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The First Eigenvalue of the Laplacian of an Isoparametric Minimal Hypersurface in a Unit Sphere

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1. Introduction

Let M^{n-1} be an $(n-1)$-dimensional compact connected Riemannian manifold without boundary and Δ its Laplacian acting on smooth functions on M. Δ has a discrete spectrum $\{0=\lambda_0(M)<\lambda_1(M)\leqq\lambda_2(M)\leqq\ldots\leqq\lambda_k(M),\ldots,\uparrow\infty\}$. Let f be a minimal immersion of M^{n-1} into the N-dimensional unit sphere $S^N(1)$ of the Euclidean space with the canonical coordinate system $(x^1, x^2, \ldots, x^{N+1})$. Then the first eigenvalue $\lambda_1(M)$ is not greater than the dimension $n-1$ of M by Takahashi's theorem (see [20]) that f is minimal in $S^N(1)$ if and only if $N+1$ functions $x^i\circ f$ satisfy $\Delta(x^i\circ f)=-(n-1)(x^i\circ f)$.

In this connection, Ogiue [15] (see also Yau [21]) posed the following problem:

Problem (A). What kind of embedded closed minimal hypersurfaces of $S^n(1)$ have $(n-1)$ as their first eigenvalue?

Hsiang [7] constructed infinitely many embedded minimal hypersurfaces in $S^n(1)$ which are diffeomorphic to $S^{n-1}(1)$. Choi and Wang [5] showed that the first eigenvalue of every embedded closed minimal hypersurface in $S^n(1)$ is not smaller than $(n-1)/2$.

In this paper, we consider a restricted problem (Problem (B)) of Problem (A) for closed minimal isoparametric hypersurfaces M^{n-1} in $S^n(1)$, that is, M^{n-1} has constant principal curvatures in $S^n(1)$.

Problem (B). Is is true that the first eigenvalue of every closed minimal isoparametric hypersuface in $S^n(1)$ is just $(n-1)$?

For an isoparametric hypersuface M^{n-1} in $S^n(1)$, Münzner [10], [11] obtained beautiful results (see section 2). Let ν be the unit normal vector field along M in $S^n(1)$. Let g be the number of distinct principal curvatures and $\cot\theta_\alpha(\alpha=0,\ldots,g-1: 0<\theta_0<\theta_1<\ldots<\theta_{g-1}<\pi)$ the principal curvatures with respect to ν. Let m_α be the multiplicity of $\cot\theta_\alpha(\alpha=0,\ldots,g-1)$. Münzner showed, for example, that (1) $m_\alpha=m_{\alpha+2}$(indices mod g)(2) $\theta_\alpha=\theta_0+\frac{\alpha\pi}{g}(\alpha=0,\ldots,g-1)$

(3) $g \in \{1, 2, 3, 4, 6\}$ and gave an extrinsically global structure of M^{n-1} in $S^n(1)$. In this paper, "a homogeneous submanifold M in $S^n(1)$" means an orbit of some closed subgroup of $O(n+1)$. Then every homogeneous hypersurface in $S^n(1)$ is isoparametric. For Problem (B), Muto, Ohnita and Urakawa [14] and Kotani [9] answered affirmatively when M is a homogeneous minimal hypersurface ($g=1, 2, 3, 6$) in $S^n(1)$ using the representation theory of groups and the classification of homogeneous hypersurfaces in $S^n(1)$ (see [8], [19]).

It is known (see [2]) that if $g \leqq 3$, then M is homogeneous and, in particular, that when $g=1$, $M=S^n(1)$ and when $g=2$, M is a generalized Clifford torus $S^p(\sqrt{p/(n-1)}) \times S^q(\sqrt{q/(n-1)})(p+q=n-1)$. But for $g=4$, Ozeki and Takeuchi [17] found two families of nonhomogeneous minimal isoparametric hypersurfaces ($(m_0, m_1)=(3, 4k), (7, 8k)(k \geqq 1)$). And Ferus, Karcher and Münzner [6] found infinite families of such hypersurfaces.

Our purpose is to show that Problem (B) is true for some families of isoparametric, nonhomogeneous hypersurfaces with $g=4$. We give a relation between $\lambda_k(M^{n-1})$ and $\lambda_k(S^n)$ (Theorem A) and obtain our main result (Theorem B) as a corollary of Theorem A.

Theorem A. *Let M be a closed isoparametric hypersurface satisfying $g \geqq 2$ and $\min(m_0, m_1) \geqq 2$. Then for any $k=0, 1, 2, \ldots$,*

$$\lambda_k(M^{n-1}) \geqq G(g: m_0, m_1, \theta_0) \quad \lambda_k(S^n(1)).$$

Here $l=g/2$ and

$$G(g: m_0, m_1, \theta_0)=\frac{G_1(m_0, m_1)}{G_2(g: m_0, m_1, \theta_0)+G_3(g: m_0, m_1, \theta_0)},$$

$$G_1(m_0, m_1)=\int_0^{\frac{\pi}{2}} \sin^{m_0} x \cos^{m_1} x \, dx,$$

$$=\frac{1}{2} B\left(\frac{m_0+1}{2}, \frac{m_1+1}{2}\right),$$

$$G_2(g: m_0, m_1, \theta_0)=\sin^2 \theta_0 \int_0^{l\theta_0} \frac{\sin^{m_0} x}{\sin^2 \frac{x}{l}} \cos^{m_1} x \, dx,$$

$$G_3(g: m_0, m_1, \theta_0)=\sin^2\left(\frac{\pi}{2l}-\theta_0\right) \int_0^{\frac{\pi}{2}-l\theta_0} \frac{\sin^{m_1} x}{\sin^2 \frac{x}{l}} \cos^{m_0} x \, dx.$$

Moreover, when M is minimal, then $l\theta_0=\operatorname{arc}\cot \sqrt{m_0/m_1}$.

Condition (C). A closed minimal isoparametric hypersurface M in $S^n(1)$ satisfies one of the following conditions for some integer $k \geqq 1$:

(1) $g=1$

(2) $g=2$

(3) $g=3$:

$$(m_0, m_1)=(4,4),(8,8).$$

(4) $g=4$:

$$\begin{aligned}(m_0, m_1)=&(3,4),(3,8),\ldots,(3,4k),\ldots\\ &(4,5)\\ &(4,7),(4,11),\ldots,(4,4k+3),\ldots\\ &(5,9),(5,10),(5,18),(5,26),(5,34)\\ &(6,9),(6,17),(6,25),(6,33)\\ &(7,8),(7,16),\ldots,(7,8k),\ldots\\ &(8,15),(8,23),(8,31),(8,39)\\ &(9,22),(9,38)\\ &(10,21),(10,53)\end{aligned}$$

Theorem B. *Let M be a closed isoparametric minimal hypersurface in $S^n(1)$ satisfying Condition (C). Then*

$$\lambda_1(M^{n-1})=n-1.$$

By the classification of homogeneous hypersurfaces (see [8], [19]), the pairs (m_0, m_1) of minimal homogeneous hypersurfaces are $(1,1)$, $(2,2)$, $(4,4)$, $(8,8)$ when $g=3$, $(1,k)$, $(2,2k-1)$, $(4,4k-1)$, $(2,2)$, $(4,5)$, $(6,9)$ when $g=4$, $(1,1)$, $(2,2)$ when $g=6$. Therefore hypersurfaces satisfying Condition (C) in some families are not homogeneous.

We summarize our results. We review Münzner's results [10], [11] in section two. He showed that every closed isoparametric hypersurface M in a unit sphere $S^n(1)$ has two smooth closed focal embedded submanifolds M_+ and M_- whose codimensions in the sphere are greater than one and M is a normal sphere bundle over M_+ and M_-. Hence M is embedded. In section three, we compare the volume elements of M^{n-1} and $S^n(1)$ and, as a corollary of this estimate, we give an estimate of Cheeger's isoperimetric constant $C(M)$ of M from below, where

$$C(M)=\inf\left\{\frac{\operatorname{vol}(\partial D)}{\operatorname{vol}(D)}\,\middle|\, D \text{ is a domain of } M \text{ with the smoth boundary } \partial D, \operatorname{vol}(D)\leqq\tfrac{1}{2}\operatorname{vol}(M)\right\}.$$

For compact manifold D with boundary ∂D, $C(D)$ is defined as follows.

$$C(D)=\inf\left\{\frac{\operatorname{vol}(\partial D')}{\operatorname{vol}(D')}\,\middle|\, D' \text{ is a domain of } D \text{ with the smooth boundary } \partial D', \partial D'\cap\partial D=\phi\right\}.$$

In section four, we extend the k-th eigenfunction on M to a suitable function satisfying the Dirichlet boundary condition on a domain in $S^n(1)$ obtained by excluding ε-neighborhoods of M_+ and M_- from $S^n(1)$. Using these extended functions, we give a relation between $\lambda_k(M^{n-1})$ and the $(k+1)$-th eigenvalue $\lambda_{k+1}(\varepsilon)$ of the Laplacian on the domain under the Dirichlet boundary condition $(k=0, 1, 2, \ldots)$. By the results of Chavel and Feldman [4] and Ozawa [16], $\lim_{\varepsilon\to 0} \lambda_{k+1}(\varepsilon)=\lambda_k(S^n(1))$. This is the outline of the proof of Theorem A. In section five, for any isoparametric hypersurface M in $S^n(1)$ satisfying the condition (C), we show that $\lambda_{n+2}(M^{n-1})>n-1=\dim M$ using Theorem A. Since every minimal hypersurface fully immersed in $S^n(1)$ which is not isometric to $S^{(n-1)}(1)$ has its dimension as eigenvalues with multiplicity $\geqq(n+1)$, the first eigenvalue of M^{n-1} must be its dimension $(n-1)$ with multiplicity $(n+1)$.

2. An Isoparametric Hypersurface in a Unit Sphere

In this section, since Müzner's extrinsically global structure theorem for an isoparametric hypersurface in $S^n(1)$ is one of our main tools, we recall his results [10, 11].

Let $f\colon M^{n-1}\to S^n(1)(\subset \mathbf{R}^{n+1})$ be an isoparametric hypersurface of $S^n(1)$ and $E^\alpha(\alpha=0, \ldots, g-1)$ be the eigenspace of the shape operator with eigenvalues $\cot\theta_\alpha(0<\theta_0<\ldots<\theta_{g-1}<\pi)$. We define $f_\theta\colon M\to S^n(1)(-\pi<\theta<\pi)$ by the following: for any $p\in M$,

$$f_\theta(p)=\exp_{f(p)}\theta v, \quad //\cos\theta f(p)+\sin\theta v.$$

Here $x//y$ means that x and y are parallel as vectors in $\mathbf{R}^{n+1}$.

Theorem 1. (*Münzner* [10], [11]) *Let M be a closed isoparametric hypersurface in $S^n(1)$. Then* (1)–(5) *hold*:

(1) $\theta_\alpha=\theta_0+\dfrac{\alpha\pi}{g}(\alpha=0, 1, \ldots, g-1)$,

(2) $m_\alpha=m_{\alpha+2}$ (*indices* mod g),

(3) $g\in\{1, 2, 3, 4, 6\}$,

(4) *Set* $M_+=f_{\theta_0}(M)$(resp. $M_-=f_{-\pi+\theta_{g-1}}(M)=f_{-\frac{\pi}{g}+\theta_0}(M)$). *Then M_+ (resp. M_-) is a smooth embedded closed submanifold of dimension $(n-m_0-1)$(resp. $(n-m_1-1)$). Moreover, set $\tilde M_+=\bigcup_{\theta\in[0,\theta_0]} f_\theta(M)$ (resp. $\tilde M_-=\bigcup_{\theta\in[\theta_0-\frac{\pi}{g},0]} f_\theta(M)$). Then $\tilde M_+$ (resp. $\tilde M_-$) is a normal disk bundle over $\tilde M_+$ (resp. $\tilde M_-$) induced by f_θ and $\tilde M_+$ and $\tilde M_-$ satisfy $\tilde M_+\cup\tilde M_-=S^n(1)$ and $\tilde M_+\cap\tilde M_-=M$.*

(5) $f_\theta(M)\left(\theta\in\left(-\dfrac{\pi}{g}+\theta_0,\ \theta_0\right)\right)$ *is an isoparametric hypersurface which is diffeomorphic to M.*

(6) $2g=\dim_R H^*(M\colon R)$, *where $R=Z$ in a case that M_+ and M_- are orientable and $R=Z_2$ in the other case.*

We prepare some formula (see Münzner [10]).

(2.1) For $X \in E^{\alpha}$,

$$f_{\theta_*} X = \frac{\sin(\theta_\alpha - \theta)}{\sin(\theta_\alpha)} \tilde{X}.$$

Here $\tilde{X} // X$.

(2.2) Set $t(p) = \mathrm{dist}(p, M_+)$ for any $p \in S^n(1)$. If $p \in f_\theta(M)\left(\theta \in \left[-\frac{\pi}{g} + \theta_0, \theta_0\right]\right)$. Then $t(p) = \theta_0 - \theta$.

(2.3) Let h be the mean curvature of M^{n-1} in $S^n(1)$ with respect to ν. Then

$$(n-1)h = \sum_{\alpha=0}^{g-1} m_\alpha \cot\left(t + \frac{\alpha\pi}{g}\right),$$

$$= \begin{cases} m_0 g \cot(gt), & (g\text{: odd or } m_0 = m_1), \\ \frac{m_0 g}{2} \cot \frac{gt}{2} - \frac{m_1 g}{2} \tan \frac{gt}{2}, & (g\text{: even or } m_0 \neq m_1). \end{cases}$$

3. An Estimate of Volume Elements and Cheeger's Constant

In this section, we compare volume elements of M^{n-1} and $S^n(1)$ and give an estimate of Cheeger's isoperimetric constant from below.

Let M^{n-1} be a closed isoparametric hypersurface in $S^n(1)$ and D be a domain in M with a smooth boundary H. We use the same notations as in section two. Set $D_\theta = f_\theta(D)$, $H_\theta = f_\theta(H)$ and

$$\tilde{D}_+ = \bigcup_{\theta \in [0, \theta_0]} D_\theta, \qquad \tilde{D}_- = \bigcup_{\theta \in [-\frac{\pi}{g} + \theta_0, 0]} D_\theta,$$

$$\tilde{H}_+ = \bigcup_{\theta \in [0, \theta_0]} D_\theta, \qquad \tilde{H}_- = \bigcup_{\theta \in [-\frac{\pi}{g} + \theta_0, 0]} H_\theta,$$

$$\tilde{D} = \tilde{D}_+ \cup \tilde{D}_-, \qquad \tilde{H} = \tilde{H}_+ \cup \tilde{H}_-.$$

Then, by the construction of $\tilde{D}$ and $\tilde{H}$, we easily have the following Lemma 1.

Lemma 1. *We have*

$$\partial \tilde{D} = \tilde{H}, \quad \tilde{H}_+ \cap \tilde{H}_- = H, \quad \text{and} \quad \tilde{D}_+ \cap \tilde{D}_- = D.$$

Lemma 2. *When $g \geq 2$, we have*

$$\mathrm{vol.}(\tilde{D}) = \frac{2\,\mathrm{vol}(D)}{g \sin^{m_0} l\theta_0 \cos^{m_1} l\theta_0} \int_0^{\frac{\pi}{2}} \sin^{m_0} x \cos^{m_1} x \, dx.$$

Here $l = g/2$.

Proof. Let $\{X_{\alpha,i} | i = 1, \ldots, m_\alpha : \alpha = 0, \ldots, g-1, X_{\alpha,i} \in E^\alpha\}$ be a local orthonormal frame field on D and $\{\omega_{\alpha,i} | i = 1, \ldots, m_\alpha : \alpha = 0, \ldots, g-1\}$ be its dual frame field on D. For $\theta_0 \in \left(-\frac{\pi}{g}+\theta_0, \theta_0\right)$, g_0 denotes the Riemannian metric of $f_\theta(D)$ induced from $S^n(1)$. Then, by (2.1), we have:

$$f_\theta^* g_\theta = \sum_{\alpha=0}^{g-1} \frac{\sin_2(\theta_\alpha-\theta)}{\sin^2\theta_\alpha} \sum_{i=1}^{m_\alpha} \omega_{\alpha,i}\otimes\omega_{\alpha,i}.$$

Let dD_θ (resp. dD) be the volume element of D_θ (resp. D). We first assume here that g is even. Then we have the following equation (3.1) using the above representation of $f_\theta^* g_\theta$ and the formula, $\prod_{\alpha=0}^{T-1} \sin\left(x+\frac{\alpha\pi}{T}\right)=2^{1-T}\sin Tx$ for any $x\in\mathbf{R}$ and any positive integer T:

$$f_\theta^* dD_\theta = F(\theta)\, dD, \tag{3.1}$$

Here

$$\begin{aligned} F(\theta) &= \prod_{\alpha=0}^{g-1} \frac{\sin^{m_\alpha}(\theta_\alpha-\theta)}{\sin^{m_\alpha}\theta_\alpha}, \\ &= \frac{\sin^{m_0} l(\theta_0-\theta)\cos^{m_1} l(\theta_0-\theta)}{\sin^{m_0} l\theta_0 \cos^{m_1} l\theta_0}. \end{aligned}$$

When g is odd ($\geqq 2$), m_0 must be equal to m_1 by Theorem 1 (2) and (3.1) is valid. Therefore we have the required result:

$$\begin{aligned} \mathrm{vol}(\tilde{D}) &= \int_{-\frac{\pi}{g}+\theta_0}^{\theta_0} \mathrm{vol}(D_\theta)\, d\theta, \\ &= \mathrm{vol}(D) \int_{-\frac{\pi}{g}+\theta_0}^{\theta_0} F(\theta)\, d\theta, \\ &= \frac{2\,\mathrm{vol}(D)}{g \sin^{m_0} l\theta_0 \cos^{m_1} l\theta_0} \int_0^{\frac{\pi}{2}} \sin^{m_0}x \cos^{m_1}x\, dx. \end{aligned}$$

Remark 1. Lemma 2 implies that $\mathrm{vol}(D)/\mathrm{vol}(M)=\mathrm{vol}(\tilde{D})/\mathrm{vol}(S^n(1))$. Therefore, by this fact and the following Lemma 3, we can estimate Cheeger's isoperimetric constant of M from below in terms of such a constant of $S^n(1)$ (see Theorem 2 and (3.3)).

Remark 2. We also show that every isoparametric hypersurface M in a unit sphere $S^n(1)$ is tight, that is, every non degenerate function $\xi_p(x)=(f(x), p)(x\in M, p\in S^n(1))$ has the minimum number of critical points required by the Morse inequalities where (,) denotes the canonical Euclidean inner product. This is verified by calculating the absolute total curvature and Theorem 1 (6). This fact was first proved by Cecil and Ryan [3] through the other method.

Lemma 3. Set $l=g/2$. *Then we have:*

$$\operatorname{vol}(\tilde{H}) \leqq \frac{\operatorname{vol}(H)}{\sin^{m_0} l\theta_0 \cos^{m_1} l\theta_0} \times \left(\sin\theta_0 \int_0^{l\theta_0} \sin^{m_0-1} x \cos^{m_1} x\, dx + \sin\left(\frac{\pi}{2l}-\theta_0\right) \int_0^{\frac{\pi}{2}-l\theta_0} \sin^{m_1-1} x \cos^{m_0} x\, dx\right).$$

Proof. Let N be the inward unit normal vector field of $H=\partial D$ in M. $\|\cdot\|$ denotes the length of a vector in $S^n(1)$. Set $N_\theta = f_{\theta *} N / \|f_{\theta *} N\|$ and $Z_\theta = N / \|f_{\theta *} N\|$. Then by the definition of H_θ and (2.1) we have that N_θ is an inward unit normal vector field of $H_\theta = \partial D_\theta$ in M_θ. Let dH_θ be the volume element of H_θ. Then we have:

$$\begin{aligned} f_\theta^* \, dH_\theta &= f_\theta^* (i_{N_\theta} dD_\theta) \\ &= f_\theta^* (i_{f_{\theta *} Z_\theta} dD_\theta) \\ &= i_{Z_\theta} (f_\theta^* \, dD_\theta) \\ &= F(\theta)\, i_{Z_\theta} dD \\ &= \frac{F(\theta)}{\|f_{\theta *} N\|} dH. \end{aligned}$$

Let $N = \sum_{\alpha,i} a_{\alpha,i} x_{\alpha,i}$ $\sum_{\alpha,i} a_{\alpha,i}^2 = 1$. Then

$$\begin{aligned} \|f_{\theta_*} N\|^2 &= \sum_{\alpha,i}^{2} \frac{\sin^2(\theta_\alpha - \theta)}{\sin^2 \theta_\alpha},\ a_{\alpha,i}^2 \\ &\geqq \varphi(\theta)^2. \end{aligned}$$

Here

$$\begin{aligned} \varphi(\theta) &= \min\{|\sin(\theta_\alpha - \theta)/\sin\theta_\alpha| : \alpha = 0, \ldots, g-1\}, \\ &= \begin{cases} \sin(\theta_0 - \theta)/\sin\theta_0, & (\theta \geqq 0), \\ \sin\left(\frac{\pi}{g} - \theta_0 + \theta\right) \Big/ \sin\left(\frac{\pi}{g} - \theta_0\right), & (\theta < 0). \end{cases} \end{aligned} \tag{3.2}$$

Therefore we have:

$$\operatorname{vol}(\tilde{H}) \leqq \frac{\operatorname{vol}(H)}{\sin^{m_0} \theta_0 \cos^{m_1} \theta_0} \times \left(\sin\theta_0 \int_0^{\theta_0} \frac{\sin^{m_0} lx \cos^{m_1} lx}{\sin x} dx + \sin\left(\frac{\pi}{2l}-\theta_0\right) \int_0^{\frac{\pi}{2}-\theta_0} \frac{\sin^{m_1} lx \cos^{m_0} lx}{\sin x} dx\right).$$

And we have required inequality using an inequality: $l \sin x \geqq \sin lx$ for any $x \in \left[0, \frac{\pi}{2l}\right]$.

Theorem 2. *Let M^{n-1} be a closed isoparametric hypersurface in $S^n(1)$ with $g \geqq 2$. Then we have:*

$$C(M^{n-1}) \geqq \frac{4}{3\pi} \frac{1}{\frac{1}{\sqrt{m_0}} + \frac{1}{\sqrt{m_1}}},$$
$$\geqq \frac{2}{3\pi}.$$

Proof. We set $l = g/2$ and for any positive integer s, t,

$$I(s, t) = \int_0^{\frac{\pi}{2}} \sin^s x \cos^t x \, dx,$$
$$J(s) = I(s, 0).$$

Namely, $2I(s, t) = B\left(\frac{s+1}{2}, \frac{t+1}{2}\right)$. By $0 < l\theta_0 < \pi/2$, Lemma 2, Lemma 3 and Remark 1, we have:

$$\operatorname{vol}(\tilde{H}) \leqq \frac{\operatorname{vol}(H)}{\sin^{m_0} l\theta_0 \cos^{m_1} l\theta_0} \times \left(\sin\theta_0 \, I(m_0 - 1, m_1) + \sin\left(\frac{\pi}{2l} - \theta_0\right) I(m_1 - 1, m_0)\right),$$

$$\operatorname{vol}(\tilde{D}) = \frac{\operatorname{vol}(D)}{l \sin^{m_0} l\theta_0 \cos^{m_1} l\theta_0} I(m_0, m_1).$$

$$\frac{\operatorname{vol}(D)}{\operatorname{vol}(M)} = \frac{\operatorname{vol}(\tilde{D})}{\operatorname{vol}(S^n(1))}.$$

Therefore, by the definition of Cheeger's isoperimetric constant, we obtain:

$$C(M^{n-1}) \geqq C(S^n(1)) \frac{1}{l} \times \frac{I(m_0, m_1)}{\sin\theta_0 \, I(m_0 - 1, m_1) + \sin\left(\frac{\pi}{2l} - \theta_0\right) I(m_1 - 1, m_0)}, \tag{3.3}$$
$$> \frac{C(S^n(1))}{l \sin \frac{\pi}{2l}} \frac{I(m_0, m_1)}{I(m_0 - 1, m_1) + I(m_1 - 1, m_0)}.$$

We notice here that any integers s, t, $s \geqq 1$, $t \geqq 0$, $I(s-1, t)/I(s, t) = J(s-1)/J(s+t)$ and that by the classical result,

$$C(S^n(1)) = \frac{2\,\mathrm{vol}(S^{n-1}(1))}{\mathrm{vol}(S^n(1)} = \frac{1}{J(n-1)}.$$

Therefore we have:

$$\begin{aligned} C(M^{n-1}) &\geqq \frac{C(S^n(1))}{l \sin\frac{\pi}{2l}} \frac{J(m_0+m_1)}{J(m_0-1)+J(m_1-1)}, \\ &= \frac{1}{l \sin\frac{\pi}{2l}} \frac{J(m_0+m_1)}{J(n-1)} \frac{1}{J(m_0-1)+J(m_1-1)}, \\ &\geqq \frac{3}{4\pi} \frac{1}{\frac{1}{\sqrt{m_0}}+\frac{1}{\sqrt{m_1}}}, \\ &\geqq \frac{2}{3\pi}. \end{aligned}$$

Here we use the facts that (1) $J(k)$ is monotonely decreasing in k, (2) $J(k) \leqq \frac{\pi}{2\sqrt{k+1}}$ $(k \geqq 0)$ and (3) $l \sin\frac{\pi}{2l} \leqq 3/2$.

When M is, moreover, minimal, we have the following theorem by the same method of the proof of Theorem 2.

Theorem 3. *Let M^{n-1} be a closed isoparametric minimal hypersurface in $S^n(1)$. Then we have:*

$$\begin{aligned} C(M^{n-1}) &\geqq \frac{2}{g\pi} C(S^n(1)) \geqq \frac{8}{g\sqrt{3}\pi^2}, \\ &\geqq \frac{4}{3\sqrt{3}\pi^2}\sqrt{n}. \end{aligned}$$

Moreover,

$$\begin{aligned} \lambda_1(M^{n-1}) &\geqq \frac{1}{\pi^2 g^2} C(S^n(1))^2 \geqq \frac{16}{3\pi^3 g^2} n, \\ &\geqq \frac{4}{27\pi^4} n. \end{aligned}$$

Proof. Since M is minimal, (2.3) implies that

$$l\theta_0 = \operatorname{arc\,cot}\sqrt{m_1/m_0},$$
$$\sin l\theta_0 = \sqrt{m_0}/\sqrt{m_0+m_1}, \qquad \cos l\theta_0 = \sqrt{m_1}/\sqrt{m_0+m_1}.$$

From (3.3), we have:

$$C(M^{n-1}) \geqq \frac{C(S^n(1))}{l} \frac{J(m_0+m_1)}{\sin l\theta_0\, J(m_0-1)+\cos l\theta_0\, J(m_1-1)}.$$

Since we easily have that (1) $J(k) \leqq \frac{\pi}{2\sqrt{k+1}} (k \geqq 0)$, (2) $J(k) \leqq \frac{\sqrt{3}\pi}{4\sqrt{k+1}} (k \geqq 2)$ and (3) $J(k) \geqq \frac{1}{\sqrt{k}} (k \geqq 1)$, we obtain the required result:

$$\begin{aligned} C(M^{n-1}) &\geqq \frac{C(S^n(1))}{l\pi}, \\ &= \frac{2}{g\pi} \frac{1}{J(n-1)}, \\ &\geqq \frac{8}{\sqrt{3}\pi^2 g} \sqrt{n}. \end{aligned}$$

In general, it is well-known that the first eigenvalue is estimated from below in terms of the Cheeger's constant, $\lambda_1 \geqq \frac{1}{4} C^2$ (see [1]). Theorem 3 is completely proved.

Next we give a stability condition for a domain of a closed isoparametric minimal hypersurface in $S^n(1)$ as a corollary of Theorem 3.

Let D be a compact minimal hypersurface with boundary ∂D in $S^n(1)$ and v be a unit normal vector field along D. For any smooth function φ on D satisfying $\varphi_{|\partial_D}=0$, $\varphi_t(-\varepsilon<t<\varepsilon)$ denotes a variation induced from φv. Then D is minimal if and only if that for any φ satisfying $\varphi_{|\partial D}=0$,

$$\frac{d}{dt} \operatorname{vol}(\varphi_t(D))_{|t=0}=0.$$

D is said to be stable if and only if for any φ satisfying $\varphi_{|\partial D}=0$,

$$\begin{aligned} \frac{d^2}{dt^2} \operatorname{vol}(\varphi_t(D))_{|t=0} &= \int_D L(\varphi)\, \varphi\, dD, \\ &\geqq 0. \end{aligned}$$

Here L is a Jacobi operator of D,

$$L(\varphi) = -\Delta \varphi - (n-1+\|B\|^2)\, \varphi,$$

and $\|B\|^2$ is the square length of the second fundamental form B of D in $S^n(1)$ (see [18]).

Lemma 4. *Let M be a closed isoparametric hypersurface in $s^n(1)$. Let h be the mean curvature of M with respect to v and set $c=(m_1-m_0)\, g^2/2$. Then*

$$\|B\|^2=\begin{cases}(n-1)(g-1)\dfrac{((n-1)\,h)^2}{m_0}, & (g\colon \text{odd or } m_0=m_1),\\[2ex] (n-1)(g-1)+\dfrac{(n-1)^3}{g\,m_0\,m_1}\,h^3 & \\[2ex] \qquad +\dfrac{c}{g^2\,m_0\,m_1}(n-1)\,h\sqrt{(n-1)^2\,h^2+m_0\,m_1\,g^2}, & \\[1ex] & (g\colon \text{even or } m_0\neq m_1).\end{cases}$$

Moreover when M is minimal, we have

$$\|B\|^2=(n-1)(g-1).$$

Proof. Set $l=g/2$. By Theorem 1, we have:

$$\begin{aligned}\|B\|^2&=\sum_{\alpha=0}^{g-1} m_\alpha \cot^2\left(\mathrm{t}+\frac{\alpha}{g}\pi\right),\\ &=m_0\sum_{\alpha=0}^{l-1}\cot^2\left(t+\frac{\alpha}{l}\pi\right)\\ &\quad+m_1\sum_{\alpha=0}^{l-1}\cot^2\left(\left(t+\frac{\pi}{2l}\right)+\frac{\alpha}{l}\pi\right).\end{aligned}$$

We first assume that g is even. Set $A=(n-1)h$, then we have

$$\begin{aligned}\tan lt&=\frac{-A+\sqrt{A^2+4m_0\,m_1\,l^2}}{2m_1\,l},\\ \cot lt&=\frac{A+\sqrt{A^2+4m_0\,m_1\,l^2}}{2m_0\,l}.\end{aligned}$$

Therefore using a formula

$$\sum_{\alpha=0}^{T-1}\cot^2\left(y+\frac{\alpha}{T}\pi\right)=T^2\cot^2 Ty+T(T-1),$$

for any $y\in\mathbb{R}$ and a positive integer T, we have:

$$\begin{aligned}\|B\|^2&=(m_0+m_1)\,l(l-1)+l^2(m_0\cot^2 lt+m_1\tan^2 lt)\\ &=(n-1)(g-1)+\frac{m_0+m_1}{2m_0\,m_1}A^2\\ &\quad+\frac{m_0-m_1}{2m_0\,m_1}A\sqrt{A^2+4m_0\,m_1\,l^2},\\ &=(n-1)(g-1)+\frac{(n-1)^3}{g\,m_0\,m_1}\,h^2\\ &\quad+\frac{c}{g^2\,m_0\,m_1}(n-1)\,h\sqrt{(n-1)^2\,h^2+m_0\,m_1\,g^2}.\end{aligned}$$

When g is odd, m_0 must be equal to m_1 by Theorem 1 (2). This lemma is proved.

Theorem 4. *Let M^{n-1} be a closed isoparametric minimal hypersurface in $S^n(1)(n \geqq 3)$ and D be a compact domain of M. If* $\operatorname{vol}(D) \leqq a_n \operatorname{vol}(M)$, *then D is stable. Here* $a_n = \frac{2^{n-1}}{\pi^{2n}} \sqrt{n} \frac{1}{(n-1)^{n/2} g^{3n/2}}$.

Proof. By the above lemma, Jacobi operator L is represented by

$$L(\varphi) = -\Delta \varphi - (n-1) g \varphi.$$

Hence D is stable if and only if that the first eigenvalue $\lambda_1^D(D)$ of the Laplacian on D under the Dirichlet boundary condition is not smaller than $(n-1)g$. By our assumption and Remark 1 of Lemma 2, we see that $\operatorname{vol}(\tilde{D})/\operatorname{vol}(S^n(1)) = \operatorname{vol}(D)/\operatorname{vol}(M) = a \leqq a_n < 1/2$. Let D^* be a geodesic ball of radius r_0 in $S^n(1)$ having the same volume as $\tilde{D}$. Then the classical result implies that $C(\tilde{D}) \geqq C(D^*)$. Since an estimate of Cheeger's isoparametric constant in Theorem 3 is valid for D and $\tilde{D}$, we have that $C(D) \geqq \frac{2}{g\pi} C(D^*)$. We define r_1 by $r_1^n = n\left(\frac{\pi}{2}\right)^{n-1} 2a\, J(n-1)$. Since

$$\begin{aligned} r_0 \sin^{n-1} r_0 &> \int_0^{r_0} \sin^{n-1} dx, \\ &> \left(\frac{2}{\pi}\right)^{n-1} \frac{1}{n} r_0^n, \end{aligned}$$

we see that $r_1 > r_0$ and that

$$\begin{aligned} C(D^*) &= \frac{\sin^{n-1} r_0}{\int_0^{r_0} \sin^{n-1} x\, dx}, \\ &> \frac{1}{r_1}, \\ &\geqq \frac{2}{\pi}\left(\frac{1}{2\sqrt{na}}\right)^{1/n}. \end{aligned}$$

The last inequality follows from the inequality $J(n-1) \leqq \frac{\pi}{2\sqrt{n}}$ $(n \geqq 1)$. Therefore we obtain that

$$\begin{aligned} \lambda_1^D(D) &\geqq \tfrac{1}{4} C(D)^2, \\ &\geqq \frac{4}{g^2 \pi^4}\left(\frac{1}{2\sqrt{n a_n}}\right)^{2/n}, \\ &= (n-1) g. \end{aligned}$$

Remark. Let M be a closed isoparametric minimal hypersurface in $S^n(1)$. Then we have by Lemma 2 that

$$\mathrm{vol}(M^{n-1})=\frac{\mathrm{vol}(S^n(1))}{2I(m_0,m_1)}\left[\frac{m_0^{m_0}m_1^{m_1}}{(m_0+m_1)^{m_0+m_1}}\right]^{1/2}g.$$

4. An Estimate of Eigenvalues

In this section, we prove Theorem A. In order to estimate eigenvalues of M, we prepare some theorem.

Theorem 4. (*Chavel and Feldman* [4], *Ozawa* [16]) *Let V be a closed, connected smooth Riemannian manifold and W a closed submanifold of V. For any sufficiently small $\varepsilon>0$, set $w(\varepsilon)=\{x\in V\colon \mathrm{dist}(x, W)\leqq\varepsilon\}$. Let $\lambda_k^D(\varepsilon)(k=1, 2, \ldots)$ be the k-th eigenvalue of the Laplacian on $V-W(\varepsilon)$ under the Dirichlet boundary condition. If* $\dim V\geqq\dim W+2$, *then for any* $k=1, 2, \ldots,$

$$\lim_{\varepsilon\to 0}\lambda_k^D(\varepsilon)=\lambda_{k-1}(V).$$

Remark 1. The condition for the codimension of W in V is essential. This is easily seen when $V=S^n(1)$ and $W=S^{n-1}(1)\subset V$.

Remark 2. Ozawa proved the above theorem and, moreover, he studied the behavior of $\lambda_k^D(\varepsilon)$ when $\varepsilon\to 0$.

Remark 3. We need the fact that $\lambda_k^D(\varepsilon)\geqq\lambda_{k-1}(S^n(1))$ for $k\geqq 1$ and this is proved by the mini-max principle.

We denote the spectrum of $S^n(1)$ by another notation, that is, $\{(\mu_k, n_k)|0=\mu_0<\mu_1<\ldots<\mu_k<\ldots,\uparrow\infty:\mu_k$ is an eigenvalue, n_k is the multiplicity of $\mu_k\}$. Then it is well-known (see [1]) that for nonnegative integer k,

$$\begin{aligned}&\mu_k=k(n+k-1),\\ (4.1)\qquad &n_0=1,\quad n_1=n+1,\\ &n_k={}_{n+k}C_n-{}_{n+k-2}C_n(k\geqq 2).\end{aligned}$$

Proof of Theorem A. For sufficiently small $\varepsilon>0$, set

$$M(\varepsilon)=\bigcup_{\theta\in[-\frac{\pi}{g}+\theta_0+\varepsilon,\theta_0-\varepsilon]}f_\theta(M).$$

Then, by Münzner's theorem (Theorem 1), $M(\varepsilon)$ is a domain of $S^n(1)$ obtained by excluding ε-neighborhood of M_+ and M_- from $S^n(1)$ and is diffeomorphic to $M\times\left[-\frac{\pi}{g}+\theta_0+\varepsilon,\ \theta_0-\varepsilon\right]$ under $f_\theta(p)$. Since $\lim_{\varepsilon\to 0}\lambda_k^D(M(\varepsilon))=\lambda_{k-1}(S^n(1))(k=1,$

2, ...), we may estimate $\lambda_k^D(M(\varepsilon))$ from above in terms of $\lambda_{k-1}(M^{n-1})$. Let $\{X_{\alpha,i}: i=1, \dots, m_\alpha,\ \alpha=0, \dots, g-1,\ X_{\alpha,i} \in E^\alpha\}$ be a local orthonormal frame field on M. Then $\left\{\frac{\partial}{\partial\theta},\ \sin\theta_\alpha/\sin(\theta_\alpha-\theta)\ X_{\alpha,i}:\ i=1, \dots, m_\alpha,\ \alpha=0, \dots, g-1,\ X_{\alpha,i} \in E^\alpha,\ \theta \in \left[-\frac{\pi}{g}+\theta_0+\varepsilon,\ \theta_0-\varepsilon\right]\right\}$ is a local orthonormal frame field on $M(\varepsilon)$ by the diffeomorphisms f_θ. (2.1) implies that the volume element $dM(\varepsilon)$ of $M(\varepsilon)$ is represented by the following:

$$dM(\varepsilon)=\frac{\sin^{m_0} l(\theta_0-\theta)\cos^{m_1} l(\theta_0-\theta)}{\sin^{m_0} l\theta_0 \cos^{m_1} l\theta_0}\,d\theta\, dM. \tag{4.2}$$

Let $f_k (k=0, 1, \dots)$ be the k-th eigenfunctions on M which are orthogonal to each other with respect to the square integral inner product on M. Let h be a nonnegative, non-decreasing smooth function on $[0, \infty])$ satisfying $h=1$ on $[2, \infty)$ and $h=0$ on $[0, 1]$. For sufficiently small $\eta>0$, let ψ_η be a nonnegative smooth function on $\left[\eta, \frac{\pi}{2}-\eta\right]$ such that (1) $\psi_\eta(\eta)=\psi_\eta\left(\frac{\pi}{2}-\eta\right)=0$ (2) ψ_η is symmetric with respect to $x=\frac{\pi}{4}$ and (3) $\psi_\eta(x)=h\left(\frac{x}{\eta}\right)$ on $\left[\eta, \frac{\pi}{4}\right]$. Let L_k be the space of functions spanned by $\{f_0, f_1, \dots, f_k\} (k \geqq 0)$. For any $\varphi \in L_k$, define a function Φ_ε on $M(\varepsilon)$ by

$$\Phi_\varepsilon(x, \theta)=\psi_\varepsilon(l(\theta_0-\theta))\,\varphi(x).$$

Then Φ_ε is a smooth function on $M(\varepsilon)$ satisfying the Dirichlet boundary condition and Φ_ε is square integrable on $M(\varepsilon)$. By (3.2), (4.2) and the condition $\min(m_0, m_1) \geqq 2$, we see that

$$\begin{aligned}
\frac{\|d\Phi_\varepsilon\|_2^2}{\|\Phi_\varepsilon\|_2^2} &\ll l^2 \frac{\int_{l\varepsilon}^{\frac{\pi}{2}-l\varepsilon} \psi_{l\varepsilon}'(x)^2 \sin^{m_0}x \cos^{m_1}x\,dx}{\int_{l\varepsilon}^{\frac{\pi}{2}-l\varepsilon} \psi_{l\varepsilon}(x)^2 \sin^{m_0}x \cos^{m_1}x\,dx} \\
&\quad + \frac{1}{\int_{l\varepsilon}^{\frac{\pi}{2}-l\varepsilon} \psi_{l\varepsilon}(x)^2 \sin^{m_0}x \cos^{m_1}x\,dx} \times \frac{\|d\varphi\|_2^2}{\|\varphi\|_2^2} \\
&\quad \times \left(\sin^2\theta_0 \int_{l\varepsilon}^{l\theta_0} \psi_{l\varepsilon}^2(x) \frac{\sin^{m_0}x}{\sin^2\frac{x}{l}} \cos^{m_1}x\,dx \right. \\
&\quad \left. + \sin^2\left(\frac{\pi}{2l}-\theta_0\right) \int_{l\theta_0}^{\frac{\pi}{2}-l\varepsilon} \psi_{l\varepsilon}^2(x) \sin^{m_0}x \frac{\sin^{m_1}x}{\sin^2\frac{1}{l}\left(\frac{\pi}{2}-x\right)}\,\mathrm{d}x\right).
\end{aligned}$$

By the mini-max principle, we have:

$$\lambda_{k+1}^{D}(M(\varepsilon)) \leqq \sup_{\varphi \in L_k} \frac{\|d\Phi_\varepsilon\|^2}{\|\Phi_\varepsilon\|^2}.$$

By the condition $\min(m_0, m_1) \geqq 2$, we see that the first term in the right hand side tends to 0 as $\varepsilon \to 0$. Therefore, we see that

$$\begin{aligned} &\lim_{\varepsilon \to 0} \sup_{\varphi \in L_k} \frac{\|d\Phi_\varepsilon\|^2}{\|\Phi_\varepsilon\|^2} \\ &\leqq \lambda_k(M^{n-1}) \frac{G_2(g\colon m_0, m_1, \theta_0) + G_3(g\colon m_0, m_1, \theta_0)}{G_1(m_0, m_1)}. \end{aligned}$$

By combining the above two inequalities and Theorem 4, we have the required inequality. Moreover, when M is minimal, $l\theta_0$ must be equal to arc tan $\sqrt{m_0/m_1}$ by (2.3).

We prove Theorem B in the next section, that is, under some condition (Condition (C)), the first eigenvalue of a closed minimal isoparametric hypersurface in a unit sphere is equal to its dimension. We here estimate $G(g\colon m_0, m_1, \theta_0)$ very roughly.

Corollary. *Let M be a closed isoparametric hypersurface of $S^n(1)$ satisfying $g \geqq 2$ and $\min(m_0, m_1) \geqq 2$. Then we have:*

$$\begin{aligned} \lambda_k(M^{n-1}) &\geqq \lambda_k(S^n(1)) \frac{1}{(n-1)\, g \sin^2(\pi/g)}, \\ &\geqq \lambda_k(S^n(1)) \frac{g}{(n-1)\pi^2}. \end{aligned}$$

Moreover, when M is minimal, we have:

$$\begin{aligned} \lambda_k(M^{n-1}) &\geqq \lambda_k(S^n(1)) \frac{4}{g^2} \frac{1}{\dfrac{m_0}{m_0-1} + \dfrac{m_1}{m_1-1}}, \\ &\geqq \frac{1}{g} \lambda_k(S^n(1)). \end{aligned}$$

Proof. By the inequality $l \sin x \geqq \sin lx$ for $x \in \left[0, \dfrac{\pi}{2l}\right]$,

$$\begin{aligned} &\frac{G_2(g\colon m_0, m_1, \theta_0) + G_3(g\colon m_0, m_1, \theta_0)}{G_1(m_0, m_1, \theta_0)} \\ &\leqq l^2 \sin^2 \frac{\pi}{2l} \frac{I(m_0-2, m_1) + I(m_0, m_1-2)}{I(m_0, m_1)}, \\ &= l^2 \sin^2 \frac{\pi}{2l} \left(\frac{m_0+m_1}{m_0-1} + \frac{m_0+m_1}{m_1-1} \right). \end{aligned}$$

Since $l(m_0+m_1)=n-1$, we have that the first inequality of the above corollary.

We assume here that M is minimal. By (2.3), $l\theta_0=\text{arc tan}\sqrt{m_0/m_1}$. Therefore we see that

$$\frac{G_2(g\colon m_0, m_1, \theta_0)+G_3(g\colon m_0, m_1, \theta_0)}{G_1(m_0, m_1)}$$
$$\leqq l^2\,\frac{\sin^2 l\theta_0\, I(m_0-2, m_1)+\cos^2 l\theta_0\, I(m_0, m_1-2)}{I(m_0, m_1)},$$
$$\leqq l^2\left(\frac{m_0}{m_0-1}+\frac{m_1}{m_1-1}\right).$$

Hence the above corollary is proved.

5. Proof of Theorem B

To prove Theorem B, we prepare some lemmas.

Set $A=A(m)=\left(1-\dfrac{3}{m+3}\right)^{1/2}$ and $B=B(m)=\left(\dfrac{3}{m+3}\right)^{1/2}$. Then we have an elementary lemma.

Lemma 5. (1) *$A(m)^{m-1}$ is monotone decreasing in m and the limit as m tends to ∞ is $e^{-3/2}$.*

(2) $A(m)^{m+1}\left(1+\dfrac{m+1}{m+2}A\right)>2\dfrac{m}{m+3}e^{-3/2}A.$

(3) $\dfrac{2m+3}{m+2}-A^{m+1}\left(1+\dfrac{m+1}{m+2}A\right)<2(1-A^{m+2}).$

(4) *$A(m)^{m+2}$ is monotone increasing in m and the limit as m tends to ∞ is $e^{-3/2}$.*

Lemma 6. (1) $(1+6/x^2)^x\leqq e^{6/x}(x\geqq 1)$.

(2) $e^{6/x}-1<8/x(x\geqq 11)$.

(3) $e^{3/x}\,2x\sqrt{x}<(3+\sqrt{x(x+2)})(\sqrt{x+2}+\sqrt{x})(x\geqq 11)$.

Proof of lemma 6. Since (1) and (2) are easily verified, we show (3). Set $y(x)=$(the left hand side)^2 – (the right hand side)^2 in (3). Then we have (the right hand side)$^2>4x^3+34x^2+76x+8$. Therefore we have $y(x)<4[(e^{6/x}-1)\,x^2-8x-19]\,x-18$. By (2), we have $y<0$.

For $m\geqq 4$, set

$$2\theta_0=2\theta_0(\mathrm{m})=\cot^{-1}\sqrt{3/m},$$
$$\alpha(m)=\int_0^{2\theta_0(m)}\sin^{m-2}x\,dx.$$

Lemma 7. *For $m \geqq 11$, we have $\alpha(m+2)<\alpha(m)$.*

Proof. Set

$$\beta(m+2)=\int_{2\theta_0(m)}^{2\theta_0(m+2)} \sin^m x\,dx.$$

Then we have $\alpha(m+2)=\frac{m-1}{m}\alpha(m)-\frac{1}{m}A(m)^{m-1}B(m)+\beta(m+2)$. We may show that $\beta(m+2)<\frac{1}{m}A(m)^{m-1}B(m)+\beta(m)$ for $m \geqq 11$.

$$\begin{aligned}\beta(m+2)&<A(m+2)^m(2\theta_0(m+2)-2\theta_0(m))\\&<A(m+2)^m\tan(2\theta_0(m+2)-2\theta_0(m))\\&=A(m)^{m-1}B(m)\left(1+\frac{6}{m(m+5)}\right)^{m/2}\frac{2\sqrt{m}}{(3+\sqrt{m(m+2)})(\sqrt{m+2}+\sqrt{m})}.\end{aligned}$$

By lemma 6 (1) and (3), $\beta(m+2)<A(m)^{m-1}B(m)\frac{1}{m}$. Therefore we have $\alpha(m+2)<\frac{m-1}{m}\alpha(m)<\alpha(m)$.

Proof of Theorem B. When $g=1$ or 2, it is known (see [2]) that $M^{n-1}=S^{n-1}(1)$ or $S^p(\sqrt{p/(n-1)})\times S^q(\sqrt{q/(n-1)})(p+q=n-1)$ and the first eigenvalue of M must be equal to its dimension. When $g=3$ and $(m_0, m_1)=(4, 4)$ or $(8, 8)$, Kotani (see [9]) first showed that $\lambda_1(M^{n-1})=n-1$. We also prove this fact by our method. To prove Theorem B, since the multiplicity of every minimal submanifold fully immersed in $S^N(1)$ which is not isometric to the unit sphere is not smaller than $N+1$ (see [20]), we may assume $m_0 \geqq m_1$ and show that $\lambda_{n+2}(M^{n-1})>n-1=\dim M$ for each closed isoparametric minimal hypersurface M^{n-1} of $S^n(1)$ which satisfies Condition (C). From Theorem A, we may show that $G(g\colon m_0, m_1, \theta_0)\ \lambda_{n+2}(S^n(1))>n-1$ in each case of Condition (C). We notice here that $l(n-1)=m_0+m_1$ by Theorem 1 (2).

Let M^{n-1} be a closed isoparametric minimal hypersurface of $S^n(1)$ with $g=4$ and satisfies one of the following: $(m_0, m_1)=(4, 3), (8, 3), \ldots, (4k, 3), \ldots, (k \geqq 1)$. We first show that $G(4\colon m, 3, \theta_0)>0.5$ for any $m \geqq 46$ and, by (4.1), that $G(4\colon m, 3, \theta_0)\ \lambda_{n+2}(S^n(1))>0.5\times 2(n+1)>n-1$ for $m \geqq 46$. And for the other cases $(g=4,\ m_1=3)$, we can verify the inequality $G(4\colon m, 3, \theta_0)\ \lambda_{n+2}(S^n(1))>n-1$ by using a computer. We assume here that $m \geqq 46$. Since M is minimal, (2.3) implies that $2\theta_0=2\theta_0(m)=\operatorname{arc\,cot}\sqrt{3/m}$. Set $\eta=2\theta_0$, $A=A(m)=\sin\eta=\sqrt{m}/\sqrt{m+3}$ and $B=B(m)=\cos\eta=\sqrt{3}/\sqrt{m+3}$. Then we have:

$$\sin^2\theta_0=\frac{m}{2(m+3)}\frac{1}{1+B},$$

$$\sin^2\left(\frac{\pi}{4}-\theta_0\right)=\frac{3}{2(m+3)}\frac{1}{1+A}.$$

Therefore we see that

$$G_2(4\colon m, 3, \theta_0)$$
$$=\frac{m}{m+3}\frac{1}{1+B}\int_0^{\eta}(1+\cos x)\sin^{m-2}\cos^3 x\,dx,$$
$$=\frac{2}{(m+1)(m+3)}\frac{1}{2(1+B)}$$
$$\times\left(\frac{m(5m+3)}{(m-1)(m+3)}A^{m-1}+\frac{3(m+1)(2m+3)}{(m+2)(m+3)}A^{m-1}B+\frac{3(m+1)}{m+2}\alpha(m)\right).$$

Similarly we see that

$$G_3(4\colon m, 3, \theta_0)=\frac{2}{(m+1)(m+3)}\frac{1}{(1+A)}\frac{3}{2}\left(\frac{2m+3}{m+3}-A^{m+1}\left(1+\frac{m+1}{m+2}A\right)\right),$$

and

$$G_1(m, 3)=\frac{2}{(m+1)(m+3)}.$$

By lemma 5, we have

$$G(4\colon m, 3, \theta_0)^{-1}$$
$$<\tfrac{1}{2}(5A^{m-1}+6A^{m-1}B+3\alpha(m))+\frac{3}{2}\frac{1}{1+A}2(1-A^{m+2}).$$

By lemmas 5 and 7, we see that the right hand side of the above inequality is decreasing for $m, m+2, \ldots$, and $G(4{:}46, 3, \theta_0)^{-1}<1.9983374\ldots<2$ and $G(4{:}47, 3, \theta_0)^{-1}<1.9933199\ldots<2$. Therefore we have the required inequality for $m\geqq 46$. For $m\leqq 45$, we can directly show by using a computer that the difference $D(g\colon m_0, m_1)=G(g\colon m_0, m_1, \theta_0)\,\lambda_{n+2}(S^n(1))-(n-1)(g=4, m_0=m, m_1=3)$ is positive. We use the double exponential formula (see [13]) and the language of the program is FORTRAN. A subroutine program using the double exponential formula is written in an appendix of a book [12]. This is a subroutine program to integrate an analytic function on $(-1, 1)$ or $(0, \infty)$ and has an absolute error 10^{-16}. But it is easy to make a partial revision of this program so that we have relatively very small errors which depend on this program and our machine. For example, $G(4\colon 4, 3, \theta_0)=0.4411526996992993$, $D(4\colon 4, 3)=0.1168863903775783>0$, $G(4\colon 45, 3, \theta_0)=0.555614098973507>0.5$. By these computations, we obtain the required inequality for $m_1=3$.

By the similar estimate, we have that $G(4\colon m, 4, \theta_0)>0.5$ for any $m\geqq 34$ and $G(4\colon m, 7, \theta_0)>0.5$ for any $m\geqq 36$ and by a computer, we have, for example, $G(4\colon 4, 4, \theta_0)=0.4846093593926227$, $D(4\colon 4, 4)=0.1445936938134416$, $G(4\colon 5, 4, \theta_0)=0.5110829726081493>0.5$, $G(4\colon 33, 4, \theta_0)=0.6146834883261047>0.5$, $G(4\colon 8, 7, \theta_0)=0.6258833686366021>0.5$, and $G(4\colon 35, 7, \theta_0)=0.7061682378135796>0.5$.

For the other cases, we directly compute G and D, for example, $G(4\colon 6,$

9, θ_0)=0.6185633191383751, $G(4:8, 15, \theta_0)=0.6856918246775244>0.5$, $G(4:10, 53, \theta_0)=0.7590277619970185>0.5$, $G(3:4, 4, \theta_0)=0.495059684\ldots$, $D(3:4, 4, \theta_0)=1.86167115\ldots$, $G(3:8, 8, \theta_0)=0.648727497\ldots>0.5$. Therefore we complete the proof of Theorem B.

Remark. We have the limits $G(4:m_1)$ of $\lim_{m_0\to\infty} G(4:m_0, m_1, \theta_0)(m_1=3, 4, 7)$ as follows: $G(4:3)=1/(1.5+e^{-3/2})=0.5803392124$, $G(4:4)>3/(4+6e^{-2}\sqrt{2/\pi})=0.64545394898$ and $G(4:7)=0.7524581288$.

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