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Common Knowledge: Relating anti-founded situation semantics to modal logic neighbourhood semantics

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Abstract. Two approaches for defining common knowledge coexist in the literature: the infinite iteration definition and the circular or fixed point one. In particular, an original modelization of the fixed point definition was proposed by Barwise (1989) in the context of a non-well-founded set theory and the infinite iteration approach has been technically analyzed within multi-modal epistemic logic using neighbourhood semantics by Lismont (1993). This paper exhibits a relation between these two ways of modelling common knowledge which seem at first quite different.

Key words: Common knowledge, multi-modal logic, neighbourhood semantics, non-well-founded sets, Scott models.

1. Introduction

The notion of common knowledge was probably first introduced by Lewis (1969). It is often defined by means of an infinite hierarchy of reciprocal knowledge. Say that φ is common knowledge among two agents a and b if the following infinite sequence of assertions is true:

$$\begin{array}{rcl} a \text{ knows } \varphi & \bullet & b \text{ knows } \varphi \\ a \text{ knows } b \text{ knows } \varphi & \bullet & b \text{ knows } a \text{ knows } \varphi \\ a \text{ knows } b \text{ knows } a \text{ knows } \varphi & \bullet & b \text{ knows } a \text{ knows } b \text{ knows } \varphi \\ \dots & \bullet & \dots \end{array}$$

Probably the first to have formalized this concept, Aumann (1976) proposed this iterative definition together with a circular or fixed point definition, making it possible to condense the infinity of assertions into a single one. His model is set-theoretic. Individual knowledge (or belief) relates to "propositions" understood as subsets of states of the world. This universe is partitioned by each individual: he cannot differentiate between the states of the world which are in the same element of

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his partition. Aumann's circular definition (in the case of two persons) is: "an event is common knowledge if it contains an event in the meet of the two partitions". He shows the equivalence between this admittedly not very intuitive definition and the infinite iterative one. Thus, the duality of the common knowledge – infinite iteration and fixed point – is visible right from the beginning of its formalization.

Following Hintikka (1962) modal epistemic logic was at first concerned with individual knowledge and belief, and used a propositional modal language in which the classical necessity and possibility operators were replaced with individual knowledge or belief operators (usually: K_a or B_a for a in the finite set of persons). Starting perhaps with Fagin, Halpern, Moses and Vardi, modal epistemic logic has been extended to the multi-agent framework [see for example Halpern and Moses (1984, 1985), Fagin, Halpern and Vardi (1984), Fagin and Vardi (1985)]. Here the duality between the infinite iterative approach of common knowledge and the fixed point one also exists, but at two different levels.

On the one hand, the duality just mentioned appears at a purely semantic level. Take the case of Kripke models. In this semantics models consist of a set of possible worlds and accessibility relations to describe each person's knowledge or belief. Using this framework, common knowledge is defined in terms of an infinite iteration. But this definition can immediately be restated in terms of the transitive closure of the union of all individual accessibility relations, and transitive closure can be seen as a fixed point of some sort.

On the other hand, we can *semantically* define common knowledge using the countable infinity of formulae

$$\begin{array}{rcl} K_a\varphi & \bullet & K_b\varphi \\ K_aK_b\varphi & \bullet & K_bK_a\varphi \\ K_aK_bK_a\varphi & \bullet & K_bK_aK_b\varphi \\ \dots & \bullet & \dots \end{array}$$

but in the context of a *finitary* logic, common knowledge cannot be characterized *syntactically* by the infinite conjunction of these formulae. This problem is solved by introducing a fixed point axiom schema and an induction rule. Halpern and Moses (1992) prove a determination theorem with respect to the class of Kripke models for a modal system that includes these two components. [In these models common knowledge is equivalently defined in terms of either infinite iteration or transitive closure.] The duality here reflects the semantics-syntax polarity.[1]

The problem of axiomatizing common knowledge in a more general modal framework than Kripke's, i.e., Scott's neighbourhood semantics, was tackled by Lismont (1993) and Lismont and Mongin (1993a and 1993b). The neighbourhood semantics makes it possible to model knowledge and belief under weak epistemic constraints on the individual knowledge or belief operators.[2] Here we shall use this semantics for the following reason: it is the suitable one to highlight the connection between modal epistemic logic and Barwise's (1989) semantics. Starting again with the infinite iteration definition in the neighbourhood framework, a fixed point neighbourhood system can be obtained. Remarkably, in contradistinction with the Kripkean case a *transfinite* iteration may be required – depending on the cardinality of the models – to reach the fixed point. These facts will be explained in detail in section 3.

Barwise (1989) also arrives at this transfinite iteration but in a totally different framework, starting as he does with a fixed point definition of common knowledge. The latter can be expressed using one ciruclar sentence only:

A situation in which σ is common knowledge
is a situation in which
each person knows σ
and in which
each person knows that situation in which σ is common knowledge.

Barwise has proposed to model this definition using a non-well-founded set theory known as ZFC$^-$AFA (AFA for Anti-Foundation Axiom). In this set theory the foundation axiom is replaced with an anti-foundation axiom, which was first introduced as axiom X_1 by Forti and Honsell (1983), and is better known today as axiom AFA [see for example Aczel (1988), Barwise and Etchemendy (1987) or Hinnion (1992)]. In section 2 we shall present this modelling in detail.

The aim of this paper is thus to show how to connect the ZFC$^-$AFA fixed point approach with the modal approach that uses neighbourhood semantics. This connection will be made explicit in section 4. Our main results are theorems 6 and 8, which state a semantic equivalence of the two approaches. Section 5 is devoted to some general comments.

2. The fixed point approach

We work here within ZFC$^-$AFA set theory. We shall mainly use two results from this theory (see Aczel, 1988):

Fixed Points: For each continuous class operator Φ, the class $\cup\{c \in V \mid c \subseteq \Phi(c)\}$, where V is the universe of all sets, is the largest fixed point of Φ.

Solution Lemma: The universe of all sets is extended considering a class X of atoms.[3] For all $x \in X$, let a_x be a set constructed on X. The equation system $x = a_x(x \in X)$ has a unique solution within the universe of pure sets.

To present Barwise's (1989) modelling of common knowledge, the important concepts are *infons* and *situations*. There is a finite set A of agents, a set C of objects, and two relation symbols H and K. The class *INFON* of infons is the largest class such that if $\theta \in INFON$, then θ is either a set:

$-\langle H, a, c\rangle$ where $a \in A$ and $c \in C$;

or a set:

$-\langle K, a, \sigma\rangle$ where $a \in A$ and $\sigma \subseteq$ *INFON* and σ is a set.

The existence and uniqueness of the class above is proved by showing it to be the largest fixed point of a continuous class operator. The infons of type $\langle H, a, c\rangle$ are called *basic infons*.[4] *SIT*, the class of all situations, can then be defined as follows:

$\sigma \in$ *SIT* iff σ *INFON* and σ is a set.

The notion of an infon can be interpreted as follows: $\langle H, a, c\rangle$ means that agent a has the object c, for example a playing card. Clearly, we could have taken a much larger class of basic infons. Infons $\langle K, a, s\rangle$ can be read: agent a sees, knows or believes the situation s.

The class *INFON* contains well-founded infons, i.e., infons that are well-founded sets, but also non-well-founded infons. Similarly, there are well-founded as well as non-well-founded situations. It can easily be shown that the class of well-founded infons is the smallest fixed point of the operator of which *INFON* is the largest fixed point.

We can then define *partial satisfaction relations* to be sub-classes $\models^*$ of *SIT* $\times$ *INFON* such that if $s \models^* \theta$, then

- if $\theta = \langle H, a, c\rangle$, then $\langle H, a, c\rangle \in s$;
- if $\theta = \langle K, a, s_0\rangle$, then there exists $s_1 \in$ *SIT* such that $\langle K, a, s_1\rangle \in s$ and for all $\sigma \in s_0, s_1 \models^* \sigma$.

The *satisfaction relation* $\models$ is the largest of all partial satisfaction relations. The existence of partial satisfaction relations and of the satisfaction relation is also proved using a continuous class operator. If s_1 and s_2 are situations, we shall write $s_1 \models s_2$ when $\forall\theta \in s_2, s_1 \models \theta$.

Let us consider a well-founded situation s_0. The corresponding *shared situation* is the situation

$$s_c = \{\langle K, a, s_c \cup s_0\rangle | a \in A\}.$$

This situation always exists and is unique from the Solution Lemma. It models the common knowledge of situation s_0; we shall say that in a situation s, there is common knowledge of s_0 iff $s \models s_c(s_0)$.

Barwise defines shared situations only for first order situations s_0, i.e., those containing only basic infons. We shall consider shared situations for any situation s_0, provided that it is well-founded.[5] To establish a connection with modal logic, we should introduce the notion of approximations of shared situations. Since we have a larger collection of shared situations than Barwise, we have to adapt Barwise's notion of approximations.

With regard to shared situations $s_c = \{\langle K, a, s_0 \cup s_c\rangle \mid a \in A\}$, where s_0 is well-founded, we shall denote the infon $\langle K, a, s_0 \cup s_c\rangle$ by θ_a. We define approximations of those infons θ_a by well-founded infons:

- $\theta_a^0 = \langle K, a, s_0\rangle$;
- for all ordinal $\eta > 0 : \theta_a^\eta = \langle K, a, s_0 \cup s_c^{<\eta}\rangle$, where $s_c^{<\eta} = \{\theta_a^\zeta \mid a \in A, \zeta < \eta\}$.

The only difference with Barwise's approximation concept is that he defines:

– $\theta_a^0 = \langle K, a, 1^{st}ord(s_0)\rangle$

where $1^{st}ord(s_0)$ is the set of all basic infons of s_0.

Having broadened the notion of shared situations and correspondingly modified the notion of approximations, we may restate Barwise's theorem 5 of chapter 9:

LEMMA 1. – *$s \models \theta_a$ iff for any ordinal $\eta, s \models \theta_a^\eta$.*

Proof: We shall first prove necessity by induction on the ordinals.

– $\eta = 0:\ s \models \langle K, a, s_0 \cup s_c\rangle \Rightarrow s \models \langle K, a, s_0\rangle$.

$$
\begin{aligned}
-\ \eta > 0:\ s \models \langle K, a, s_0 \cup s_c\rangle &\Rightarrow \exists\langle K, a, s_a\rangle \in s, s_a \models s_0 \cup s_c \\
&\Rightarrow \exists\langle K, a, s_a\rangle \in s, s_a \models s_0 \text{ and } \forall\zeta < \eta, \\
&\quad\ \forall b \in A, s_a \models \theta_b^\zeta \text{ (induction hypothesis)} \\
&\Rightarrow \exists\langle K, a, s_a\rangle \in s, s_a \models s_0 \cup s_c^{<\eta} \\
&\Rightarrow s \models \langle K, a, s_0 \cup s_c^{<\eta}\rangle.
\end{aligned}
$$

To prove sufficiency, we define a relation $\models^*$ adding to $\models$ all pairs $\langle s, \theta_a\rangle$ such that for all ordinals $\eta, s \models \theta_a^\eta$. We shall show that $\models^*$ is a partial satisfaction relation. To do this, we must check that for the added pairs, the relation $\models^*$ satisfies the condition

$$s \models^* \langle K, a, \tau\rangle \Rightarrow \exists\langle K, a, s'\rangle \in s \text{ such that } s' \models^* \tau.$$

Let s be a situation in which all approximations of an infon $\theta_a = \langle K, a, s_0 \cup s_c\rangle$ hold. For any ordinal η, there exists $\langle K, a, s_\eta\rangle \in s$ such that $s_\eta \models s_0 \cup s_c^{<\eta}$. We shall later show that this implies the existence of a situation s' such that $\langle K, a, s'\rangle \in s$ and $s' \models s_0 \cup s_c^{<\eta}$ for any ordinal η. Consequently, $\forall b \in A, \forall\eta, s' \models \theta_b^\eta$ and thus $\forall b \in A, s' \models^* \theta_b$. So we have $s' \models^* s_c$. Because $s' \models s_0$, we have $s' \models^* s_0$ and thus $s' \models^* s_0 \cup s_c$.

Hence, $\models^*$ is a partial satisfaction relation. Since $\models$ is the largest partial satisfaction relation, the pairs that we have added to $\models$ were already in $\models$.

To conclude, we only have to show that a situation s' exists. Let us assume that there is no such situation. We inductively show that, for all ordinals ζ, there is an ordinal $\eta(\zeta)$ such that

$$\forall\xi < \zeta, s_{\eta(\xi)} \not\models s_0 \cup s_c^{<\eta(\zeta)}.$$

Define $\eta(0) = 0$. For $\zeta > 0$, assume that we have defined $\eta(\xi)$ for all $\xi < \zeta$. From the nonexistence of s', we know that for each $\xi < \zeta$ there is an ordinal ξ^* such that $s_{\eta(\xi)} \not\models s_0 \cup s_c^{<\xi^*}$. Define next $\eta(\zeta) = \sup\{\xi^* \mid \xi < \zeta\}$. For all $\xi < \zeta, s_{\eta(\xi)} \not\models$

$s_0 \cup s_c^{<\eta(\zeta)}$ since $s_0 \cup s_c^{<\xi*} \subseteq s_c^{<\eta(\zeta)}$. By definition of $s_{\eta(\zeta)}, s_{\eta(\zeta)} \models s_0 \cup s_c^{<\eta(\zeta)}$; consequently, $\forall \xi < \eta, s_{\eta(\xi)} \neq s_{\eta(\zeta)}$. We thus have a sequence of situations $s_{\eta(\zeta)}$ such that $\xi \neq \zeta \Rightarrow s_{\eta(\xi)} \neq s_{\eta(\zeta)}$. From the Substitution Schema, the collection of $s_{\eta(\zeta)}$ is not a set. This contradicts the fact that all $s_{\eta(\zeta)}$ come from the set $\{s'' \mid \langle K, a, s'' \rangle \in s\}$. ■

3. The iterative approach within neighbourhood semantics

In this section we work with a modal language constructed on a set *PV* of propositional variables, a finite set A of agents, a modal operator K_a for each agent $a \in A$, modal operators E and C, and the usual logical connectives. The formula $K_a\varphi$ will informally be interpreted as "agent a knows or believes[6] φ", $E\varphi$ as "everybody knows or believes φ" and $C\varphi$ as "there is common knowledge or common belief that φ".

The models of the neighbourhood semantics, which will be called *Scott models*, are tuples

$$\mathcal{M} = \langle I, \mathcal{V}, N_a, N_E, N_C \rangle_{a \in A}$$

where I is a set of possible worlds, $\mathcal{V}$ a valuation function, i.e., a mapping from $I \times PV$ to $\{0, 1\}$, N_a, N_E and N_C are mappings from I to 2^{2^I} (each of which associates a set of subsets of I to each $\alpha \in I$). We call N_a, N_E and N_C *neighbourhood systems*. The idea of this semantics is that a proposition can be semantically identified with a subset of possible worlds, namely those in which that proposition is true. So if $\alpha \in I$, $N_a(\alpha)$ will contain all the propositions that agent a thinks are true in world α.

In this semantics we define the satisfaction of formula φ in world α of model $\mathcal{M}$, which we denote $\mathcal{M} \models^\alpha \varphi$, by induction on logical complexity:

- if $p \in PV$ then $\mathcal{M} \models^\alpha p \Leftrightarrow \mathcal{V}(\alpha, p) = 1$;
- $\mathcal{M} \models^\alpha \neg\varphi \Leftrightarrow \mathcal{M} \not\models^\alpha \varphi$;
- $\mathcal{M} \models^\alpha \varphi \wedge \psi \Leftrightarrow \mathcal{M} \models^\alpha \varphi$ and $\mathcal{M} \models^\alpha \psi$;
- $\mathcal{M} \models^\alpha \varphi \vee \psi \Leftrightarrow \mathcal{M} \models^\alpha \varphi$ or $\mathcal{M} \models^\alpha \psi$;
- $\mathcal{M} \models^\alpha \varphi \rightarrow \psi \Leftrightarrow$ if $\mathcal{M} \models^\alpha \varphi$ then $\mathcal{M} \models^\alpha \psi$;
- $\mathcal{M} \models^\alpha \mu\varphi \Leftrightarrow \|\varphi\| \in N_\mu(\alpha)$ where
  $\mu = K_a, E$ or C and $\|\varphi\| = \{\beta \in I | \mathcal{M} \models^\beta \varphi\}$.[7]

This is a general framework for studying knowledge and common knowledge. What we need now is to establish relations between the systems N_a representing individual knowledge and systems N_E and N_C.

This is easily done for N_E: taking N_E as $\bigcap_{a \in A} N_a$,[8] we see that

$$\mathcal{M} \models^\alpha E\varphi \Leftrightarrow \forall a \in A, \mathcal{M} \models^\alpha K_a\varphi.$$

It is not so obvious how the system N_E should be related to common knowledge. Let us consider for example the formula $K_a K_b \varphi$. We have

$$\mathcal{M} \models^\alpha K_a K_b \varphi \Leftrightarrow \{\beta \in I \mid \|\varphi\| \in N_b(\beta)\} \in N_a(\alpha).$$

Such expressions are not easy to manipulate, and since studying common knowledge requires formulae of arbitrary length, it is convenient to introduce an algebraic operation on neighbourhood systems. Let us define[9]

$$P \in N_1 \circ N_2(\alpha) \Leftrightarrow \{\beta \in I \mid P \in N_2(\beta)\} \in N_1(\alpha).$$

where N_1 and N_2 are neighbourhood systems on I. It is easy to check that

$$\mathcal{M} \models^\alpha K_a K_b \varphi \Leftrightarrow \|\varphi\| \in N_a \circ N_b(\alpha).$$

If we write $E^n \varphi$ for $\overbrace{E \ldots E}^{n\ times} \varphi$ and N_E^n for $\overbrace{N_E \circ \ldots \circ N_E}^{n\ times}$, it is also easy to check that

$$\mathcal{M} \models^\alpha E^n \varphi \Leftrightarrow \|\varphi\| \in N_E^n(\alpha).$$

This operation has various interesting properties. Let *Ind* be any set of indices. Let N and, for all $i \in Ind$, N_i be neighbourhood systems on I.

LEMMA 2. – *1. The operation* $\circ$ *is associative.*
2. The system $\mathcal{E}$ *defined by*

$$P \in \mathcal{E}(\alpha) \Leftrightarrow \alpha \in P$$

is neutral for this operation.
3. $N_1 \subseteq N_2 \Rightarrow N_1 \circ N \subseteq N_2 \circ N$.
4. $(\bigcup_{i \in Ind} N_i) \circ N = \bigcup_{i \in Ind}(N_i \circ N)$.
5. $(\bigcap_{i \in Ind} N_i) \circ N = \bigcap_{i \in Ind}(N_i \circ N)$.

Assuming systems that are closed under supersets, i.e., if $P \subseteq Q \subseteq I$ and $P \in N(\alpha)$ then $Q \in N(\alpha)$, additional properties follow:

LEMMA 3. – *If N is closed under supersets, then:*
1. $N_1 \subseteq N_2 \Rightarrow N \circ N_1 \subseteq N \circ N_2$.
2. $\bigcup_{i \in Ind}(N \circ N_i) \subseteq N \circ (\bigcup_{i \in Ind} N_i)$.
3. $N \circ (\bigcap_{i \in Ind} N_i) \subseteq \bigcap_{i \in Ind}(N \circ N_i)$.

Note also that $\mathcal{E}$ is closed under supersets and that if N_1 and N_2 are closed under superstes then $N_1 \circ N_2$ is also closed under supersets.

To have enough algebraic properties, we shall consider only systems N_a closed under supersets. Obviously the system N_E will also be closed under supersets. A consequence of this is the validity in our models of the monotonicity rule

$$\frac{\varphi \to \psi}{\mu\varphi \to \mu\varphi},$$

where μ is either K_a or E.

The following points will prove useful for the sequel:

(i) To define common knowledge of φ we shall consider the infinite hierarchy of crossed knowledges as follows:
- everybody knows φ;
- everybody knows that everybody knows φ;
- everybody knows that everybody knows that everybody knows φ;
- ...

(Note that this is not quite the same as the infinite hierarchy of crossed knowledges discussed in the introduction.)

(ii) Within our models it is not the same to say: "agent a knows or believes φ and ψ" and: "agent a knows or believes φ and agent a knows or believes φ". This is because our neighbourhood systems are not necessarily closed under intersections. Thus $E_\varphi \wedge EE\varphi$ and $E(\varphi \wedge E\varphi)$ are not equivalent. What we really need is the second formula. This leads us to modify the infinite hierarchy once again. To be able to say that φ is common knowledge in world α of model $\mathcal{M}$, we want the following infinite sequence of formulae to be true in world α:
- $E\varphi$;
- $E(\varphi \wedge E\varphi)$;
- $E(\varphi \wedge E\varphi \wedge E(\varphi \wedge E\varphi)$;
- ...

We can easily check that these formulae will be true in world α if, respectively,
- $\|\varphi\| \in N_E(\alpha)$;
- $\|\varphi\| \in N_E \circ (N_E \cap \mathcal{E})(\alpha)$;
- $\|\varphi\| \in N_E \circ ((N_E \circ (N_E \cap \mathcal{E})) \cap N_E \cap \mathcal{E})(\alpha)$;
- ...

(iii) To conclude these remarks, let us point out that if φ is common knowledge among the agents, then every agent knows that it is. Relating this to the fact that if there is common knowledge of φ, then everybody knows φ, the formulae

$$C_\varphi \to E(C\varphi \wedge \varphi)$$

should be valid in our models. These formulae are called *fixed point formulae*.

From the point (ii), define by induction on the natural numbers the sequence of neighbourhood systems that occur in the iteration:
- $N_0 = N_E$;
- $N_n = N_E \circ (\bigcap_{k<n} N_k \cap \mathcal{E})$.

We could then define

$$N_C = \bigcap_{n<\omega} N_n. \tag{1}$$

The problem with this definition is that the fixed point formula is not valid in the class of models where N_C would be so defined:

PROPOSITION 4. – *There exists a Scott model $\mathcal{M}$ where N_C is defined as in (1) and a formula φ such that $\mathcal{M} \not\models C\varphi \rightarrow E(C\varphi \wedge \varphi)$.*

Proof. Define $\mathcal{M}$ as follows. Let I be the real interval $[-2, 2]$. For each real r, define $\bar{r}$ to be the smallest integer strictly greater than r and $\underline{r}$ to be the largest integer strictly less than r. For each $a \in A$, define then N_a by saying that, for each $P \subseteq I$ and $\alpha \in I$,

$$P \in N_a(\alpha) \Leftrightarrow \frac{\bar{r} + r}{2} \leq \alpha \leq \frac{\underline{s} + s}{2},$$

where $r = \inf(P)$ and $s = \sup(P)$. The fact that N_a is closed under supersets is trivial. Define next N_C as in (1). Let $\top$ be an abbreviation for $p \vee \neg p$ where $p \in PV$. We have that $\mathcal{M} \not\models C\top \rightarrow E(C\top \wedge \top)$:

– We show by induction that, for $n < \omega$,

$$I \in N_n(\alpha) \Leftrightarrow \alpha \in \left[-1 - \frac{1}{2^{n+1}}, 1 + \frac{1}{2^{n+1}}\right]:$$

$$\begin{aligned}
I \in N_E(\alpha) &\Leftrightarrow \frac{\overline{(-2)} + (-2)}{2} \leq \alpha \leq \frac{\underline{2} + 2}{2} \\
&\Leftrightarrow -\frac{3}{2} \leq \alpha \leq \frac{3}{2}. \\
I \in N_n(\alpha) &\Leftrightarrow I \in N_E \circ (\bigcap_{k<n} N_k \cap \mathcal{E})(\alpha) \\
&\Leftrightarrow \{\beta \in I | I \in \bigcap_{k<n} N_k \cap \mathcal{E}(\beta)\} \in N_E(\alpha) \\
&\Leftrightarrow [-1 - \frac{1}{2^n}, 1 + \frac{1}{2^n}] \in N_E(\alpha) \\
&\Leftrightarrow \frac{\overline{(-1 - \frac{1}{2^n})} + (-1 - \frac{1}{2^n})}{2} \leq \alpha \leq \frac{\underline{(1 + \frac{1}{2^n})} + (1 + \frac{1}{2^n})}{2} \\
&\Leftrightarrow -1 - \frac{1}{2^{n+1}} \leq \alpha \leq 1 + \frac{1}{2^{n+1}}.
\end{aligned}$$

– Thus,

$$\begin{aligned}
I \in N_C(\alpha) &\Leftrightarrow \forall n < \omega, I \in N_n(\alpha) \\
&\Leftrightarrow \forall n < \omega, \alpha \in [-1 - \frac{1}{2^{n+1}}, 1 + \frac{1}{2^{n+1}}] \\
&\Leftrightarrow \alpha \in \bigcap_{n<\omega} [-1 - \frac{1}{2^{n+1}}, 1 + \frac{1}{2^{n+2}}] \\
&\Leftrightarrow \alpha \in [-1, 1].
\end{aligned}$$

– It is easy to see that [-1,1] $\in N_E(\alpha) \Leftrightarrow \alpha \in [-\frac{1}{2}, \frac{1}{2}]$.

– We are now able to see that $\mathcal{M} \not\models^1 C\top \rightarrow E(C\top \wedge \top)$:

· $\mathcal{M} \models^1 C\top$: $\|\top\| = I$ and $I \in N_C(1)$.
· $\mathcal{M} \not\models^1 E(C\top \wedge \top)$: $\|C\top \wedge \top\| = \|C\top\| \cap \|\top\| = [-1,1]$ and $[-1,1] \notin N_E(1)$. ■

The idea is then to proceed with a transfinite iteration and to broaden the construction of N_C by induction on the ordinals. Let η be any ordinal:

– $N_0 = N_E$;
– $\eta > 0$: $N_\eta = N_E \circ (\bigcap_{\zeta<\eta} N_\zeta \cap \varepsilon)$.

LEMMA 5. – *Let η and ξ be two ordinals. If $\xi < \eta$, then $N_\eta \subseteq N_\xi$.*

This lemma ensures that for each $\alpha \in I$, the sequence $N_\eta(\alpha)$ is a decreasing sequence of sets. Thus, there exists a smallest ordinal $\min_\alpha$ such that for all ordinals $\eta \geq \min_\alpha, N_\eta(\alpha) = N_{\min_\alpha}(\alpha)$. Taking $\min = \sup\{\min_\alpha \mid \alpha \in I\}$, we get $N_\eta = N_{\min}$ for all $\eta \geq \min$.

Next define $\mathcal{S}$ to be the class of Scott models

$$\mathcal{M} = \langle I, \mathcal{V}, N_a, N_E, N_C\rangle_{a\in A}$$

where the systems N_a are closed under supersets, $N_E = \bigcap_{a\in A} N_a$ and $N_C = N_{\min}$. This class of models satisfies all of our criteria concerning the iterative approach to common knowledge. The only condition that we imposed in addition is the closure under supersets of the systems N_a. As a consequence we get the same closure of N_E and N_C.

Lismont (1993) proves that the theory of the above models, i.e., the set of all valid formulae in that class $\mathcal{S}$, is axiomatized by a modal logic system called MC. This system consists of the axiom schemata and inference rules of propositional logic, and the following axiom schemata and rules:

$$\text{Definition of } E: \; E\varphi \leftrightarrow \bigwedge_{a\in A} K_a\varphi.$$

$$\text{Fixed Point Axiom}: \; C\varphi \rightarrow E(C\varphi \wedge \varphi).$$

$$\text{Induction Rule}: \; \frac{\varphi \rightarrow E\varphi}{E\varphi \rightarrow C\varphi}.$$

$$\text{Monotonicity Rules}: \; \frac{\varphi \rightarrow \psi}{\mu\varphi \rightarrow \mu\varphi}(\mu \neq E).$$

4. Relating the two approaches to each other

We start with a set S of situations that have closure properties and a modal language constructed on a set PV associated with all basic infons in the transitive closure of S. Using these elements, we shall construct a Scott model that is related to S and a mapping from well-founded infons and situations occurring in S to the set of modal formulae. This mapping will enable us to compare the satisfaction relations defined in each semantics respectively, hence the two modellings of common knowledge.

To secure the coherence of further definitions we should make sure that the sets of situations that we are considering have sufficient "stability". We want that for any such set S, if $s \in S$ and $\langle K, a, s' \rangle \in s$, then $s' \in S$ – obviously this is not always the case. Furthermore, if some well-founded situation s_0 is in S, then the corresponding shared situation $s_c(s_0)$ should also be in S. Let us begin by showing that sets with the desirable properties exist.

Let S_0 be a set of well-founded situations. We shall construct a set S that contains S_0 and the set of shared situations associated with the situations occurring in S_0, in such a way that S will satisfy the required stability property. We define

$$C(S_0) = \{s \in SIT \mid \exists a \in A \text{ and } s_0 \in S_0 \text{ such that } \langle K, a, s \rangle \in s_0\},$$

and

$$H(S_0) = \bigcup_{n<\omega} C^n(S_0).$$

We then define

$$S = H(S_0) \cup \{s_c(s) \mid s \in H(S_0)\} \cup \{s \cup s_c(s) \mid s \in H(S_0)\}.$$

It can easily be shown that if $s \in S$ and $\langle K, a, s' \rangle \in s$, then $s' \in S$. There are three possible cases:

- $s \in H(S_0)$: $\exists n < \omega, s \in C^n(S_0)$; if $\langle K, a, s' \rangle \in s$, then $s \in C^{n+1}(S_0)$ and $s' \in H(S_0)$.
- $s = s_c(s_0)$ with $s_0 \in H(S_0)$: if $\langle K, a, s' \rangle \in s$, then s' is $s_0 \cup s_c(s_0)$, which is in S.
- $s = s_0 \cup s_c(s_0)$ with $s_0 \in H(S_0)$: if $\langle K, a, s' \rangle \in s$, then $\langle K, a, s' \rangle \in s_0$ or $\langle K, a, s' \rangle \in s_c(s_0)$; the conclusion then follows from the two first points.

From now on we shall work with such a stable set S containing at least all shared situations constructed from any well-founded situation of S.

Before introducing further notions, we have to define the *rank* of well-founded infons or situations.

For basic infons $\theta = \langle H, a, c \rangle$:

$$\text{rank}(\theta) = 0.$$

For a well-founded infon $\theta = \langle K, a, s \rangle$:

$$\text{rank}(\theta) = \sup\{\text{rank}(\sigma) + 1 \mid \sigma \in s\}.$$

In the same way, if s is a well-founded situation:

$$\text{rank}(s) = \sup\{\text{rang}(\sigma) + 1 \mid \sigma \in s\}.$$

Let us consider a modal language based upon a set PV of propositional variables that is large enough to have one variable associated with any basic infon occurring in S. Let $p_{\langle H,a,c\rangle}$ stand for the variable associated with basic infon $\langle H, a, c\rangle$. By induction on the rank of well-founded infons, we can then define a mapping φ from the set of well-founded infons occurring in S to the set of formulae of our modal language.

- $\varphi(\langle H, a, c\rangle) = p_{\langle H,a,c\rangle}$;
- $\varphi(\langle K, a, s\rangle) = K_a \bigwedge_{\sigma \in s} \varphi(\sigma)$.

This definition can be extended straightforwardly to well-founded situations: $\varphi(s) = \bigwedge_{\sigma \in s} \varphi(\sigma)$. This enables us to write

- $\varphi(\langle K, a, s\rangle) = K_a \varphi(s)$.

The finiteness of situations is not required in the definition of this mapping. Infinite formulae would be obtained in the case of infinite situations. This is not a problem since we are only considering a semantic context.

Let us now construct a Scott model $\mathcal{M}$. We take I, the set of possible worlds, equal to S.[10]

We next define the valuation and neighbourhood systems. We are looking for the equivalence between satisfaction of an infon in terms of situation semantics and satisfaction of the associated formula in the corresponding Scott model. The valuation is then defined by

$$\mathcal{V}(s, p_{\langle H,a,c\rangle}) = 1 \Leftrightarrow \langle H, a, c\rangle \in s.$$

The neighbourhood systems N_a are defined as follows:

$$N_a(s) = \{P \in \mathcal{P}(I) \mid \exists s' \in P \text{ such that } \langle K, a, s'\rangle \in s\}.$$

These definitions immediately yield the followign result:

THEOREM 6. – *Let σ be a well-founded infon and s_0 a well-founded situation.*
1. $\mathcal{M} \models^s \varphi(\sigma) \Leftrightarrow s \models \sigma$.
2. $\mathcal{M} \models^s \varphi(s_0) \Leftrightarrow s \models s_0$.

Proof:

1. By induction on the rank of a well-founded infon. We assume that the property is true for all well-founded infons the rank of which is strictly less than the rank of σ.
   - If $\sigma = \langle H, a, c\rangle$, the property holds trivially from the definition of $\mathcal{V}$.
   - $\sigma = \langle K, a, s_0\rangle$. We know that $\varphi(\sigma) = K_a \varphi(s_0)$.

$$\begin{aligned} \mathcal{M} \models^s K_a \varphi(s) &\Leftrightarrow \|\varphi(s_0)\| \in N_a(s) \\ &\Leftrightarrow \exists s_1 \in \|\varphi(s_0)\| \text{ such that } \langle K, a, s_1\rangle \in s \end{aligned}$$

$$\Leftrightarrow \exists s_1 \in \|\bigwedge_{\tau \in s_0} \varphi(\tau)\| \text{ such that } \langle K, a, s_1\rangle \in s$$
$$\Leftrightarrow \exists \langle K, a, s_1\rangle \in s \text{ such that } \forall \tau \in s_0, \mathcal{M} \models^{s_1} \varphi(\tau)$$
$$\Leftrightarrow \exists \langle K, a, s_1\rangle \text{ such that } \forall \tau \in s_0, s_1 \models \tau$$
$$\text{(induction hypothesis)}$$
$$\Leftrightarrow s \models \langle K, a, s_0\rangle.$$

The proof uses that the rank of all infons of s_0 is strictly less than the rank of s_0, which is the same as that of σ. It also uses the fact that, in situation semantics, $\models$ was defined as a fixed point and that the implications in the definition of $\models$ can thus be replaced with equivalences.

2. $s \models s_0 \Leftrightarrow \forall \sigma \in s_0, s \models \sigma \Leftrightarrow \forall \sigma \in s_0, \mathcal{M} \models^s \varphi(\sigma) \Leftrightarrow \mathcal{M} \models^s \bigwedge_{\sigma \in s_0} \varphi(\sigma) \Leftrightarrow \mathcal{M} \models^s \varphi(s_0)$. ■

Notice that the neighbourhood systems that we have defined are closed under supersets: let $P \in N_a(s)$ and $P \subseteq Q$; if there exists $s_1 \in P$ such that $\langle K, a, s_1\rangle \in s$, then $s_1 \in Q$ as well. Using the previous theorem we see that the monotonicity rule is implicitly valid in Barwise's situation semantics. In the sequel, $\sigma \overset{S}{\Rightarrow} \sigma'$ corresponds to Barwise's *entailment*, as restricted to the set S:

$$\sigma \overset{S}{\Rightarrow} \sigma' \text{ iff } \forall s \in S, s \models \sigma \Rightarrow s \models \sigma'.$$

The following theorem confirms that monotonicity underlies Barwise's anti-founded situation semantics:

THEOREM 7. – *If* $\sigma \overset{S}{\Rightarrow} \tau$, *then* $\langle K, a, \{a\}\rangle \overset{S}{\Rightarrow} \langle K, a, \{\tau\}\rangle$.

Proof. Let $s \in S$ such that $s \models \langle K, a, \{\sigma\}\rangle$. We know that there exists $\langle K, a, s_1\rangle \in s$ such that $s_1 \models \sigma$. As $s_1 \in S$, we have $s_1 \models \tau$. Consequently, $s \models \langle K, a, \{\tau\}\rangle$. ■

What about common knowledge?

THEOREM 8. – *Let s_0 be a well-founded situation and* $s_c = \{\langle K, a, s_0 \cup s_c\rangle | a \in A\}$. *Then*

$$s \models s_c \Leftrightarrow \mathcal{M} \models^s C\varphi(s_0).$$

Proof. We inductively show that, for any ordinal η,

$$(\forall a \in A, s \models \theta_a^\eta) \Leftrightarrow \|\varphi(s_0)\| \in N_\eta(s),$$

where (as before) $\theta_a = \langle K, a, s_0 \cup s_c\rangle$.

- $\eta = 0$: $(\forall a \in A, s \models \langle K, a, s_0\rangle) \Leftrightarrow \|\varphi(s_0)\| \in N_E(s) = N_0(s)$ (theorem 6).
- $\eta > 0 : \forall a \in A, s \models \theta_a^\eta$

$$\Leftrightarrow \forall a \in A, s \models \langle K, a, s_0 \cup s_c^{<\eta}\rangle$$
$$\Leftrightarrow \forall a \in A, \exists\langle K, a, s_a\rangle \in s \text{ such that } s_a \models s_0 \cup s_c^{<\eta}$$
$$\Leftrightarrow \forall a \in A, \exists\langle K, a, s_a\rangle \in s \text{ such that } s_a \models s_0 \text{ and } \forall b \in A, \forall\zeta < \eta, s_a \models \theta_b^\zeta$$
$$\Leftrightarrow \forall a \in A, \exists\langle K, a, s_a\rangle \in s \text{ such that } s_a \in \|\varphi(s_0)\| \text{ and } \|\varphi(s_0)\| \in \bigcap_{\zeta<\eta} N_\zeta(s_a)$$

(theorem 6 and induction hypothesis)

$$\Leftrightarrow \forall a \in A, \exists\langle K, a, s_a\rangle \in s \text{ such that } \|\varphi(s_0)\| \in \bigcap_{\zeta<\eta} N_\zeta \bigcap \varepsilon(s_a)$$

(definition of $\mathcal{E}$)

$$\Leftrightarrow \forall a \in A, \{s' \in I \mid \|\varphi(s_0)\| \in \bigcap_{\zeta<\eta} N_\zeta \cap \mathcal{E}(s')\} \in N_a(s)$$
$$\Leftrightarrow \{s' \in I \mid \|\varphi(s_0)\| \in \bigcap_{\zeta<\eta} \cap\mathcal{E}(s')\} \in N_E(s)$$
$$\Leftrightarrow \|\varphi(s_0)\| \in N_E \circ (\bigcap_{\zeta<\eta} N_\zeta \cap \mathcal{E})(s) = N_\eta(s).$$

The conclusion then immediately follows from lemma 1. ■

5. Final comments

To summarize, theorems 6 and 8 tell us that the link between the two semantics preserves their validation relation – most importantly, their knowledge clauses. Theorem 6 demonstrates the similarity between the two epistemic structures. We highlighted the role of monotonicity in the situation semantics (theorem 7), whereas it was explicit from the beginning in the neighbourhood semantics of this paper. Theorem 8 shows that the inductive definition of the neighbourhood system N_C exactly corresponds to the notion of common knowledge in anti-founded situation semantics.

We shall here analyze the correspondence between the two semantics in some more detail. We will look first at the languages, second at the semantics themselves.

The two languages differ in their modelling of the "objective level": the objective information units in situations are given by basic infons, which are structured objects. The corresponding units in propositional modal languages – i.e., propositional variables – are left unanalyzed. This sets us thinking that the situational language would be more adapted to deal with problems such as the following one. Suppose that we impose the requirement that whenever somebody has a playing card, he knows that he has it. Such a property can be expressed by a *single* schema in the situational language but not in the modal language. In the former, one solves the problem by requiring the validity of the formulae $\langle H, a, c\rangle \Rightarrow \langle K, a, \{\langle H, a, c\rangle\}\rangle$. The latter language calls for a slightly more complicated analysis: if p is the variable corresponding to "agent a has the card c", then for any $a \in A$ and any $\alpha \in I$, whenever $\mathcal{V}(\alpha, p) = 1$, it should be the case that $\{\beta \in I | \mathcal{V}(\beta, p) = 1\} \in N_a(\alpha)$. This exemplifies the general difficulty that some semantic requirements cannot be adequately expressed in the given formal language: they can only be rendered as properties of a relevant class of models.[11]

Let us give another example that has to see with contradictory objective pieces of information. Within classical modal languages it is impossible to validate p and $\neg p$ at the same time. But this clearly does not exclude contradictions that are external to the language itself. For example suppose that the playing card we are considering has only one ace of spades. Any situation having two infons indicating that two different players have an ace of spades would be contradictory and thus eliminated by the modeller. Similarly, within the modal language he would forbid any valuation that gives the value 1 to the corresponding propositional variables p and q. This is once again an example of the fact that the modeller can overcome the limits of the formal languages. Notice, however, that it clearly is the modeller's role to establish what is such an external contradiction.

Another problematic situation would be to express the requirement that whenever somebody knows that a person has the ace of spades, then he knows that this person has an ace. This is the well-known problem in computer science of how to deal with "general" facts when some "particular" facts are available, a problem which is reminiscent of the current discussion on non-monotonic logics. Given the languages used here, the only answer seems to go as follows. Suppose that we are working within the modal framework. Let the propositional variables p and q stand for "a has the ace of spades" and "a has an ace". We want that the semantics describe the reality; thus we have to restrict the valuations so that whenever p is true, q also is. That is to say, the implication $p \rightarrow q$ is valid in our class of models. By the closure of the neighbourhood systems under supersets, for any $b \in A$, $K_b p \rightarrow K_b q$ will also be valid. To formalize the above requirement in the situational language, we should be able to express facts such as "a has an ace", and therefore enrich the class of basic infons. From there the analysis would parallel that made in the modal case.

Now, let us consider the semantics themselves. A difference is that the semantics-syntax distinction is explicit in modal logic but not in the situation semantics. We chose here to take infons as syntactical items and situations as semantic items. This seems a natural choice given the validation relation defined between situations and infons. In this connection, the anti-founded semantics turns out to be related to the "knowledge structures" introduced by Fagin, Halpern and Vardi (1984) and the "belief worlds" introduced by Vardi (1986). Both the knowledge structures and the belief worlds are constructed from subsets of propositional variables – in effect from syntactical items.

There is one more way in which Barwise's anti-founded situations can be connected with alternative epistemic structures. Barwise's construction is close to the hierarchical structures just mentioned as well as to the hierarchical construction of the "universal probabilistic belief spaces" (Mertens and Zamir, 1985). In all of these structures, worlds are constructed inductively, adding at each level epistemic information on the preceding level. To be specific, define situations of level zero (or equivalently of rank zero) as all possible sets containing only basic infons. Then, the situations of rank $\eta > 0$ are all possible unions of situations of rank zero with any set containing infons of the kind $\langle K, a, s \rangle$, where s is a situation of rank less than η. This construction delivers all well-founded Barwise situations. The main difference here[12] is that Barwise's construction is *co-inductive* rather than inductive. In the same spirit Lismont (1992) offered a co-inductive construction of belief spaces *à la* Mertens and Zamir. The present construction can be seen to be the co-inductive analogue of Vardi's (1986) already-mentioned structures,[13] once the following two conditions are met: one has to (i) make the same correspondence as we made here between situations and neighbourhood systems, and (ii) add a monotonicity requirement to Vardi's construction.

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Notes

[1] Notice also that there is a relation between Aumann's approach and these Kripke models. The partitional models can be seen as particular case of Kripke models where the individual accessibility relations are equivalence relations [see for example Bacharach (1993), Binmore (1991) or Lismont and Mongin (1983b)].

[2] More should be said on the properties of neighbourhood semantics. On their connection with Kripke semantics, hence indirectly with Aumann's partitional model, the reader is referred to Chellas (1980, ch. 7–9). As an example of their flexibility in epistemic applications Mongin (1993) offers a neighbourhood semantics interpretation of the sets having probability 1 or belief function (in Shafer's sense) 1.

[3] This extension is useful only to simplify the lemma. As shown by Aczel, the Solution Lemma can in fact be proved within the universe of pure sets.

[4] Barwise calls these infons *facts*.

[5] This condition seems required to relate shared situations to Scott semantics within classical modal logic.

[6] We make no formal difference here between knowing or believing something. Many authors seem to consider that knowing something is believing something that is true. In our opinion, this is a philosophically weak consideration: someone can believe something true but for wrong reasons and then not know it. Formally, these authors consider a modal operator K_a to be a knowledge

operator if all formulae $K_a\varphi \rightarrow \varphi$ are valid. In the semantics proposed here such formulae will not be automatically valid. One can always add a semantic condition to the models and make the formulae valid.

[7] We should write $||\varphi||^{\mathcal{M}}$ since this set is related to a particular model. For the sake of readability, however, we will not do so.

[8] Formally this means that for each $\alpha \in I$, $N_E(\alpha) = \bigcap_{a \in A} N_a(\alpha)$.

[9] This operation was first introduced by Th. Lucas and R. Lavendhomme, Université Catholique de Louvain (unpublished work).

[10] One could object that this set is not well-founded. We could then take as the set of possible worlds the cardinal of S, which is in the well-founded universe. This choice is not a problem since we are assuming the axiom of choice.

[11] Notice that the requirement that if an agent has a card, he knows that, has not a universal epistemic character unlike the requirement that if an agent knows something, then he knows that he knows it. The latter can be easily expressed by the modal schema $K_a\varphi \rightarrow K_aK_a\varphi$.

[12] Another minor difference is that the above mentioned inductive constructions of knowledge and belief structures involve an induction on the natural numbers only. It can be seen (but will not be proved here) that these constructions could have been extended to the transfinite unproblematically.

[13] It is also possible to rely directly Mertens and Zamir's construction to Vardi's one: see Mongin (forthcoming).

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