Math. Z. 199, 55–60 (1988)

Mathematische Zeitschrift

Isotropic Totally Real Submanifolds

Sebastian Montiel* and Francisco Urbano*
Departamento de Geometría y Topología, Universidad de Granada, 18071-Granada, Spain

1. Introduction

The notion of an isotropic submanifold of a Riemannian manifold was introduced by B. O'Neill ([2]), who studied the general properties of such class of submanifolds.

These submanifolds, which can be considered as a generalization of the totally geodesic submanifolds, have been nearly always studied under the additional hypothesis of parallelism of the second fundamental form. When the ambient space is a sphere, this study was made by K. Sakamoto ([3]); in the case of the complex projective space by H. Naitoh ([1]).

In this paper, we study n-dimensional isotropic totally real submanifolds of a complex n-dimensional Kaehler manifold without assumptions about the parallelism of the second fundamental form. So, we prove that a strong restriction exists on the dimension so that a Kaehler manifold admits non-zero isotropic minimal totally real submanifolds (Theorem 1). Also, we prove that under the assumption that the submanifold is curvature-invariant, the isotropic condition implies that the second fundamental form is parallel (Proposition 1). Then, using the results given by H. Naitoh ([1]), we classify the isotropic totally real submanifolds in a complex space form (Corollaries 2 and 3).

2. Preliminaries

Let M^n be an n-dimensional totally real submanifold isometrically immersed in a complex n-dimensional Kaehler manifold $\bar{M}^n$. We denote by $\langle , \rangle$ the metric of $\bar{M}^n$ as well as that induced on M^n. If σ is the second fundamental form of the immersion and A_ξ the Weingarten endomorphism associated to a normal vector ξ, then it is well known that

$$A_{JX}\,Y = -J\sigma(X, Y), \tag{2.1}$$

* Research partially supported by CAICYT Grant PR84-1242-CO2-02

for vectors X, Y tangent to M^n, where J denotes the complex structure of $\bar{M}^n$. From (2.1) we have that $F(X, Y, Z) = \langle \sigma(X, Y), JZ \rangle$ is a symmetric tensor. We denote by R the curvature tensor of M^n.

Now we suppose that M^n is a curvature-invariant submanifold of $\bar{M}^n$, i.e., $\bar{R}(X, Y)Z \in T_p M^n$ for $X, Y, Z \in T_p M^n$, where $\bar{R}$ is the curvature tensor of $\bar{M}^n$. Then, if $\nabla\sigma$ and $\nabla^2\sigma$ denote the first and second covariant derivatives of σ, respectively, we have that $\nabla\sigma$ is symmetric and $\nabla^2\sigma$ satisfies the relation

$$(\nabla^2\sigma)(X, Y, Z, W) = (\nabla^2\sigma)(Y, X, Z, W) + JR(X, Y) A_{JZ} W \\ - \sigma(R(X, Y) Z, W) - \sigma(Z, R(X, Y) W). \tag{2.2}$$

We recall the notion of isotropic submanifold. A submanifold M in a Riemannian manifold $\bar{M}$ is called *isotropic* if there exists a positive function λ: $M \to R$ such that $|\sigma(v, v)| = \lambda(p)$ for any unit tangent vector $v \in T_p M$ and for any $p \in M$. If λ is constant we say that M is *constant isotropic*.

Finally we prove a Lemma which will be used later.

Lemma 1. *Let M^n be a totally real submanifold in a Kaehler manifold $\bar{M}^n$. If p is a point of M^n, S_p the unit sphere in $T_p M^n$ and f: $S_p \to R$ the function given by $f(v) = \langle \sigma(v, v), Jv \rangle$, then there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_p M^n$ satisfying*

a) $\sigma(e_1, e_i) = \lambda_i J e_i$, $i = 1, \ldots, n$, *where λ_1 is the maximum of f.*
b) $\lambda_1 \geqq 2\lambda_i$, $i = 2, \ldots, n$, *and if $\lambda_1 = 2\lambda_j$ for some $j \in \{2, \ldots, n\}$, then $f(e_j) = 0$.*

Proof. Let e_1 be a vector of S_p where f attains its maximum. Then for any unit vector v orthogonal to e_1, we have,

$$0 = d f_{e_1}(v) = 3 \langle \sigma(e_1, e_1), J v \rangle, \tag{2.3}$$

and

$$0 \geqq d^2 f_{e_1}(v, v) = 6 \langle \sigma(v, v), J e_1 \rangle - 3 f(e_1). \tag{2.4}$$

From (2.3), we obtain that $\sigma(e_1, e_1) = \lambda_1 J e_1$, where $\lambda_1 = f(e_1)$. Using (2.1), this implies that e_1 is an eigenvector of A_{Je_1}. So we can choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_p M^n$ which diagonalizes A_{Je_1}, i.e., $A_{Je_1} e_i = \lambda_i e_i$. So using (2.1) we prove a).

Now, using (2.4) one has that $\lambda_1 \geqq 2\lambda_i$ for $i \in \{2, \ldots, n\}$. If $\lambda_1 = 2\lambda_j$, for some $j \in \{2, \ldots, n\}$, then $d^2 f_{e_1}(e_j, e_j) = 0$, and so $d^3 f_{e_1}(e_j, e_j, e_j) = 0$. But using (2.3), $d^3 f_{e_1}(e_j, e_j, e_j) = 6 f(e_j)$. So, we prove b).

3. Statement of Results

Let M be a minimal totally real surface in a Kaehler manifold $\bar{M}^2$. Then using Lemma 1, we have at any point p of M an orthonormal basis $\{e_1, e_2\}$ of $T_p M$ such that $\sigma(e_1, e_1) = \lambda_1 J e_1$, $\sigma(e_1, e_2) = -\lambda_1 J e_2$ and $\sigma(e_2, e_2) = -\lambda_1 J e_1$. Then

$|\sigma(v,v)|=\lambda_1$, for any unit vector v tangent to M ap p. So, we have that "*Every minimal totally real surface in a Kaehler manifold* $\bar{M}^2$ *is isotropic*".

For higher dimension, we can prove the following:

Theorem 1. *Let* $M^n (n \geqq 3)$ *be a minimal totally real submanifold isometrically immersed in a Kaehler manifold* $\bar{M}^n$. *If* M^n *is isotropic, then either* M^n *is totally geodesic or* $n=5, 8, 14$ *or* 26.

Proof. We suppose that M^n is not totally geodesic. Then there exists a point p of M^n such that $|\sigma(v,v)|=\lambda \neq 0$ for any unit vector v tangent to M^n at p.

Then, using that the submanifold is isotropic, we have for any orthonormal vectors v and w of $T_p M^n$ that

$$\langle \sigma(v,v), \sigma(v,w) \rangle = 0, \tag{3.1}$$

$$2|\sigma(v,w)|^2 + \langle \sigma(v,v), \sigma(w,w) \rangle = \lambda^2. \tag{3.2}$$

Now, if $\{e_1, \ldots, e_n\}$ is the orthonormal basis given in Lemma 1, then using (3.1) and (3.2) we have that $\lambda_1=\lambda$, and λ_i, $i=2, \ldots n$, satisfies the following equation $2\lambda_i^2 + \lambda \lambda_i = \lambda^2$. So, λ_i is either $-\lambda$ or $\lambda/2$. Let V_1 and V_2 be the eigenspaces of A_{Je_1} corresponding to the eigenvalues $-\lambda$ and $\lambda/2$ respectively. As M^n is a minimal submanifold we have that Trace $A_{Je_1}=0$, and so it is trivial to see that dim V_2 is even, i.e., dim $V_2 = 2p$, where $p+1=\dim V_1$.

Let x be any unit vector of V_1. Then from Schwartz inequality one has

$$\lambda^2 = \langle \sigma(x,x), Je_1 \rangle^2 \leqq |\sigma(x,x)|^2 = \lambda^2$$

and then $\sigma(x,x) = -\lambda J e_1$. So

$$\sigma(z,w) = -\lambda \langle z, w \rangle J e_1 \tag{3.3}$$

for any vectors z, w of V_1.

As dim $V_1 \geqq 1$, let x_0 be a fix unit vector of V_1. Then using (2.1) and (3.3) we have that A_{Jx_0} applies V_2 into itself, and so there exists an orthonormal basis $w_1, \ldots, w_{2p}$ of V_2 such that $A_{Jx_0} w_i = \rho_i w_i$, $i=1, \ldots, 2p$. But Trace $A_{Jx_0}=0$, and then using (3.3) we obtain $\rho_1 + \ldots + \rho_{2p} = 0$. Again, using (2.1) and (3.2) we have $\rho_i^2 = 3\lambda^2/4$ and so $\rho_i = \pm\sqrt{3}\lambda/2$, $i=1, \ldots, 2p$. So, if V_2^+ and V_2^- are the subspaces of V_2 corresponding to the eigenvalues $\sqrt{3}\lambda/2$ and $-\sqrt{3}\lambda/2$ respectively, we get dim $V_2^+ = \dim V_2^- = p$. Then if V' is the orthogonal complement of $\langle x_0 \rangle$ in V_1, we have that dim V' is also p.

In order to finish the proof, we are going to define a map μ

$$\mu\colon V_2^+ \times V_2^- \to V'$$

which will be bilineal and will satisfy the following condition

$$|\mu(x^+, x^-)| = |x^+| \cdot |x^-|$$

for any x^+ and x^- in V_2^+ and V_2^- respectively. Then, using a very well-known result, we get that $p=1, 2, 4$ or 8, and then the Theorem will follow.

First, using Lemma 1, we have that for any vector x of V_2, $f(x)=0$, and so

$$\langle \sigma(x, y), Jz \rangle = 0 \tag{3.4}$$

for vectors $x, y, z \in V_2$. Then, if $x^+ \in V_2^+$ and $x^- \in V_2^-$, using (3.4) we obtain

$$\sigma(x^+, x^-) \in JV' \tag{3.5}$$

and

$$\begin{aligned} \sigma(x^+, x^-) &= (\lambda/2)|x^+|^2 Je_1 + (\sqrt{3}\lambda/2)|x^+|^2 Jx_0 \\ \sigma(x^-, x^-) &= (\lambda/2)|x^+|^2 Je_1 - (\sqrt{3}\lambda/2)|x^-|^2 Jx_0. \end{aligned} \tag{3.6}$$

From (3.2) and (3.6) we get

$$|\sigma(x^+, x^-)|^2 = (3/4)\lambda^2 |x^+|^2 |x^-|^2. \tag{3.7}$$

Finally, using (3.5), we can define μ as

$$\mu(x^+, x^-) = (2/\sqrt{3}\lambda) J\sigma(x^+, x^-).$$

It is clear that μ is bilineal and from (3.7) also satisfies $|\mu(x^+, x^-)| = |x^+| \cdot |x^-|$. So, the Theorem is proved.

When $\bar{M}^n$ is a complex space form of constant holomorphic sectional curvature c, the assumption in Theorem 1 implies, using (3.2), that

$$\mathrm{Ric}(v, w) = \left(\frac{(n-1)c}{4} - \frac{n+2}{2}\lambda^2\right)\langle v, w \rangle$$

where Ric is the Ricci tensor of M^n. So, as $n \geqq 3$, λ^2 is constant and M^n is constant isotropic. In general, it is not true when $\bar{M}^n$ is any Kaehler manifold.

Now, we study constant isotropic totally real submanifolds. First, we prove the following:

Proposition 1. *Let M^n be a curvature-invariant totally real submanifold in a Kaehler manifold $\bar{M}^n$. If M^n is constant isotropic, then M^n has parallel second fundamental form.*

Proof. As $|\sigma(v, v)|^2$ is a constant function on the unit tangent bundle, we have

$$\langle (\nabla \sigma)(x, v, v), \sigma(v, v) \rangle = 0 \tag{3.8}$$

for any tangent vector $x \in T_p M^n$, $p \in M^n$. In particular

$$\langle (\nabla \sigma)(v, v, v), \sigma(v, v) \rangle = 0 \tag{3.9}$$

for $v \in S_p$, $p \in M^n$. So, if w is a unit vector at p orthogonal to v, we have using the symmetry of $\nabla \sigma$

$$0 = 3\langle (\nabla \sigma)(w, v, v), \sigma(v, v) \rangle + 2\langle (\nabla \sigma)(v, v, v), \sigma(v, w) \rangle,$$

and then, (3.8) and (3.9) imply

$$0=\langle(\nabla\sigma)(v,v,v),\sigma(v,x)\rangle \tag{3.10}$$

for vectors $v\in S_p$, $x\in T_pM^n$, $p\in M^n$. Taking in (3.10) $x=A_{Jv}v$ and using (2.1) and (3.1) we obtain

$$0=\lambda^2\langle(\nabla\sigma)(v,v,v),Jv\rangle$$

for $v\in S_p$, $p\in M^n$. So, either $\lambda=0$ and M^n is totally geodesic, or $\langle(\nabla\sigma)(v,v,v),Jv\rangle=0$ for all $v\in S_p$, $p\in M^n$, and using again the symmetry of $\nabla\sigma$ we obtain that $\nabla\sigma=0$.

Corollary 1. *Let M^n be a curvature-invariant totally real submanifold in a non-positively curved Kaehler manifold $\bar{M}^n$. If M^n is constant isotropic, then M^n is totally geodesic.*

Proof. From Proposition 1, M^n has parallel second fundamental form. Then, at any point p of M^n, let $\{e_1,\ldots,e_n\}$ be the basis of T_pM^n given in Lemma 1. Then using (2.2) we get

$$0=K(e_1\wedge e_i)(\lambda_1-2\lambda_i),\qquad i=2,\ldots,n, \tag{3.11}$$

where $K(e_1\wedge e_i)$ is the sectional curvature of the two plane spanned by $\{e_1,e_i\}$.

Now, using the same argument as in the proof of the Theorem 1, we have that λ_i is either $-\lambda_1$ or $\lambda_1/2$.

If $\lambda_j=-\lambda_1$ for some $j\in\{2,\ldots,n\}$, then from (3.11) we have that $K(e_1\wedge e_j)=0$, and using Gauss equation we get $\bar{K}(e_1\wedge e_j)=2\lambda_1^2$, being $\bar{K}$ the sectional curvature of $\bar{M}^n$. As $\bar{K}$ is non-positive, we have that $\lambda_1=0$ and then M^n is totally geodesic.

On the other hand, if $\lambda_i=\lambda_1/2$ for all $i\in\{2,\ldots,n\}$, then from Lemma 1, we obtain that $f(x)=0$ for any vector x orthogonal to e_1. Then $\sigma(x,x)=(\lambda_1/2)\,Je_1$ for x orthonormal to e_1. As the submanifold is isotropic, $\lambda_1=0$ and M^n is totally geodesic.

Now, let $\bar{M}^n(c)$ be a complex space form with constant holomorphic sectional curvature c. If M^n is a totally real submanifold of $\bar{M}^n(c)$, then M^n is curvature invariant. So, from Corollary 1, we have

Corollary 2. *Every n-dimensional constant isotropic totally real submanifold of $\bar{M}^n(c)$, with $c\leqq 0$ is totally geodesic.*

Finally, using the classification of the constant isotropic totally real submanifolds with parallel second fundamental form of a complex projective space given by H. Naitoh ([1]), we obtain

Corollary 3. *Let M^n be a complete constant isotropic totally real submanifold of $CP^n(c)$. Then either M^n is totally geodesic or M^n is locally isometric to $S^1\times S^{n-1}(n\geqq 2)$; $SU(3)/SO(3)$, $n=5$; $SU(3)$, $n=8$; $SU(6)/\mathrm{Sp}(3)$, $n=14$; E_6/F_4, $n=26$.*

Except $S^1\times S^{n-1}(n\geqq 3)$, all these examples are minimal submanifolds. So, Corollary 3 proves that the result given in Theorem 1 is the best possible.

References

1. Naitoh, H.: Isotropic submanifolds with parallel second fundamental form in $P^m(c)$. Osaka J. Math. **18**, 427–464 (1981)
2. O'Neill, B.: Isotropic and Kaehler immersions. Can. J. Math. **17**, 905–915 (1965)
3. Sakamoto, K.: Planar geodesic immersions. Tôhoku Math. J. **29**, 25–56 (1977)

Received July 27, 1987; in final form January 21, 1988