On the pathwise computation of derivatives with respect to the rate of a point process: The phantom RPA method

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Received 12 December 1990; revised 24 June 1991

The Rare Perturbation Analysis (RPA) method is presented using two approaches: a direct one and an indirect one via a pathwise interpretation of the Likelihood Ratio Method (LRM). These two approaches give a new point of view for the Smoothed Perturbation Analysis (SPA) discussed in Gong [4] and extend the validity of the formulas therein, in particular to the estimation of derivatives of quantities that can be computed over a busy cycle. A heuristic comparison with LRM is given and simulation results are presented to compare the performance of LRM, RPA, and a finite difference RPA in a simple system.

Keywords: Sensitivity analysis; queues; point processes; perturbation analysis; likelihood ratios; routing.

1. Introduction

The computation of derivatives with respect to a parameter of operational characteristics of either finite horizon or stationary performance measures of queueing systems by means of simulation has recently been the subject of a number of articles, under the general heading of Perturbation Analysis. This research was motivated by the work of Ho, who invented the Infinitesimal Perturbation Analysis (IPA) method (see, for instance, Ho and Cao [6] and Suri [11]).

Let λ be the varying parameter – for instance, the intensity of a Poisson process N^{λ} – and denote by Ψ_{λ} the corresponding trajectory value of a performance index $E(\Psi_{\lambda})$ when the intensity is set to the value λ.

IPA seeks conditions under which the following holds:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}E(\Psi_\lambda) = E\left(\lim_{\Delta\lambda \to 0} \frac{1}{\Delta\lambda}(\Psi_{\lambda+\Delta\lambda} - \Psi_\lambda)\right).$$

The applicability of IPA depends on the continuity and differentiability of Ψ_λ, and on the validity of the interchange of limits ($\Delta\lambda \to 0$) and expectations in the expression $\lim_{\Delta\lambda \to 0} E\{(\Psi_{\lambda+\Delta\lambda} - \Psi_\lambda)/\Delta\lambda\}$ (see theorem 1 in L'Ecuyer [7]). As is well known, a number of interesting systems do not verify this condition, and this has led Gong and Ho [5] to devise a variant of IPA which they called *smoothed perturbation analysis* (SPA). Basically, SPA applies when there exists a smoothing σ-field $\mathscr{G}$ such that the interchange of expectations and limits below is valid:

$$\lim_{\Delta\lambda \to 0} E\left[E\left(\frac{\Psi_{\lambda+\Delta\lambda} - \Psi_\lambda}{\Delta\lambda}\middle|\mathscr{G}\right)\right] = E\left[\lim_{\Delta\lambda \to 0} E\left(\frac{\Psi_{\lambda+\Delta\lambda} - \Psi_\lambda}{\Delta\lambda}\middle|\mathscr{G}\right)\right].$$

Let $\Delta\lambda > 0$. In the Rare Perturbation Analysis (RPA) method, $N^{\lambda+\Delta\lambda}$ can be generated by adding to N^λ a Poisson process $N^{\Delta\lambda}$. The $N^{\lambda+\Delta\lambda}$ process thus obtained is rarely different from the initial N^λ process on finite intervals, hence the terminology "rare perturbation". The RPA method just described is the *positive RPA method*. In the *negative RPA method*, which we shall describe in this note, a Poisson process $N^{\lambda-\Delta\lambda}$ is created by thinning a Poisson process N^λ with the thinning probability $\Delta\lambda/\lambda$. Here again, on finite intervals, N^λ and $N^{\lambda-\Delta\lambda}$ seldom differ.

In [14], Vázquez-Abad and Kushner give a detailed description of the construction of a finite difference estimator (for a particular performance index) from the simulation of the path N^λ using these processes.

A related, but different idea was proposed by Suri and Cao [12], who studied the effect of the removal or the addition of one customer in a closed queueing network. In the case of a removal they called their method the *marked* customer method, and in the case of an addition, they called it the *phantom* customer method (this was also the terminology used in Vázquez-Abad and Kushner [14]). The system perturbation they studied is finite whereas in the present article we are aiming at the computation of derivatives. However, in the limit $\Delta\lambda = 0$ we end up with phantom and virtual customers.

Our terminology will differ from that of Suri and Cao [12]: for us their marked or tagged customer will be called a *phantom*, whereas their phantom will be called a *virtual* customer. The negative RPA method that we study here is associated with phantoms, for which reason we call it the phantom RPA method. Analogously, the positive RPA method deals with virtual customers and could be called the virtual customer RPA method.

The positive RPA method may be attributed to Reiman and Simon [8] and Simon [10], and is also revisited by Brémaud and Baccelli [1]. The basic idea of rare perturbation analysis is to keep the trajectories of the two systems that one wishes to compare *identical* with maximum probability, that is, keeping the probability of a perturbation small and then evaluating the limiting expectations. See Brémaud [2] for more details on the method. This is in contrast with IPA, where the trajectories are kept different but close when the parameter increment $\Delta\lambda$ is small.

Another well known method for estimating sensitivities is the Likelihood Ratio Method (LRM), studied in Reiman and Weiss [9]. For more references on the likelihood ratio technique, see also Glynn [3] and L'Ecuyer [7]. In section 3 below an interesting connection between LRM and RPA will be given.

This article is organized as follows. In section 2 we derive a formula for the derivative of the expected value of a quantity that can be computed over a busy period of a queueing system, with respect to the intensity of the input Poisson process. This leads to a formula that is similar to the formula obtained by Gong in [4]. However, the formula that we obtain is relative to a random horizon, in contrast with Gong [4] who considered fixed times or arrival times of a fixed customer.

For technical reasons an assumption, which is not essential to the problem, had to be introduced, restricting the validity of our proof to a class of problems. Instead of attempting to remove this assumption, we adopt a radically different approach to obtain a different type of formula concerning the derivative with respect to a routing probability, where the routing is applied to an arbitrary point process. This formula will be called the two-sided RPA formula. Its relation with the one-sided version of Gong [4] is explained in section 3. The proof of section 3 allows us to extend the domain of validity of the SPA formulae to stopping times and performance measures verifying a very mild condition. (The stopping times considered in [4] were either a fixed time or the arrival time of a given customer.)

The proof in section 3 is obtained via a pathwise interpretation of the LRM formulae for this problem. In section 4 we give a heuristic comparison of the LRM and RPA approaches in the situation considered in section 2. We also present the results from a simulation designed to compare RPA, LRM and a finite difference RPA.

2. The phantom RPA

Consider a queueing system defined for non-negative times and starting empty at the origin of time. The arrival times of the customers form a Poisson process, homogeneous with intensity λ, which is denoted N^{λ}. The sequence of arrival times is denoted $\{T_n, n \in \mathbb{N}_+\}$, with $T_0 \equiv 0 < T_1 < T_2 < \dots$ and customer number n arriving at time T_n is endowed with an attribute random variable Z_n representing for example its service requirement σ_n, its route through the network or its priority class. The sequence $\{Z_n, n \in \mathbb{N}_+\}$ is assumed i.i.d. and independent of N^λ.

Now let $\{X_n, n \in \mathbb{N}_+\}$ be an i.i.d. sequence, independent both of N^λ and $\{Z_n\}$, of $\{0, 1\}$-valued random variables with:

$$P(X_n = 1) = \frac{\Delta\lambda}{\lambda},$$

with $\lambda > \Delta\lambda > 0$. A new queueing system is formed by deleting customer number n if and only if $X_n = 1$, for each $n \geqslant 0$. Clearly the resulting arrival process forms a Poisson process with intensity $\lambda - \Delta\lambda$. The original system will be called the λ-system and the one derived from it by cancellation of customers the $(\lambda - \Delta\lambda)$-system.

It will be assumed that the λ-system is recurrent in the sense that it becomes empty infinitely often with probability 1. The time of termination of the first busy period (defined as the epoch of the next arrival that finds an empty system) will be denoted by S_1 (thus $S_1 < \infty$, a.s.). We also assume that the $(\lambda - \Delta\lambda)$-system is dominated by the λ-system, that is, whenever the λ-system is empty, the $(\lambda - \Delta\lambda)$-system is empty too. This hypothesis will later be removed.

Let Ψ be some operating characteristics that can be computed on one cycle of a queueing system, for instance the number of customers served in one cycle or the time spent at level k (k customers in the system) during one cycle. For the λ-system, this quantity is denoted:

$$\Psi_\lambda = \Psi(N^\lambda, \{Z_n\}, \emptyset), \tag{1}$$

thus expressing dependency upon N^λ and $\{Z_n\}$. Here the empty set $\emptyset$ plays only a notational role in order to distinguish Ψ_λ from the expression for the $(\lambda - \Delta\lambda)$-system:

$$\Psi_{\lambda-\Delta\lambda} = \Psi(N^\lambda, \{Z_n\}, \{X_n\}). \tag{2}$$

In Vázquez-Abad and Kushner [14], a finite difference estimator is constructed using these two processes, so that the bias of the estimator depends on the chosen value of $\Delta\lambda$, which is kept positive. The purpose of the present work is to compute the left hand derivative:

$$\frac{\mathrm{d}^-}{\mathrm{d}\lambda} E[\Psi] = \lim_{\Delta\lambda \downarrow 0} \frac{1}{\Delta\lambda} E[\Psi_\lambda - \Psi_{\lambda-\Delta\lambda}]. \tag{3}$$

In order to do so, we need more notations. Call $N_{S_1}^\lambda$ the number of points of N^λ in $(0, S_1)$, call M_{S_1} the number of points in $(0, S_1)$ for which $X_n = 1$, and call $A(n, k)$ a random set of k integers among $\{1, 2, \ldots, n\}$ without repetition (of course, here $k \leqslant n$). It is clear that, for some function g:

$$\Psi_{\lambda-\Delta\lambda} = g\left(N^\lambda, \{Z_n\}, A\left(N_{S_1}^\lambda, M_{S_1}\right)\right), \quad \text{and} \tag{4}$$

$$\Psi_\lambda = g\left(N^\lambda, \{Z_n\}, \emptyset\right) \tag{5}$$

In order to simplify notation, we discard the superscript λ in what follows. Taking expectations, we get:

$$E[\Psi_{\lambda-\Delta\lambda}-\Psi_\lambda] = \sum_{k=0}^{\infty} E\Big[\big(g\big(N, \{Z_n\}, A(N_{S_1}, k)\big) - g(N, \{Z_n\}, \emptyset)\big)\mathbf{1}_{\{M_{S_1}=k\}}\Big]$$
$$= \sum_{k=1}^{\infty} E\Big[\big(g\big(N, \{Z_n\}, A(N_{S_1}, k)\big) - g(N, \{Z_n\}, \emptyset)\big)\mathbf{1}_{\{M_{S_1}=k\}}\Big],$$

since $g(N, \{Z_n\}, A(N_{S_1}, 0)) = g(N, \{Z_n\}, \emptyset)$. Observing that:

$$P\big(M_{S_1}=k \mid N, \{Z_n\}, A(N_{S_1}, k)\big) = P\big(M_{S_1}=k \mid N_{S_1}\big)$$
$$= \binom{N_{S_1}}{k}\left(\frac{\Delta\lambda}{\lambda}\right)^k\left(1-\frac{\Delta\lambda}{\lambda}\right)^{N_{S_1}-k},$$

we have

$$\frac{1}{\Delta\lambda}E[\Psi_{\lambda-\Delta\lambda}-\Psi_\lambda] = E\left[\big(g\big(N, \{Z_n\}, A(N_{S_1}, 1)\big) - g(N, \{Z_n\}, \emptyset)\big)\times\frac{N_{S_1}}{\lambda}\right]$$
$$+\sum_{k\geqslant 2} E\left[\big(g\big(N, \{Z_n\}, A(N_{S_1}, k)\big) - g(N, \{Z_n\}, \emptyset)\big)\right.$$
$$\left.\times\frac{1}{\lambda}\binom{N_{S_1}}{k}\left(\frac{\Delta\lambda}{\lambda}\right)^{k-1}\left(1-\frac{\Delta\lambda}{\lambda}\right)^{N_{S_1}-k}\right]$$
$$-E\left[\big(g\big(N, \{Z_n\}, A(N_{S_1}, 1)\big) - g(N, \{Z_n\}, \emptyset)\big)\right.$$
$$\left.\times\frac{N_{S_1}}{\lambda}\left(1-\left(1-\frac{\Delta\lambda}{\lambda}\right)^{N_{S_1}-1}\right)\right].$$

Let us observe now that:

$$\left[\sum_{k\geqslant 2}\binom{N_{S_1}}{k}\left(\frac{\Delta\lambda}{\lambda}\right)^{k-1}\left(1-\frac{\Delta\lambda}{\lambda}\right)^{N_{S_1}-k}\right]+\frac{N_{S_1}}{\lambda}\left(1-\left(1-\frac{\Delta\lambda}{\lambda}\right)^{N_{S_1}-1}\right)$$
$$\leqslant\frac{\Delta\lambda}{\lambda}N_{S_1}+\frac{\Delta\lambda}{\lambda^2}N_{S_1}\big(N_{S_1}-1\big).$$

Therefore under certain integrability conditions, we obtain the following *differentiation formula*:

$$\lim_{\Delta\downarrow 0}\frac{1}{\lambda}E[\Psi_\lambda-\Psi_{\lambda-\Delta\lambda}] = E\left[\big(g(N, \{Z_n\}, \emptyset) - g\big(N, \{Z_n\}, A(N_{S_1}, 1)\big)\big)\frac{N_{S_1}}{\lambda}\right]. \tag{6}$$

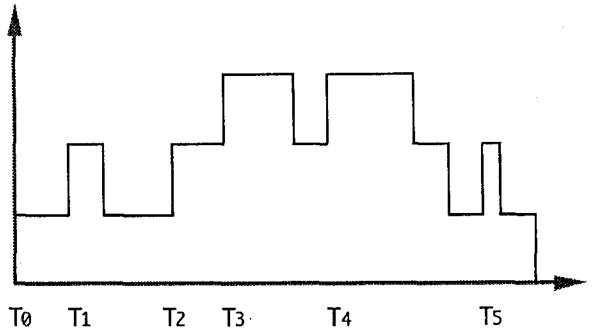

Fig. 1. A busy cycle of the N^{λ}-system.

We shall not give the sharpest conditions under which (6) is valid. Instead, we shall consider two particular cases of interest:

$$|g(N, \{Z_n\}, A(N_{S_1}, k))| \leqslant CN_{S_1} \text{ (resp. } CS_1), \tag{7}$$

where C is a positive constant. These cases occur when Ψ_λ is a sum over the points of N in $[0, S_1)$, or an integral over $[0, S_1)$ of a bounded quantity. In that case, a sufficient condition for (6) to hold is:

$$E\left[N_{S_1}^3\right] < \infty \qquad \left(\text{resp. } E\left[S_1, N_{S_1}\right] < \infty.\right) \tag{8}$$

An elementary example will help to clarify the notations.

Example: M / GI / 1 / ∞ / FIFO

Consider a single queue with Poisson arrival stream. The object is to compute by simulation the derivative of the average number of customers served in one busy period. Let us construct the congestion process $\{X(t)\}$.

Figure 1 shows one busy period of the trajectory, starting exactly at time $T_0 = 0$, where an arrival to an empty queue has occurred. We have labelled this customer as customer number zero, thus what we call the kth arrival (or customer) actually corresponds to the kth arrival *after* the start of the busy period (recall that the definitions of N_{S_1} and M_{S_1} are based on the open interval (T_0, S_1)).

Here $g(N, \{\sigma_n\}, \emptyset) = 6$. Since the number N_{S_1} of arrivals in the interval $(0, S_1)$ is 5, the quantity $A(N_{S_1}, 1)$ can take five values: $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, and $\{5\}$. With $A(N_{S_1}, 1) = \{1\}$ the first customer after the start of the cycle is a phantom and the congestion process becomes as shown in fig. 2 and thus $g(N, \{\sigma_n\}, \{1\}) = 1$.

If the phantom customer is the last one arriving to the busy period, that is $A(N_{S_1}, 1) = \{5\}$, then the congestion process becomes as depicted in fig. 3 and $g(N, \{Z_n\}, \{5\}) = 5$.

Remark

In formula (6), the quantity $g(N, \{Z_n\}, A(N_{S_1}, 1))$ related to choosing one of

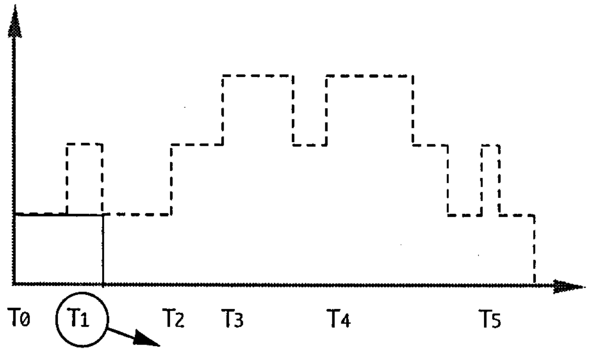

Fig. 2. Process $N^{\lambda-\Delta\lambda}$ when the first customer is a phantom.

the customers of the busy period at random, can be replaced by:

$$\frac{1}{N_{S_1}}\sum_{k=1}^{N_{S_1}} g(N, \{Z_n\}, \{k\}),$$

which represents the average over all possible phantoms. Indeed, we have:

$$\begin{aligned}
&E\left[g\left(N, \{Z_n\}, A(N_{S_1}, 1)\right)\right]\\
&\quad = E\left[\sum_{k=1}^{\infty} g(N, \{Z_n\}, \{k\})\mathbf{1}_{\{k \leqslant N_{S_1}\}}\mathbf{1}_{\{A(N_{S_1},1)=k\}}\right]\\
&\quad = E\left[\sum_{k=1}^{\infty} g(N, \{Z_n\}, \{k\})E\left[\mathbf{1}_{\{k \leqslant N_{S_1}\}}\mathbf{1}_{\{A(N_{S_1},1)=k\}} \mid N, \{Z_n\}\right]\right].
\end{aligned}$$

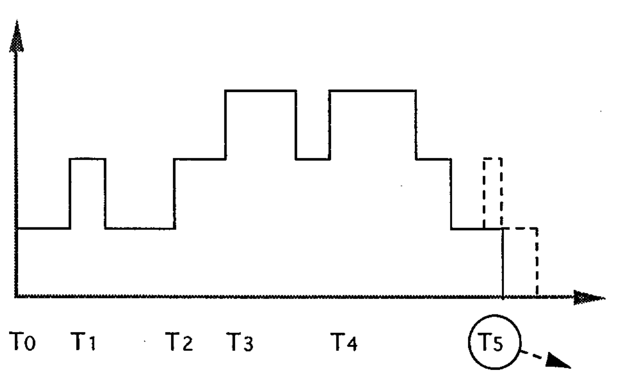

Fig. 3. Process $N^{\lambda-\Delta\lambda}$ when the last customer is a phantom.

The inner conditional expectation can be readily evaluated, obtaining:

$$E\left[g\left(N, \{Z_n\}, A\left(N_{S_1}, 1\right)\right)\right] = E\left[\sum_{k=1}^{\infty} g(N, \{Z_n\}, \{k\})\left\{\mathbf{1}_{\{k \leqslant N_{S_1}\}} \frac{1}{N_{S_1}}\right\}\right]$$

$$= E\left[\frac{1}{N_{S_1}} \sum_{k=1}^{N_{S_1}} g(N, \{Z_n\}, \{k\})\right],$$

so that formula (6) can be rewritten as:

$$\lim_{\Delta\lambda \downarrow 0} \frac{1}{\Delta\lambda} E[\Psi_\lambda - \Psi_{\lambda-\Delta\lambda}] = \frac{1}{\lambda} E\left[\sum_{k=1}^{N_{S_1}} \{g(N, \{Z_n\}, \emptyset) - g(N, \{Z_n\}, \{k\})\}\right]. \tag{9}$$

The above modification introduces a gain in simulation at the expense of an extra computational effort (which may depend on the traffic intensity and will be discussed in section 4) since the quantity of which the expectation is taken in the right hand side of (9) has smaller variance than the corresponding one in (6). In order to see this, notice that:

$$E\left[\left\{\sum_{k=1}^{N_{S_1}} [g(N, \{Z_n\}, \emptyset) - g(N, \{Z_n\}, \{k\})]\right\}^2\right]$$

$$= E\left[\left\{N_{S_1}\left[g(N, \{Z_n\}, \emptyset) - g\left(N, \{Z_n\}, A\left(N_{S_1}, 1\right)\right)\right]\right\}\right.$$

$$\left.\times\left\{N_{S_1}\left[g(N, \{Z_n\}, \emptyset) - g\left(N, \{Z_n\}, A'\left(N_{S_1}, 1\right)\right)\right]\right\}\right],$$

where the random sets $A(N_{S_1}, 1)$ and $A'(N_{S_1}, 1)$ are independent given N_{S_1}. By Schwarz's inequality, the above quantity is smaller than or equal to:

$$E\left[\left\{N_{S_1}\left[g(N, \{Z_n\}, \emptyset) - g\left(N, \{Z_n\}, A\left(N_{S_1}, 1\right)\right)\right]\right\}^2\right].$$

Equality holds only if $g(N, \{Z_n\}, A(N_{S_1}, 1))$ and $g(N, \{Z_n\}, A'(N_{S_1}, 1))$ are mutually proportional and this is generally not the case.

Remark

Suppose that the rate λ was obtained by thinning a Poisson process with rate $\lambda_0 > \lambda$ with thinning probability θ, so that $\lambda = \theta\lambda_0$ and that the rate $\lambda - \Delta\lambda$ is obtained by the thinning probability $\theta - \Delta\theta$ so that $\lambda - \Delta\lambda = \lambda_0(\theta - \Delta\theta)$ and therefore:

$$\frac{\Delta\lambda}{\Delta\theta} = \frac{\lambda}{\theta}.$$

Thus if θ is the parameter with respect to which the derivative is computed, then:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}E\left[\tilde{\Psi}_\theta\right] = \frac{\mathrm{d}}{\mathrm{d}\lambda}E[\Psi_\lambda]\frac{\mathrm{d}\lambda}{\mathrm{d}\theta} = \frac{\mathrm{d}}{\mathrm{d}\lambda}E[\Psi_\lambda]\frac{\lambda}{\theta},$$

where $\tilde{\Psi}_\theta = \Psi_{\lambda_0\theta} = \Psi_\lambda$. We therefore recover Gong's formula:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}E\left[\tilde{\Psi}_\theta\right] = \frac{1}{\theta}E\left[\sum_{k=1}^{N_{S_1}}\{g(N, \{Z_n\}, \phi) - g(N, \{Z_n\}, \{k\})\}\right].$$

HIGHER ORDER DERIVATIVES

Since the expression of the derivative in (6) is also a function of type (1), one can reiterate the procedure just described in order to compute the second order derivative of Ψ with respect to λ. From (9), we have:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}E[\Psi_\lambda] = \frac{1}{\lambda}E[h(N, \{Z_n\}, \emptyset)], \tag{9'}$$

where

$$h(N, \{Z_n\}, \emptyset) = \sum_{k=1}^{N_{S_1}}\{g(N, \{Z_n\}, \emptyset) - g(N, \{Z_n\}, \{k\})\}, \tag{10}$$

and therefore the second derivative is of the form:

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}E[\Psi] = &-\frac{1}{\lambda^2}E[h(N, \{Z_n\}, \emptyset)]\\ &+\frac{1}{\lambda^2}E\left[\sum_{l=1}^{N_{S_1}}\{h(N, \{Z_n\}, \emptyset) - h(N, \{Z_n\}, \{l\})\}\right],\end{aligned}$$

which yields:

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}E[\Psi] = &-\frac{1}{\lambda^2}E\left[\sum_{k=1}^{N_{S_1}}\{g(N, \{Z_n\}, \emptyset) - g(N, \{Z_n\}, \{k\})\}\right]\\ &+\frac{1}{\lambda^2}E\left[\sum_{k=1}^{N_{S_1}}\sum_{l=1}^{N_{S_1}}\{g(N, \{Z_n\}, \emptyset) - 2g(N, \{Z_n\}, \{k\})\right.\\ &\left.+g(N, \{Z_n\}, \{k, l\})\}\right].\end{aligned} \tag{11}$$

From the arguments just presented it is clear that the computational effort involved in estimating the second (or higher) derivatives involves evaluation of the effects of two (or more) phantoms within the same busy period.

3. Derivation of RPA via likelihood ratios

The hypothesis of domination of the λ-system over the $(\lambda - \Delta\lambda)$-system is in fact restrictive for a number of systems of interest. However, removal of this assumption using the method presented in the previous section is technically difficult. We shall consider a different approach, less direct but more general, and with the additional advantage that it connects the LRM of Reiman and Weiss [9] with the RPA method. We shall illustrate the method with an example considered by Gong [4]. This example fits the general framework given below.

THE GENERAL FRAMEWORK

Let $\{(T_n, Z_n), n \geqslant 0\}$ and $\{X_n, n \geqslant 1\}$ be two sequences of random elements, and let P_θ, $\theta \in [0, 1]$ be a family of probability measures defined on the common probability space where the random elements live. The following assumptions are made:

$$\left.\begin{array}{l} \{(T_n, Z_n)\} \text{ and } \{X_n\} \text{ and } P_\theta - \text{independent.} \\ X_n \in \{0, 1\}, \{X_n, n \geqslant 1\} \text{ is i.i.d. with } P_\theta(X_n = 1) = \theta. \\ \text{The law of } \{(T_n, Z_n)\} \text{ is independent of } \theta. \end{array}\right\} \tag{12}$$

Define $X_0 \equiv 1$ and let $\mathscr{F}_k$ be the σ-field generated by $\{(X_i, T_i, Z_i), 0 \leqslant i \leqslant k\}$. Let N be a random variable with values in $\mathbb{N}$ such that $\{N = k\} \in \mathscr{F}_k$ for all $k \geqslant 0$, that is, N is an $\mathscr{F}_k$-stopping time.

Let Ψ be an $\mathscr{F}_N$-measurable functional and suppose that:

$$E_\theta[\,|\Psi|\,] < \infty \quad \text{for all } \theta \in [0, 1]. \tag{13}$$

Example

We consider an example from Gong [4]. Here $T_0 \equiv 0$ and the sequences $\{T_{n+1} - T_n, n \geqslant 0\}$ are i.i.d. with values in $(0, \infty)$; $\{Z_n, n \geqslant 0\}$ are i.i.d. with values in $(0, \infty)$. The random variable T_n is interpreted as the arrival time of customer number n in a queueing system and Z_n is the service requirement of this customer, denoted σ_n as before. If $X_n = 1$ then the nth customer is accepted in the system, otherwise it is rejected. We have just described a GI/G input flow with random Bernoulli acceptance. Let the queueing system be, for instance, one with K servers at unit speed, an infinite capacity waiting room and first-come-first-served service discipline. Assume the system to be stable for all $\theta \in [0, 1]$, that is to say:

$$\frac{E[\sigma_1]}{E[T_1]} < K.$$

Suppose the queue is empty at time $T_0^- = 0^-$. In view of the stability assumption the queue eventually empties (a.s.). We call S the first beginning of a busy period after $T_0 = 0$ and N the number of customers that arrived within $(0, S)$, which is also the number of customers served in the first busy cycle. For the functional Ψ take, for instance:

$$\Psi = \int_0^S \mathbf{1}_{\{Q(s) > C\}} \, \mathrm{d}s$$

for a fixed constant C, where $\{Q(t), t \geqslant 0\}$ is the congestion process, i.e. $Q(\cdot)$ is the number of customers in the waiting line (accepted but not receiving service). This particular choice counts the time during which the congestion process exeeds a particular value C. As another example, take:

$$\Psi = \sum_{i=0}^{\infty} W_i X_i \mathbf{1}_{\{T_i < S\}},$$

where W_i is the time customer i waits in line. This functional Ψ is the cumulative waiting time of all customers accepted in the first busy period.

Remark

Since the phantom method involves computation of the resulting trajectory when customers are removed, the pathwise domination assumption depends on how we model the system. In particular, for a single server FIFO queue this property follows if we associate the service time with the customers. This would be the case in the present example for $K = 1$.

THE LRM PERTURBATION ANALYSIS

The objective is to find an expression for the derivative $\mathrm{d}E_\theta[\Psi]/\mathrm{d}\theta$ of the form $E_\theta[\Phi]$. To obtain this we start with the likelihood ratio approach: we assume first without loss of generality that $P_{\theta+\Delta\theta}$ is constructed from P_θ in such a way that for all $k \geqslant 1$, $P_{\theta+\Delta\theta} \ll P_\theta$ on F_k and:

$$\begin{aligned}
\left.\frac{\mathrm{d}P_{\theta+\Delta\theta}}{\mathrm{d}P_\theta}\right|_{\mathscr{F}_k} &= \prod_{i=1}^{k}\left[\frac{\theta+\Delta\theta}{\theta}X_i\right]\prod_{i=1}^{k}\left[\frac{1-\theta-\Delta\theta}{1-\theta}(1-X_i)\right] \\
&= \left(1+\frac{\Delta\theta}{\theta}\right)^{\sum_{i=1}^{k}Xi}\left(1-\frac{\Delta\theta}{1-\theta}\right)^{k-\sum_{i=1}^{k}X_i} \\
&\equiv L_k(\theta, \Delta\theta),
\end{aligned} \tag{14}$$

and we assume that

$$\left.\frac{\mathrm{d}P_{\theta+\Delta\theta}}{\mathrm{d}P_\theta}\right|_{\mathscr{F}_N} = L_N(\theta, \Delta\theta). \tag{15}$$

Since N is a stopping time, (15) in general requires a proof (and special conditions) that will be given later.

We have:

$$\frac{1}{\Delta\theta}\left(E_{\theta+\Delta\theta}[\Psi]-E_\theta[\Psi]\right)=E_\theta\left[\Psi\left(\frac{L_N(\theta,\Delta\theta)-1}{\Delta\theta}\right)\right].$$

An easy computation yields:

$$\lim_{\Delta\theta\to 0}\frac{L_N(\theta,\Delta\theta)-1}{\Delta\theta}=\sum_{i=1}^{N}\left[\frac{1}{\theta}X_i-\frac{1}{1-\theta}(1-X_i)\right]. \tag{16}$$

Let us assume the following limit holds (a proof and conditions for this are given below):

$$\lim_{\Delta\theta\to 0}E_\theta\left[\Psi\left(\frac{L_N(\theta,\Delta\theta)-1}{\Delta\theta}\right)\right]=E_\theta\left[\Psi\left(\sum_{i=1}^{N}\left[\frac{1}{\theta}X_i-\frac{1}{1-\theta}(1-X_i)\right]\right)\right], \tag{17}$$

then it follows that:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}E_\theta[\Psi]=E_\theta\left[\Psi\left(\sum_{i=1}^{N}\left[\frac{1}{\theta}X_i-\frac{1}{1-\theta}(1-X_i)\right]\right)\right]. \tag{18}$$

INTEGRABILITY CONDITIONS

We shall now give conditions under which (15) and (17) are satisfied. For (15), the condition:

$$P_\theta(N<\infty)=1 \quad \text{for all } \theta\in[0,1] \tag{19}$$

is enough. In order to see this, we have to show that for all $A\in\mathscr{F}_N$

$$E_\theta[\mathbf{1}_A L_N(\theta,\Delta\theta)]=E_{\theta+\Delta\theta}[\mathbf{1}_A].$$

But if $N<\infty$ ($P_{\theta+\Delta\theta}$-a.s.), then $\sum_{k=0}^{\infty}\mathbf{1}_{\{N=k\}}=1$ ($P_{\theta+\Delta\theta}$-a.s.), and therefore, taking into account the fact that $A\cap\{N=k\}\in\mathscr{F}_k$ for all $k\geqslant 0$,

$$\begin{aligned}E_{\theta+\Delta\theta}[\mathbf{1}_A]&=E_{\theta+\Delta\theta}\left[\sum_{k=0}^{\infty}\mathbf{1}_{A\cap\{N=k\}}\right]=\sum_{k=0}^{\infty}E_{\theta+\Delta\theta}[\mathbf{1}_{A\cap\{N=k\}}]\\&=\sum_{k=0}^{\infty}E_\theta[\mathbf{1}_{A\cap\{N=k\}}L_k]=E_\theta\left[\sum_{k=0}^{\infty}\mathbf{1}_{A\cap\{N=k\}}L_k\right]\\&=E_\theta[\mathbf{1}_A L_N],\end{aligned}$$

where we have used for the last equality the assumption $N<\infty$ (P_θ-a.s.), which guarantees that $\sum_{k=0}^{\infty}\mathbf{1}_{A\cap\{N=k\}}=\mathbf{1}_A$ (P_θ-a.s.).

We now turn to the expression (17). Call $M_1=\Sigma X_i$, $M_2=\Sigma(1-X_i)$, then it follows from (14) that the second derivative of $L_N(\theta,h)$ with respect to h is:

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}h^2}L_N(\theta,h)=L_N(\theta,h)&\left\{\frac{M_1(M_1-1)}{(\theta+h)^2}+\frac{M_2(M_2-1)}{(1-\theta-h)^2}\right.\\&\left.-2\frac{M_1M_2}{(\theta+h)(1-\theta-h)}\right\}.\end{aligned}$$

By Taylor's formula, we have:

$$\Psi \frac{L_N(\theta, h) - 1}{h} = \Psi\left(\frac{M_1}{\theta} - \frac{M_2}{1-\theta}\right) + h\Psi L_N(\theta, \alpha h) \times \left\{ \frac{M_1(M_1-1)}{(\theta+\alpha h)^2} + \frac{M_2(M_2-1)}{(1-\theta-\alpha h)^2} - 2\frac{M_1 M_2}{(\theta+\alpha h)(1-\theta-\alpha h)} \right\} \tag{20}$$

for some (random) $\alpha \in (0, 1)$. The quantity inside brackets in (20) is bounded by KN^2 for some $K > 0$ (not random but depending upon θ) as long as h is sufficiently close to 0. Also, for any $\epsilon > 0$ and sufficiently small h, $|L_n(\theta, \alpha h)| < \mathrm{e}^{\epsilon N}$. Therefore, if:

$$E_\theta\left[\,|\Psi|\mathrm{e}^{\epsilon N}\right] < \infty \quad \text{for some } \epsilon > 0, \tag{21}$$

then (18) holds true, as one can see from (20).

Remark

If $|\Psi|$ is bounded by some polynomial in N, condition (21) is implied by $E_\theta[\mathrm{e}^{\epsilon N}] < \infty$ for some $\epsilon > 0$.

Remark

There are situations in which a condition weaker than (21) is sufficient. Indeed, using Lagrange's residual

$$\Psi \frac{L_N(\theta, h) - 1}{h} = \Psi\left(\frac{M_1}{\theta} - \frac{M_2}{1-\theta}\right) + \frac{1}{h}\int_0^h (h-t)\frac{\mathrm{d}^2}{\mathrm{d}h^2} L_N(\theta, t)\Psi \,\mathrm{d}t$$

in absolute value, the expression for the residual term is bounded by:

$$\frac{1}{h}\int_0^h (h-t)E_\theta\left[L_N(\theta, t)KN^2|\Psi|\right]\mathrm{d}t = \frac{K}{h}\int_0^h (h-t)E_{\theta+t}\left[\,|\Psi|N^2\right]\mathrm{d}t,$$

which goes to 0 as $h \to 0$ if $E_{\theta+t}[\,|\Psi|N^2]$ is bounded in a t-neighborhood of 0.

Suppose for instance that $|\Psi| < \gamma N$ for some $\gamma > 0$; it then suffices that $E_{\theta+t}[N^3]$ be bounded in a t-neighborhood of 0. The domination assumption of section 2 precisely guarantees such a condition.

MODIFICATION OF LRM

We now turn to the analysis that yields the two-sided RPA estimators. Define for each $i \in \mathbb{N}_+$ the probability measures P_{+i} and P_{-i} by:

$$\frac{\mathrm{d}P_{+i}}{\mathrm{d}P_\theta} = \frac{1}{\theta}\mathbf{1}_{\{X_i=1\}}, \quad \frac{\mathrm{d}P_{-i}}{\mathrm{d}P_\theta} = \frac{1}{1-\theta}\mathbf{1}_{\{X_i=0\}}. \tag{22}$$

P_{+i} gives to $\{(T_n, Z_n), n \in \mathbb{N}; X_n, n \in \mathbb{N}_+ - \{i\}\}$ the same distribution as P_θ, the only difference between P_θ and P_{+i} being that $P_{+i}(X_i = 1) = 1$. An analogous statement is true for P_{-i}, with $P_{-i}(X_i = 0) = 1$.

Therefore, denoting by Ψ_{+i} the functional Ψ computed with the ith customer accepted (even if $X_i = 0$), and by Ψ_{-i} the functional Ψ calculated with the ith customer always rejected, we have:

$$\begin{aligned} E_\theta\left[\Psi \mathbf{1}_{\{i \leqslant N\}} \frac{1}{\theta} \mathbf{1}_{\{X_i=1\}}\right] &= E_\theta\left[\Psi_{+i} \mathbf{1}_{\{i \leqslant N_{+i}\}}\right], \\ E_\theta\left[\Psi \mathbf{1}_{\{1 \leqslant N\}} \frac{1}{1-\theta} \mathbf{1}_{\{X_i=0\}}\right] &= E_\theta\left[\Psi_{-i} \mathbf{1}_{\{i \leqslant N_{-i}\}}\right], \end{aligned} \tag{23}$$

where N_{+i} and N_{-i} receive the obvious interpretation (in the example above, they represent the number of customers served within the first busy cycle when customer number i is either accepted or rejected with probability 1). Observe that if $i \geqslant N$ then $N_{+i} = N_{-i} = N$, and $\Psi_{+i} = \Psi_{-i} = \Psi$ and therefore:

$$\begin{aligned} &E_\theta\left[\Psi_{+i} \mathbf{1}_{\{i \leqslant N_{+i}\}} - \Psi_{-i} \mathbf{1}_{\{i \leqslant N_{-i}\}}\right] \\ &\quad = E_\theta\left[\left(\Psi_{+i} \mathbf{1}_{\{i \leqslant N_{+i}\}} - \Psi_{-i} \mathbf{1}_{\{i \leqslant N_{-i}\}}\right) \mathbf{1}_{\{i \leqslant N\}}\right] \\ &\qquad + E_\theta\left[\left(\Psi_{+i} \mathbf{1}_{\{i \leqslant N_{+i}\}} - \Psi_{-i} \mathbf{1}_{\{i \leqslant N_{-i}\}}\right) \mathbf{1}_{\{i > N\}}\right] \\ &\quad = E_\theta\left[\left(\Psi_{+i} \mathbf{1}_{\{i \leqslant N_{+i}\}} - \Psi_{-i} \mathbf{1}_{\{i \leqslant N_{-i}\}}\right) \mathbf{1}_{\{i \leqslant N\}}\right], \end{aligned}$$

since the term involving $\mathbf{1}_{\{i > N\}}$ vanishes. We shall assume that $i \leq N$ implies $i \leqslant N_{+i}$ and $i \leqslant N_{-i}$, which is not a restrictive assumption. Indeed, if N is the entrance time (measured in customer number) to a set G of an $\mathscr{F}_n$-adapted process, then the situation $N_{\pm i} < i \leqslant N$ is impossible, because for all customers before i the (T_n, Z_n, X_n) process was undisturbed and thus it could not have reached the set G at $N_{\pm i} < i \leqslant N$. Notice that this condition is fulfilled in the case that N is the beginning of the next busy cycle. Under this assumption, we have:

$$\begin{aligned} &E_\theta\left[\Psi \mathbf{1}_{\{i \leqslant N\}} \frac{1}{\theta} \mathbf{1}_{\{X_i=1\}}\right] - E_\theta\left[\Psi \mathbf{1}_{\{i \leqslant N\}} \frac{1}{1-\theta} \mathbf{1}_{\{X_i=0\}}\right] \\ &\quad = E_\theta\left[(\Psi_{+i} - \Psi_{-i}) \mathbf{1}_{\{i \leqslant N\}}\right]. \end{aligned} \tag{24}$$

Adding up over all possible i we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}\theta} E_\theta[\Psi] = E_\theta\left[\sum_{i=1}^{N} (\Psi_{+i} - \Psi_{-i})\right]. \tag{25}$$

A natural question arises: What is the relation between the two-sided RPA estimate (25) and the one-sided RPA estimate (9) of our previous section? In order to answer this question, we shall state the problem considered in section 2 in the general framework given at the beginning of the present section. The sequence $\{T_n\}$ is now a Poisson process with intensity λ_0, which upon thinning with acceptance probability $\theta = \lambda/\lambda_0$, becomes a Poisson process with rate λ. Since $d\theta = d\lambda/\lambda_0$, (25) can be written for this case as:

$$\frac{d}{d\lambda} E_\lambda[\Psi] = \frac{1}{\lambda_0} E_\lambda\left[\sum_{i=1}^{N} (\Psi_{+i} - \Psi_{-i})\right]. \tag{26}$$

According to the general framework, the thinning of the process with rate λ_0 is achieved through the random variables X_i, which depend on θ, thus obtaining a Poisson process with rate λ. $\lambda \leqslant \lambda_0$ ensures that θ is well defined as a probability. Equation (26) uses the original process with rate λ_0 in order to estimate the derivative at λ, but the actual estimates presented in section 2 were evaluated at $\lambda = \lambda_0$, which corresponds to $\theta = 1$. The likelihood ratio technique cannot be used at extreme points of the parameters (in this case, eqs. (22) do not make sense for $\theta = 0$ or $\theta = 1$), since absolute continuity of P_θ ($\theta \in (0, 1)$) with respect to P_{θ_0} ($\theta_0 = 1$ or 0) is lost.

However, we can still give a meaningful interpretation of the expression (26) when λ tends to λ_0 without using explicitly the parametrized familiy of probabilities $\{P_\theta\}$. Indeed, the functional Ψ_{+i} is the resulting value of the functional Ψ given that $X_i = 1$. In the limiting case when $\lambda = \lambda_0$, $X_i \equiv 1$, so that Ψ_{+i} becomes Ψ, whereas Ψ_{-i} becomes the functional evaluated when we phantomize the ith customer ($X_i = 0$). This interpretation yields exactly formula (9).

4. Performance of the phantom RPA method

COMPARISON BETWEEN RPA AND LRM

We shall compare the formula (6), which we rewrite here for convenience:

$$\frac{d}{d\lambda} E[\Psi_\lambda] = E\left[\left(g(N, \{Z_n\}, \emptyset) - g\big(N, \{Z_n\}, A(N_{S_1}, 1)\big)\right)\frac{N_{S_1}}{\lambda}\right], \tag{27}$$

to the formula that can be obtained from the likelihood ratio method (LRM) of Reiman and Weiss [9]:

$$\frac{d}{d\lambda} E[\Psi_\lambda] = E\left[\left(\frac{(N_{S_1} + 1)}{\lambda} - S_1\right) g\big(N, \{Z_n\}, \emptyset\big)\right]. \tag{28}$$

It is not easy to make a direct comparison between the RPA and the LRM methods of simulation based on formulas (27) and (28). Both methods yield essentially unbiassed estimators, since both would compute the empirical means of the quantities under expectation in the right hand side of their respective definitions. The comparisons must be based on their variance, and therefore one must work with the two quantities:

$$E\left[\left(g(N, \{Z_n\}, \emptyset) - g\left(N, \{Z_n\}, A(N_{S_1}, 1)\right)\right)^2 \frac{N_{S_1}^2}{\lambda^2}\right], \quad \text{(RPA)} \tag{29}$$

$$E\left[g(N, \{Z_n\}, \emptyset)^2 \left(\frac{(N_{S_1}+1)}{\lambda} - S_1\right)^2\right]. \quad \text{(LRM)} \tag{30}$$

This comparison is not obvious. We will show the advantage of RPA over LRM on a simulation example. Of course there is no claim of generality in these comparisons and one must remain cautious about the conclusions, but we believe these results do show the general behavior of the methods.

Before we discuss the simulation experiments, we shall give a heuristic argument in favor of RPA in the case where:

$$g(N, \{Z_n\}, \emptyset) \geqslant g(N, \{Z_n\}, \{k\}) \quad \text{for all } 1 \leqslant k \leqslant N_{S_1} \tag{31}$$

in which case, naturally

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} E[\Psi_\lambda] \geqslant 0. \tag{32}$$

The quantity for which we compute the mean with the right hand side of (27) is always non-negative, whereas in (28) it can be negative. Since they both have the same mean, there is room for the belief that RPA will give smaller variance. Let us consider a somewhat useless, but illustrative example, where $\Psi_\lambda \equiv 1$ is a constant. In this case RPA will yield exactly the correct value (0 with variance 0). In contrast, LRM gives an estimator with correct zero mean, but with variance $E[((N_{S_1}+1)/\lambda - S_1)^2] = E[S_1]/\lambda$.

Although this example is outrageously *ad hoc*, it points out a possibility that is actually verified in the simulations: the variance of the LRM method gets worse as the traffic intensity increases. This phenomenon is also experienced with the RPA method, but its amplitude is smaller.

RESULTS FROM THE SIMULATIONS

The simulations are made for an $M/M/1/\infty$ queue and the quantity of interest is the derivative of the expected number of customers served in a busy period. This simple example allows us to evaluate the exact expressions of the stationary averages of interest, which makes the qualitative analysis easier.

We shall compare three methods for simulations, the LRM, the RPA and the finite difference RPA of [14], consisting in the computation of the first order approximation of the derivative:

$$\frac{1}{\Delta\lambda} E[\Psi_\lambda - \Psi_{\lambda - \Delta\lambda}]$$

Table 1
The systems for the simulations

Simulation	μ	λ	ρ
1	1.0	0.4	0.400
2	1.2	0.8	0.666
3	1.0	0.8	0.800

for fixed $\Delta\lambda > 0$. It gives a bias that can be reduced by a proper choice of the parameter $\Delta\lambda$.

We ran the simulation for three different systems, whose parameter values are given in table 1, where:

λ = mean arrival rate of Poisson stream,
μ = mean service rate (μ^{-1} is the mean service time),
$\rho = \lambda/\mu$ = traffic intensity.

Let A_i be the number of services completed during the ith busy period (that is, $A_i = N_{S_i} + 1$). Then the problem is to estimate the stationary expectation of A_1 as well as its derivative with respect to λ.

From the expressions of the stationary probabilities of the $M/M/1/\infty$ queue, we have:

$$E_\lambda[A_1] = \frac{\mu}{\mu - \lambda}, \tag{33.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} E_\lambda[A_1] = \frac{\mu}{(\mu - \lambda)^2}, \tag{33.2}$$

$$\Delta_\lambda E_\lambda[A_1] \equiv \frac{E_\lambda[A_1] - E_{(\lambda - \Delta\lambda)}[A_1]}{\Delta\lambda} = \frac{1}{\Delta\lambda}\left(\frac{\mu}{\mu - \lambda} - \frac{\mu}{\mu - \lambda + \Delta\lambda}\right), \tag{33.3}$$

where the subscript λ denotes expectations with respect to the stationary measure of the process for parameter λ.

In order to study the properties of the estimators of the derivative constructed according to the different methods considered, we designed a Monte Carlo procedure. The process is simulated and every N busy periods the different estimators are calculated in the form of a sample average over those N cycles, evaluating a sample of K derivative estimators for each method. Specifically, for $k = 0, \dots, K-1$ we calculate:

$$Z_k^{(A)} = \frac{1}{N} \sum_{i=Nk+1}^{N(k+1)} A_i \left(\frac{A_i}{\lambda} - S_i\right), \tag{34.1}$$

$$Z_k^{(B)} = \frac{1}{N} \sum_{i=Nk+1}^{N(k+1)} \left[\frac{1}{\lambda} \sum_{j=2}^{A_i} (A_i - A_i(j)) \right], \tag{34.2}$$

$$Z_k^{(C)} = \frac{1}{N} \sum_{i=Nk+1}^{N(k+1)} \left[\frac{N_{S_i}}{\lambda} \left(A_i - A_i(j) \mathbf{1}_{\{j-2/N_{S_i} \leqslant U_i \leqslant j-1/N_{S_i}\}} \right) \right],$$

$$\text{for } j = 2, \ldots, A_i, \tag{34.3}$$

$$Z_k^{(D)} = \frac{1}{\Delta\lambda} \left[\frac{1}{N} \sum_{i=Nk+1}^{N(k+1)} A_i - \frac{1}{N_k^p} \sum_{i=Nk+1}^{N(k+1)} \sum_{j=1}^{A_i} \mathbf{1}_{\{X_i(j)=1\}} \right], \tag{34.4}$$

where $A_i(j)$, $2 \leqslant j \leqslant A_i$ denotes the quantity $g(N^\lambda, \{Z_n\}, \{j-1\})$: in this case the number of services in the ith busy period when the jth customer is a phantom. In (34.3), the random variable $0 \leqslant U_i \leqslant 1$ is uniformly distributed; that is, we choose at random one of the customers $2 \leqslant j \leqslant A_i$ within the ith busy period to be phantomized.

In the last formula, $\{X_i(j);\ j = 1, \ldots, A_i\}$ are i.i.d. random variables with $P(X_i(j) = 1) = 1 - P(X_i(j) = 0) = \Delta\lambda / \lambda$, and N_k^p denotes the number of busy periods that result in the phantom queue during the whole time interval $[S_{Nk}, S_{N(k+1)})$ over which the average $Z_k^{(D)}$ is evaluated ($S_0 = T_0 \equiv 0$ is the start of the simulation with the first arrival to an idle server). The number of total arrivals in the phantom queue is divided by this number to get the sample average of the number of customers per busy period in the $(\lambda - \Delta\lambda)$-system. By construction, then, the stationary expectation of $Z_1^{(D)}$ coincides with (33.3).

The quantity $E_\lambda[A_1]$ was estimated – naturally – through the sample averages

$$X_k = \frac{1}{N} \sum_{i=Nk+1}^{N(k+1)} A_i.$$

Each simulation (see table 2) consisted of $K = 50$ runs of $N = 10{,}000$ busy cycles, computing the sample mean and variance of the estimators according to:

$$\hat{Z} = \frac{1}{K} \sum_{k=1}^{K} Z_k, \tag{35.1}$$

$$\widehat{\operatorname{Var} Z} = \frac{1}{K} \sum_{k=1}^{K} \left(Z_k - \hat{Z} \right)^2, \tag{35.2}$$

Table 2

Methods for simulations	
A	LRM
B	Averaged RPA
C	Randomized RPA
D	Finite difference RPA

Table 3
Results from Monte Carlo simulation

Quantity	System 1 $\Delta\lambda = 0.005$		System 2 $\Delta\lambda = 0.010$		System 3 $\Delta\lambda = 0.010$	
$E_\lambda[A_i]$	1.67		3.00		5.00	
$\hat{X}$	1.6662	(0.0003)	2.9947	(0.0030)	5.0093	(0.0208)
$\frac{\mathrm{d}}{\mathrm{d}\lambda} E_\lambda[A_i]$	2.78		7.50		25.00	
$\Delta_\lambda E_\lambda[A_i]$	2.75		7.32		23.81	
$\hat{Z}^{(A)}$	2.7475	(0.0453)	7.2416	(0.5569)	24.6303	(10.6829)
$\hat{Z}^{(B)}$	2.7686	(0.0133)	7.3843	(0.1877)	25.0325	(4.7038)
$\hat{Z}^{(C)}$	2.7560	(0.0217)	7.3467	(0.4032)	23.5059	(8.7154)
$\hat{Z}^{(D)}$	2.5679	(0.2429)	7.1132	(0.5813)	23.3346	(4.1962)

for the different estimators in (34). It is important to mention that we simulated the system for only one trajectory, which was used in the evaluation of all the estimators. The results are summarized in table 3, where the means (35.1) and the sample variances (35.2) are reported for each system (the variances are shown in parentheses). Equations (33) were used to evaluate the theoretical values for each system.

As already mentioned, the finite difference RPA depends on a parameter $\Delta\lambda > 0$ which has to be fixed. We experimented with three different values of $\Delta\lambda$ for each system simulated. Our results coincide with the expected behavior: as $\Delta\lambda$ decreases, the finite difference (33.3) gets closer to the derivative (33.2) as does the average $\widehat{Z^{(D)}}$, at the expense of an increase in variance. We have chosen just one among these experiments to report in table 3.

For the RPA method, the simulation method B of the algorithm based on formula (9) yields less variance than the version of the algorithm based on formula (6) (method C), at the expense of an increase in computational complexity for heavier traffic conditions. Indeed, in both algorithms we keep track of the quantities $A_i(j)$ within each busy period, then in (34.2) we average them, thus the number of operations required depends on the random variable A_i, which increases with increasing ρ. In (34.3), however, a random variable is generated every time the busy period has finished, and the corresponding term is added to the estimator, so the operation count per busy cycle does not depend on the traffic.

It is clear from the above results that the phantom RPA method outperforms both the LRM and the finite difference RPA. However, the averaged RPA method, which exhibits the greatest advantage in terms of variance reduction, requires more computations than LRM for instance. Randomized RPA is less consuming, but as the traffic intensity ρ increases its advantage over LRM seems to diminish. In our example, it seems competitive when $\rho < 0.5$.

5. Concluding remarks

We have presented the Rare Perturbation Method for sensitivity analysis with two approaches: the first one is direct and explains the terminology used, the second one, via likelihod ratios, allowed us to give the RPA formulas with weak assumptions on the moments of the stopping times involved (ends of busy cycles, for instance). The formulas obtained in section 2 are similar to those obtained by Gong [4], whereas the formulas obtained in section 3 differ in that they are "two-sided".

As we have discussed, the method is applicable to GI/GI queues with randomly deleted arrivals. Such problems may be of interest for routing in queueing networks, using the surrogate estimation approach introduced in Vázquez-Abad [13]. The applicability of the phantom method to optimization of general queueing networks may represent an important alternative to currently available methods. In [13] and Vázquez-Abad and Kushner [14], the scheme uses a Monte Carlo estimation approach over time intervals of fixed length in order to calculate derivatives of stationary averages. A second order finite difference RPA was used to construct a controlling automaton for routing in data networks. It was shown in [14] that this sensitivity estimator worked better for that problem than an IPA method. It follows from the construction of the methods that whenever a finite difference RPA is applicable, we can readily implement the phantom RPA, and in view of the results shown here, it is our belief that the phantom RPA method could yield even better performance in such systems, although more experimentation is obviously required.

It should be mentioned that we have not included an IPA estimator in our simulations because of the problem we study: in the infinitesimal IPA, one generally assumes that a small value of $\Delta\lambda$ can be chosen so that the order of events is the same in the nominal and perturbed queues before passing to the limit. Under this assumption, the number of services within busy periods coincides in both systems, so the prediction from the IPA estimator in this case is zero.

It is further explained in Brémaud [2] why we have chosen to speak of rare perturbations in opposition to infinitesimal perturbations. Indeed, IPA changes *all* the perturbed random variables of the system very slightly, whereas RPA maintains all the random variables unperturbed except for rare exceptions. RPA applies when IPA does not, and most likely efficient perturbation methods will be a mixture of RPA and IPA.

The conditions under which RPA formulas are obtained allow the consideration of cycles and extend the validity of the analysis given in Gong and Ho [5]. This is of especial interest when one considers estimating derivatives of systems in equilibrium. Indeed, the cycle estimates can be used to compute stationary derivatives using the regenerative method. For instance in the $M/GI/1/\infty$

queue one has:

$$E_\lambda\{f[X(0)]\} = \frac{E_\lambda\left[\int_0^{S_1} f[X(s)]\,\mathrm{d}s\right]}{E_\lambda[S_1]} = \lambda\left(1-\frac{\lambda}{\mu}\right)E_\lambda[\Psi],$$

where $\Psi = \int_0^{S_1} f[X(s)]\,\mathrm{d}s$. Therefore:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}E_\lambda\{f[X(0)]\} = (1-2\rho)E_\lambda[\Psi] + \lambda(1-\rho)E_\lambda[\Phi],$$

with the quantity Φ defined by:

$$\Phi = \frac{1}{N_{S_1}}\left(\int_0^{S_1}\{f[X(s)] - f[X^k(s)]\}\,\mathrm{d}s\right),$$

where $X^k(\cdot)$ is the quantity corresponding to $X(\cdot)$ when customer number k has been phantomized at random.

References

[1] F. Baccelli and P. Brémaud, Virtual customers in sensitivity and light traffic analysis via Campbell's formula for point processes, submitted to AAP (1990).

[2] P. Brémaud, Maximal coupling and rare perturbation analysis, in preparation (1990).

[3] P. Glynn, Likelihood ratio gradient estimation for stochastic systems, Commun. ACM 33 (1990) 76–84.

[4] W.B. Gong, Smoothed perurbation analysis algorithm for a $G/G/1$ routing problem, in: *Proc. 1988 Winter Simulation Conf.*, eds. M. Abrams, P. Haigh and J. Comfort (1988) pp. 525–531.

[5] W.B. Gong and Y.C. Ho, Smoothed perturbation analysis of discrete event dynamical systems, IEEE Trans. Auto. Control AC-10 (1987) 858–866.

[6] Y.C. Ho and X.R. Cao, Perturbation analysis and optimization of queueing networks, J. Optim. Theory Appl. 40 (1983) 559–582.

[7] P. L'Ecuyer, A unified version of the IPA, SF, and LR gradient estimation techniques, Manag. Sci. 36 (1990) 1364–1383.

[8] M.I. Reiman and B. Simon, Open queueing systems in light traffic, Math. Oper. Res. 14 (1989) 26–59.

[9] M.I. Reiman and A. Weiss, Sensitivity analysis for simulations via likelihood ratios, Oper. Res. 37 (1989) 830–844.

[10] B. Simon, A new estimator of sensitivity measures for simulation based on light traffic theory, ORSA J. Comput. 1 (1989) 172–180.

[11] R. Suri, Perturbation analysis: the state of the art and research issues explained via the $G/G/1$ queue, Proc. IEEE 77 (1989) 114–137.

[12] R. Suri and X.R. Cao, The phantom customer and the marked customer methods for optimization of closed queueing networks with blocking and general service times, *AGM-Sigmetrics Conf.*, Minneapolis (1983).

[13] F. Vázquez-Abad, Stochastic recursive algorithms for optimal routing in queueing networks, Ph.D. Thesis, Applied Mathematics, Brown University (1989).

[14] F. Vázquez-Abad and H. Kushner, A surrogate estimation approach for adaptive routing in communication networks, Brown University Report LCDS/CCS No. 90-2 (May 24, 1990).