L^1(\mathcal{M})_{sa}$, we choose sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{M}_{sa}$ such that $\|x_n - x\|_1 \to 0$ and $\|y_n - y\|_1 \to 0$. By Lemma 2.2,

$\lambda(x_n)$, $\lambda(y_n)$ and $\lambda(x_n-y_n)$ converge in L^1-norm to $\lambda(x)$, $\lambda(y)$ and $\lambda(x-y)$, respectively, and hence $(\lambda(x_n)-\lambda(y_n))^*$ converges in L^1-norm to $(\lambda(x)-\lambda(y))^*$. By passing to the limit of $\lambda(x_n)-\lambda(y_n)\prec\lambda(x_n-y_n)$, we obtain the desired conclusion.

The second majorization is the same as $\lambda(x+y)\prec\lambda(x)+\lambda(y)$ when y is replaced by $-y$. We can show the latter by replacing x by $x+y$ in the first majorization. (Also this is seen from the submajorization $\mu(x+y)\prec_w\mu(x)+\mu(y)$ as in the proof of the first.) □

By Theorems 3.1 and 3.2 (together with Propositions 1.2, 1.3 and (1°), (2°) in Sect. 1), we obtain the following L^p-norm inequalities for $\mu(x)$ and $\lambda(x)$.

Corollary 3.3. *Let* $1\leqq p\leqq\infty$.

(1) *If* $x, y\in L^p(\mathcal{M})$, *then*

$$\|\mu(x)-\mu(y)\|_p\leqq\|x-y\|_p.$$

(2) *If* $\tau(1)<\infty$ *and* $x, y\in L^p(\mathcal{M})_{sa}$, *then*

$$\|\lambda(x)-\lambda(y)\|_p\leqq\|x-y\|_p\leqq\|\lambda(x)-\check{\lambda}(y)\|_p.$$

Kosaki [9] introduced the noncommutative Lorentz space $L^{pq}(\mathcal{M})$ for $1\leqq p$, $q\leqq\infty$ (in particular, $L^{p\infty}(\mathcal{M})$ is called the noncommutative weak L^p-space). As in the commutative case (see [2]), when $1<p<\infty$ and $1\leqq q\leqq\infty$, $L^{pq}(\mathcal{M})$ is exactly the real interpolation Banach space $(L^1(\mathcal{M}), \mathcal{M})_{\theta,q}$ with $\theta=1-1/p$ possessing the interpolation norm:

$$\|x\|_{pq}=\left\{\int_0^\infty (t^{-\theta}K(t,x))^q\frac{dt}{t}\right\}^{1/q} \quad \text{if } q<\infty,$$

$$\|x\|_{p\infty}=\sup_{t>0} t^{-\theta}K(t,x) \quad \text{if } q=\infty,$$

where $K(t,x)=\int_0^t \mu_s(x)\,ds$ as remarked before the proof of Theorem 3.1.

Corollary 3.4. *Let* $1<p<\infty$ *and* $1\leqq q\leqq\infty$.

(1) *If* $x, y\in L^{pq}(\mathcal{M})$, *then*

$$\|\mu(x)-\mu(y)\|_{pq}\leqq\|x-y\|_{pq}.$$

(2) *If* $\tau(1)<\infty$ *and* $x, y\in L^{pq}(\mathcal{M})_{sa}$, *then*

$$\|\lambda(x)-\lambda(y)\|_{pq}\leqq\|x-y\|_{pq}\leqq\|\lambda(x)-\check{\lambda}(y)\|_{pq}.$$

Proof. (1) is immediate from Theorem 3.1 and the definition of L^{pq}-norm.

(2) For each $x\in L^1(\mathcal{M})_{sa}$ where $\tau(1)<\infty$, we get $|\lambda(x)|^*=\mu(x)$ by considering in the commutative von Neumann subalgebra of $\mathcal{M}$ generated by all spectral projections of x (cf. [3, Corollary 2.7]). Hence Theorem 3.2 gives

$$|\lambda(x)-\lambda(y)|\prec_w\mu(x-y)\prec_w|\lambda(x)-\check{\lambda}(y)|$$

for every $x, y\in L^1(\mathcal{M})_{sa}$. This implies the desired conclusion. □

We finally give a remark concerning a submajorization in relation to doubly substochastic maps. The relation between (sub)majorizations and doubly (sub-) stochastic maps was discussed in [7] in semifinite von Neumann algebras. A linear map φ of $\mathcal{M}$ into itself is called to be *doubly substochastic* if φ is positive, $\varphi(1) \leqq 1$ and $\tau(\varphi(x)) \leqq \tau(x)$ for all $x \in \mathcal{M}_+$. Such a map φ is canonically extended to a linear map of $L^1(\mathcal{M}) + \mathcal{M}$ into itself so that φ is $\|\cdot\|$-contractive on $\mathcal{M}$ and $\|\cdot\|_1$-contractive on $L^1(\mathcal{M})$ (cf. [7, Proposition 4.1]). Thus, using the K-functionals as in the proof of Theorem 3.1, we see that if $\varphi: \mathcal{M} \to \mathcal{M}$ is doubly substochastic, then $|\varphi(x)| \prec |x|$ for all $x \in L^1(\mathcal{M}) + \mathcal{M}$. This was proved in [7, Theorem 4.5] only when $x \in (L^1(\mathcal{M}) + \mathcal{M}) \cap \tilde{\mathfrak{S}}$.

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