JACKSON-TYPE THEOREMS FOR BEST ONE-SIDED APROXIMATIONS BY TRIGONOMETRICAL POLYNOMIALS AND SPLINES

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1. Many articles have recently appeared connected with one-sided approximation to functions, originating with the works of Freud [1] and Ganelius [2]. We formulate a result of Ganelius [2]: let T_n be the set of all trigonometrical polynomials of order n, f a 2π-periodic function with k-th derivative $f^{(k)}$ of bounded variation, and let $\tilde{E}_n^T(f)L_p$ be the best one-sided approximation in L_p, $1 \leqslant p \leqslant \infty$, to the function f by trigonometrical polynomials of order n:

$$\tilde{E}_n^T(f)_{L_p} = \inf\left(\int_0^{2\pi}(P(x) - Q(x))^p\,dx\right)^{1/p} : P,\ Q \in T_n, Q(x) \leqslant f(x) \leqslant P(x)$$

for any x. Then (see [2]),

$$\tilde{E}_n^T(f)_{L_1} \leqslant c_1(k)\, V_0^{2\pi} f^{(k)}/n^{k+1},$$

where $V_a^b g$ denotes the variation of the function g on the interval $[a, b]$, and the constant $c_1(k)$ depends only on k (Ganelius introduced the precise constant $c_1(k)$, but we shall not be interested in this).

Meir and Sharma [3] consider one-sided approximation by first and third degree splines, and Freud and Popov [4], having obtained an analog of Freud and Ganelius' results, generalized Meir and Sharma's result for splines of any degree.

Denote by S_{k,Σ_n} the set of all k-th degree splines on the interval [0, 1] with nodes at the points $\Sigma_n = \{0 = x_0 < \ldots < x_n = 1\}$, i.e., $s \in S_{k,\Sigma_n}$, if $s \in C^{k-1}[0, 1]$ and s is an algebraic polynomial of degree k on the interval $[x_{i-1}, x_i]$, i = 1, 2, ..., n. The best one-sided approximation $\tilde{E}_{k,\Sigma_n}(f)_{L_p}$ in L_p, bounded on the interval [0, 1], to the function f by splines in S_{k,Σ_n} is defined by the formula

$$\tilde{E}_{k,\Sigma_n}(f)_{L_p} = \inf\left(\int_0^1 (S(x) - s(x))^p\,dx\right)^{1/p} : S,\quad s \in S_{k,\Sigma_n},$$
$$s(x) \leqslant f(x) \leqslant S(x),\ x \in [0, 1].$$

Freud and Popov [4] obtained the following result:

$$\tilde{E}_{k,\Sigma_n}(f)_{L_1} \leqslant c_2(k)\,\Delta_n^{k+1} V_0^1 f^{(k)},$$

where $\Delta_n = \max|x_i - x_{i-1}|,\ i = 1, \ldots, n.$

Using this estimate and the estimates from [1], o-small type estimates were obtained in [4] for one-sided approximation by algebraic polynomials and splines.

Babenko, Doronin, and Ligun (see [5-7]) considered one-sided approximations in L_p, $1 < p < \infty$, for certain classes of functions by trigonometrical polynomials and splines. The fundamental result of [5] for the class $W^r L_p$ ($f \in W^r L_p$ if $f^{(r-1)}$ is absolutely continuous and $\|f^{(r)}\|_{L_p} \leqslant 1$) is the following:

<u>THEOREM A.</u> For any $p, 1 \leqslant p \leqslant \infty$, we have the relations

$$\sup_{f \in W^r L_p} \tilde{E}_n^T(f)_{L_p} = Q(n^{-r}),\quad r = 1, 2, \ldots,$$

$$\sup_{f \in W^r L_p} \tilde{E}_{r-1,\bar{\Sigma}_n}(f)_{L_p} = O(n^{-r}),\quad r = 1, 2, \ldots,\ \bar{\Sigma}_n = \{x_i = i\pi/n,\ i = 0, 1, \ldots, 2n\}.$$

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As a corollary we have the following estimate, which we shall use in the future:

If $f^{(r-1)}$ is an absolutely continuous function, then

$$\tilde{E}_n^T (f)_{L_p(0,2\pi)} \leqslant c_3 \| f^{(r)} \|_{L_p(0,2\pi)}/n^r. \qquad (1)$$

The aim of this article is to obtain analogs for $\tilde{E}_n^T(f)_{Lp}$ and $\tilde{E}_{k,\Sigma_n}(f)_{L_p}$ of Jackson's well-known theorem for best one-sided approximations. We obtain these analogs by using the following modulus:

$$\tau (f; \delta)_{L_p} = \| \omega (f; x; \delta) \|_{L_p},$$

where $\omega (f; x; \delta) = \sup | f (t) - f (t') |$, $| t - x | \leqslant \delta/2$, $| t' - x | \leqslant \delta/2$, and t, t' belong to the domain of definition of the function f.

As far as we know, moduli of this type were first used by Sendov [8] and Korovkin [9]. Dolzhenko and Sevast'yanov [10] used this modulus for p = 1 for Hausforff approximations by pointwise monotonic functions, and established several of its basic properties. Sendov [11] obtained estimates for the convergence of linear positive operators in L_p using this modulus.

The results of this article were published at the Conference on the Constructive Theory of Functions in Blagoevgrad in 1977, and announced in [12].

2. We note several properties of the modulus $\tau (f; \delta)_{L_p}$, $1 \leqslant p \leqslant \infty$.

Let

$$\omega (f; \delta) = \sup | f (x) - f (t) |: \ | x - t | \leqslant \delta,$$

where x, t belong to the domain of definition of the function f, be the continuity modulus of f.

LEMMA 1. $\tau (f; \delta)_{L_\infty} = \omega (f; \delta)$.

In connection with this lemma, we note that the case of uniform approximation of functions essentially coincides with one-sided approximation in L_∞.

LEMMA 2 (See [10]). $\tau (f; \delta)_L \underset{\delta\to 0}{\to} 0$ if and only if f is a Riemann-integrable function.

LEMMA 3. If f and g are bounded functions, then

$$\tau (f + g; \delta)_{L_p} \leqslant \tau (f; \delta)_{L_p} + \tau (g; \delta)_{L_p}.$$

LEMMA 4. For any 2π-periodic Riemann-integrable function f, we have the inequality

$$\omega (f; \delta)_{L_p(0,2\pi)} \leqslant \tau (f; \delta)_{L_p(0,2\pi)}.$$

Proof.

$$\omega (f; \delta)_{L_p} = \sup_{0<h\leqslant\delta} \left(\int_0^{2\pi} | f (x + h) - f (x) |^p dx\right)^{1/p} = \sup_{0<h\leqslant\delta} \left(\int_0^{2\pi} \left| f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)\right|^p dx\right)^{1/p} \leqslant$$
$$\leqslant \left(\int_0^{2\pi} \left(\sup_{0<h\leqslant\delta} \left| f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)\right|\right)^p dx\right)^{1/p} \leqslant \left(\int_0^{2\pi} (\omega (f; x; \delta))^p dx\right)^{1/p} = \tau (f; \delta)_{L_p}.$$

LEMMA 5. For any Riemann-integrable function f and any $\lambda \geqslant 0$ we have

$$\tau (f; \lambda\delta)_{L_p} \leqslant (\lambda + 1) \tau (f; \delta)_{L_p}.$$

If k is an integer, then

$$\tau (f; k\delta)_{L_p} \leqslant k\tau (f; \delta)_{L_p}.$$

The proof of this lemma is essentially the same as the proof by Dolzhenko and Sevast'yanov in [10] for the case p = 1.

LEMMA 6. If f is a function bounded on [0, 1], then

$$\tau (f; n^{-1})_L \leqslant 3\varkappa (f; n)/n,$$

where

$$\varkappa(f;n)=\sup_{\Sigma_n}\sum_{i=2}^{n}|f(x_i)-f(x_{i-1})|,$$

$\Sigma_n=\{0=x_1<\ldots<x_n=1\}$ is the variation modulus of the function f (see [13, 14]).

Proof. Write

$$S(f;\ x;\ \delta)=\sup f(t);\ \ |t-x|\leqslant\delta/2,$$
$$J(f;\ x;\ \delta)=\inf f(t);\ \ |t-x|\leqslant\delta/2.$$

It follows from the definition of $\tau(f;\delta)_L$ that

$$\tau(f;n^{-1})_L=\int_0^1\omega(f;x;n^{-1})\,dx=\int_0^1(S(f;x;n^{-1})-J(f;x;n^{-1}))\,dx=$$

$$=\sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}(S(f;x;n^{-1})-J(f;x;n^{-1}))\,dx\leqslant\frac{1}{n}\sum_{i=0}^{n}(S(f;\xi_i;n^{-1})-J(f;\xi_i;n^{-1})),\tag{2}$$

where $\xi_i\in[i/n,\ (i+1)/n]$. Let $\varepsilon>0$ and $\bar{\xi}_i$, $\underline{\xi}_i$ be such that

$$S(f;\ \xi_i;\ n^{-1})\leqslant f(\bar{\xi}_i)+\varepsilon/(2n),\ J(f;\ \xi_i;\ n^{-1})\geqslant f(\underline{\xi}_i)-\varepsilon/(2n).$$

Then from (2) we have

$$\tau(f;n^{-1})_L\leqslant\frac{1}{n}\sum_{i=1}^{n}|f(\bar{\xi}_i)-f(\underline{\xi}_i)|+\varepsilon.\tag{3}$$

We split up the sum in (3) as follows:

$$\sum_{i=1}^{n}|f(\bar{\xi}_i)-f(\underline{\xi}_i)|=\sum_{j=0}^{2}\sum_i|f(\bar{\xi}_{3i-j})-f(\underline{\xi}_{3i-j})|.$$

For $n\geqslant4$,

$$\sum_i|f(\bar{\xi}_{3i-j})-f(\underline{\xi}_{3i-j})|\leqslant\varkappa(f;n),$$

since the number of points occurring in the last sum is not greater than $[2n/3]+2\leqslant n$, and

$$\max(\bar{\xi}_{3i-j},\ \ \underline{\xi}_{3i-j})\leqslant\min(\bar{\xi}_{3i+3-j},\ \underline{\xi}_{3i+3-j}).$$

Thus

$$\sum_{i=1}^{n}|f(\bar{\xi}_i)-f(\underline{\xi}_i)|\leqslant3\varkappa(f;n).$$

LEMMA 7 (See [10]). If $V_0^1f<\infty$, then for any $\delta>0$ we have the inequality $\tau(f;\ \delta)_L\leqslant\delta V_0^1f.$

LEMMA 8. Let f be a 2π-periodic and absolutely continuous function. Then

$$\tau(f;\delta)_{L_p}\leqslant\delta\,\|f'\|_{L_p}.$$

Proof. Since

$$f(x)-f(y)=\int_y^x f'(x)\,dt,$$

we obtain

$$\omega(f;x;\delta)=\max_{\substack{|t-x|\leqslant\delta/2\\|t'-x|\leqslant\delta/2}}|f(t)-f(t')|=\max_{\substack{|t-x|\leqslant\delta/2\\|t'-x|\leqslant\delta/2}}\left|\int_t^{t'}f'(u)\,du\right|\leqslant\int_{x-\delta/2}^{x+\delta/2}|f'(t)|\,dt=\int_{-\delta/2}^{\delta/2}|f'(x-u)|\,du.$$

Thus

$$\tau(f;\delta)_{L_p}=\|\omega(f;x;\delta)\|_{L_p}\leqslant\left\|\int_{-\delta/2}^{\delta/2}|f'(x-u)|\,du\right\|_{L_p}\leqslant\int_{-\delta/2}^{\delta/2}\|f'\|_{L_p}\,du=\delta\,\|f'\|_{L_p}.$$

3. We now consider one-sided approximations. We first prove two lemmas.

LEMMA 9. Let f be a 2π-periodic function, $f \in L_p(0, 2\pi)$, $\int_0^{2\pi} f(t)\,dt = 0$, and let there exist a polynomial $T \in T_n$ such that $T(x) \geqslant f(x)$ for $x \in [0, 2\pi]$. Then there exists a polynomial $R \in T_n$ such that $R(x) \geqslant \int_0^x f(t)\,dt$, $x \in [0, 2\pi]$ and

$$\left(\int_0^{2\pi} \left| R(x) - \int_0^x f(t)\,dt \right|^p dx\right)^{1/p} \leqslant \frac{c\eta}{n},$$

where $\eta = \|f - T\|_{L_p(0, 2\pi)}$.

Proof. Let

$$T(x) = a_0 + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx).$$

Set $\tilde{T}(x) = T(x) - a_0$, $g(x) = \int_0^x (f(t) - \tilde{T}(t))\,dt$. The function g(x) can be written

$$g(x) = A + \frac{1}{\pi}\int_0^{2\pi} D_1(x-t)\,g'(t)\,dt =$$

$$= A + \frac{1}{\pi}\int_0^{2\pi} D_1(x-t)(f(t) - \tilde{T}(t))\,dt =$$

$$= A + \frac{1}{\pi}\int_0^{2\pi} D_1(u)(\tilde{T}(x-u) - f(x-u))\,du,$$

where $D_1(u) = u - \pi$, $u \in [0, 2\pi)$, $D_1(u + 2\pi) = D_1(u)$.

Using [2], we see that there exists $\gamma \in T_n$ such that $\gamma(u) \geqslant D_1(u)$ for $u \in [0, 2\pi]$ and

$$\int_0^{2\pi} (\gamma(u) - D_1(u))\,du \leqslant \frac{c}{n}, \tag{4}$$

where c is an absolute constant.

Set

$$Q(x) = A + \frac{1}{\pi}\int_0^{2\pi} \gamma(x-t)(f(t) - T(t))\,dt,$$

$$R(x) = \int_0^x \tilde{T}(t)\,dt + Q(x).$$

Clearly, $Q \in T_n$, $R \in T_n$. We have

$$R(x) - \int_0^x f(t)\,dt = \int_0^x (\tilde{T}(t) - f(t))\,dt + Q(x) = Q(x) - g(x) = \frac{1}{\pi}\int_0^{2\pi} \gamma(x-t)(f(t) - T(t))\,dt -$$

$$- \frac{1}{\pi}\int_0^{2\pi} D_1(x-t)(f(t) - \tilde{T}(t))\,dt = \frac{1}{\pi}\int_0^{2\pi} \gamma(u)(T(x-u) - f(x-u))\,du -$$

$$- \frac{1}{\pi}\int_0^{2\pi} D_1(u)(\tilde{T}(x-u) - f(x-u))\,du = \frac{1}{\pi}\int_0^{2\pi} (\gamma(u) - D_1(u))(T(x-u) - f(x-u))\,du \geqslant 0.$$

Here we are using the fact that $\int_0^{2\pi} a_0 D_1(u)\,du = 0$, and thus

$$\int_0^{2\pi} D_1(u)\tilde{T}(x-u)\,du = \int_0^{2\pi} D_1(u)\,T(x-u)\,du.$$

On the other hand, using [4] we obtain

$$\left(\int_0^{2\pi}\left|R(x)-\int_0^x f(t)\,dt\right|^p dx\right)^{1/p}=\left(\int_0^{2\pi}|Q(x)-g(x)|^p dx\right)^{1/p}=\frac{1}{\pi}\left(\int_0^{2\pi}\left|\int_0^{2\pi}(\gamma(u)-D_1(u))(T(x-u)-\right.\right.$$

$$\left.\left.-f(x-u))\,du\right|^p dx\right)^{1/p}\leqslant\frac{1}{\pi}\int_0^{2\pi}\left(\int_0^{2\pi}|\gamma(u)-D_1(u)|^p|T(x-u)\;-f(x-u)|^p dx\right)^{1/p}du=\frac{1}{\pi}\int_0^{2\pi}|\gamma(u)-D_1(u)|\,\cdot$$

$$\cdot\left(\int_0^{2\pi}|T(x-u)-f(x-u)|^p dx\right)^{1/p}du\leqslant\frac{C\|T-f\|_{L_p}}{\pi\cdot n}.$$

Thus R satisfies the requirements of the lemma.

LEMMA 10. Let the function f have integrable bounded derivative f' on the interval [0, 1]. Then for any $k,\ k\geqslant 1$, we have the inequality

$$\bar{E}_{k,\Sigma_n}(f)_{L_p}\leqslant(k+1)\Delta_n\bar{E}_{k-1,\Sigma_n}(f')_{L_p},$$

$$\Sigma_n=\{0=x_0<\ldots<x_n=1\},\quad \Delta_n=\max_{1\leqslant i\leqslant n}|x_i-x_{i-1}|.$$

Proof (See [4]). Clearly we may assume that f(0) = 0. Let $s\in S_{k-1,\Sigma_n},\ l\in S_{k-1,\Sigma_n}$ be such that

$$s(x)\geqslant f'(x)\geqslant l(x),\ x\in[0,\ 1],$$
$$\|s-l\|_{L_p(0,1)}\leqslant\bar{E}_{k-1,\Sigma_n}(f')_{L_p}+\varepsilon,\ \varepsilon>0. \tag{5}$$

Set

$$\varphi_i(x)=\sum_{j=i}^{i+k}\frac{k(x_j-x)_+^{k-1}}{\omega_i'(x_j)},$$

$$(x-t)_+^{k-1}=\begin{cases}(x-t)^{k-1}, & x\geqslant t,\\ 0, & x<t,\end{cases}$$

$$i=0,1,\ldots,\ n-k;\ \omega_i(x)=(x-x_i)(x-x_{i+1})\ldots(x-x_{i+k}).$$

It is known (see, e.g., [4]) that $\varphi_i(x)>0$ for $x\in(x_i,x_{i+k})$, $\varphi_i(x)=0$ for $x\bar{\in}(x_i,x_{i+k})$, and $\int_{-\infty}^{\infty}\varphi_i(x)\,dx=1$. Clearly, $\varphi_i\in S_{k-1,\Sigma_n}$. Set

$$A_i=\int_{x_i}^{x_{i+1}}(s(x)-f'(x))\,dx\geqslant 0,\ B_i=\int_{x_i}^{x_{i+1}}(f'(x)-l(x))\,dx\geqslant 0$$

and consider the splines

$$s^*(x)=\int_0^x s(t)\,dt-\sum_{i=0}^{n-k-1}A_i\int_0^x\varphi_{i+1}(t)\,dt,$$

$$l^*(x)=\int_0^x l(t)\,dt+\sum_{i=0}^{n-k-1}B_i\int_0^x\varphi_{i+1}(t)\,dt.$$

We have $s^*\in S_{k,\Sigma_n}$, $l^*\in S_{k,\Sigma_n}$. If $x_{i_0}<x<x_{i_0+1}$, then since f(0) = 0, we have

$$s^*(x)-\int_0^x f'(t)\,dt=s^*(x)-f(x)=\int_0^x(s(t)-f'(t))\,dt-\sum_{i=0}^{n-k-1}A_i\int_0^x\varphi_{i+1}(t)\,dt=$$

$$=\sum_{i=0}^{i_0-1}\int_{x_i}^{x_{i+1}}(s(t)-f'(t))\,dt+\int_{x_0}^{x}(s(t)-f'(t))\,dt-\sum_{i=0}^{i_0-k-1}A_i-\sum_{i=i_0-k}^{i_0-1}A_i\int_0^x\varphi_{i+1}(t)\,dt=$$

$$=\int_{x_0}^{x}(s(t)-f'(t))\,dt+\sum_{i=i_0-k}^{i_0-1}A_i\left(1-\int_0^x\varphi_{i+1}(t)\,dt\right)\geqslant 0 \tag{6}$$

$$\left(A_i\geqslant 0,\ 0\leqslant\int_0^x\varphi_{i+1}(t)\,dt\leqslant 1\right).$$

We many prove analogously that $l^*(x)\leqslant f(x)$.

Set $s(x) = 0$, $l(x) = 0$ for $x < 0$. Then

$$\| s^* - l^* \|_{L_p(0,1)} = \left\{\int_0^1 \left| \int_0^x (s(t) - l(t))\,dt - \sum_{i=0}^{n-k-1} (A_i + B_i) \int_0^x \varphi_{i+1}(t)\,dt \right|^p dx\right\}^{1/p} \leqslant$$

$$\leqslant \left\{\int_0^1 \left| \int_{x-(k+1)\Delta_n}^x (s(t) - l(t))\,dt \right|^p dx\right\}^{1/p} = \left\{\int_0^1 \left| \int_0^{(k+1)\Delta_n} (s(x-(k+1)\Delta_n + t) - l(x-(k+1)\Delta_n + t)) \cdot\right.\right.$$

$$\left.\left. \cdot dt \right|^p dx\right\}^{1/p} \leqslant \int_0^{(k+1)\Delta_n} \left\{\int_0^1 | s(x-(k+1)\Delta_n + t) -\right.$$

$$\left. - l(x-(k+1)\Delta_n + t)|^p dx\right\}^{1/p} dt \leqslant \int_0^{(k+1)\Delta_n} (E_{k-1,\Sigma_n}(f')_{L_p} + \varepsilon)\,dt = (k+1)\Delta_n (E_{k-1,\Sigma_n}(f')_{L_p} + \varepsilon). \quad (7)$$

The statement of the Lemma follows from (6) and (7).

THEOREM 1. If f is a bounded function on the interval [0, 1], then

$$E_{0,\Sigma_n}(f)_{L_p} \leqslant 2\tau(f; \Delta_n)_{L_p}, \quad 1 \leqslant p \leqslant \infty.$$

Proof. Let $\Sigma_n = \{0 = x_0 < \ldots < x_n = 1\}$, $\Delta_n = \max_{1 \leqslant i \leqslant n} |x_i - x_{i-1}|$. Set

$$\begin{aligned} s_{\Sigma_n}(x) &= \sup_{t \in [x_{i-1}, x_i]} f(t), \ x \in [x_{i-1}, x_i), \ s_{\Sigma_n}(1) = \lim_{x \to 1} s_{\Sigma_n}(x), \\ l_{\Sigma_n}(x) &= \inf_{t \in [x_{i-1}, x_i]} f(t), \ x \in [x_{i-1}, x_i), \ l_{\Sigma_n}(1) = \lim_{x \to 1} l_{\Sigma_n}(x). \end{aligned} \quad (8)$$

It follows from (8) (see the notation of Lemma 6) that

$$\begin{aligned} f(x) &\leqslant s_{\Sigma_n}(x) \leqslant S(f, x; 2\Delta n), \\ f(x) &\geqslant l_{\Sigma_n}(x) \geqslant J(f, x; 2\Delta_n). \end{aligned} \quad (9)$$

Since $s_{\Sigma_n} \in S_{0,\Sigma_n}$, $l_{\Sigma_n} \in S_{0,\Sigma_n}$, and

$$\omega(f, x; \delta) = S(f, x; \delta) - J(f, x; \delta),$$

using Lemma 5, from (9) we have

$$E_{0,\Sigma_n}(f)_{L_p} \leqslant \left\{\int_0^1 | s_{\Sigma_n}(x) - l_{\Sigma_n}(x)|^p dx\right\}^{1/p} \leqslant$$

$$\leqslant \left\{\int_0^1 | S(f, x; 2\Delta_n) - J(f, x; 2\Delta_n)|^p dx\right\}^{1/p} = \tau(f; 2\Delta_n)_{L_p} \leqslant 2\tau(f; \Delta_n)_{L_p}.$$

THEOREM 2. Let f have integrable bounded k-th derivative $f^{(k)}$ on the interval [0, 1]. Then $(1 \leqslant p \leqslant \infty)$

$$E_{k,\Sigma_n}(f)_{L_p} \leqslant 2(k+1)!\ (\Delta_n)^k \tau(f^{(k)}; \Delta_n)_{L_p}.$$

Proof. In fact, using Lemma 10 k times in succession and applying Theorem 1 to $f^{(k)}$, we obtain

$$E_{k,\Sigma_n}(f)_{L_p} \leqslant (k+1)\Delta_n E_{k-1,\Sigma_n}(f')_{L_p} \leqslant \cdots \leqslant (k+1)!\,(\Delta_n)^k E_{0,\Sigma_n}(f^{(k)})_{L_p} \leqslant 2(k+1)!\,(\Delta_n)^k \tau(f^{(k)}; \Delta_n)_{L_p}.$$

From the properties of the modulus $\tau(f; \delta)_{L_p}$ and Theorem 2, we have the following.

COROLLARY 1. If f has integrable bounded k-th derivative $f^{(k)}$, then

a) $E_{k,\Sigma_n}(f)_C \leqslant 2(k+1)!\,(\Delta_n)^k\,\omega(f^{(k)}; \Delta_n)$,

b) (Freud–Popov Theorem [4])

$$E_{k,\Sigma_n}(f)_L \leqslant 2(k+1)!\ (\Delta_n)^{k+1} V_0^1 f^{(k)}.$$

COROLLARY 2 (Babenko–Ligun Theorem [5]). If $\|f^{(k+1)}\|_{L_p} < \infty$, then

$$E_{k,\bar{\Sigma}_n}(f)_{L_p} \leqslant c_3(k) \| f^{(k+1)} \|_{L_p} n^{-k-1}, \quad \bar{\Sigma}_n = \left\{\frac{i2\pi}{n}, i = 0, \ldots, n\right\}.$$

THEOREM 3. Lef f be a 2π-periodic bounded function. Then

$$E_n^T(f)_{L_p} \leqslant c\tau(f; n^{-1})_{L_p}, \quad 1 \leqslant p \leqslant \infty,$$

where c is an absolute constant.

Proof. Set $x_i = i\pi/n$, $i = 0, \ldots, 2n$, $y_i = (x_{i-1} + x_i)/2$, $i = 1, \ldots, 2n$, $y_{2n+1} = y_1$ and define 2π-periodic functions S_n and J_n as follows:

$$S_n(x) = \begin{cases} \sup\limits_{t \in [x_{i-1}, x_i]} f(t) \text{ for } x = y_i, \ i = 1, \ldots, 2n, \\ \max\{S_n(y_i), S_n(y_{i+1})\} \text{ for } x = x_i, \ i = 1, \ldots, 2n, \\ S_n(0) = S_n(2\pi), \\ \text{linear and continuous for } x \in [x_{i-1}, y_i] \\ \text{and } x \in [y_i, x_i], \quad i = 1, \ldots, 2n, \end{cases}$$

$$J_n(x) = \begin{cases} \inf\limits_{t \in [x_{i-1}, x_i]} f(t) \text{ for } x = y_i, \ i = 1, \ldots, 2n, \\ \min\{J_n(y_i), J_n(y_{i+1})\} \text{ for } x = x_i, \ i = 1, \ldots, 2n, \\ J_n(0) = J_n(2\pi), \\ \text{linear and continuous for } x \in [x_{i-1}, y_i] \\ \text{and } x \in [y_i, x_i], \ i = 1, \ldots, 2n. \end{cases}$$

Clearly, we have

$$J_n(x) \leqq f(x) \leqslant S_n(x), \quad x \in [0, 2\pi]. \tag{10}$$

The derivatives $S_n'(x)$ and $J_n'(x)$ of S_n and J_n exist at each point of the interval $[0, 2\pi]$ except the points x_i, $i = 0, \ldots, 2n$, y_i, $i = 1, \ldots, 2n$. Moreover, using the definitions of the functions S_n and J_n, we immediately have

$$\begin{aligned} |S_n'(x)| &\leqslant 2n\pi^{-1}\omega(f, x; 4\pi n^{-1}), \quad x \neq x_i, y_i, \\ |J_n'(x)| &\leqslant 2n\pi^{-1}\omega(f, x; 4\pi n^{-1}), \quad x \neq x_i, y_i \end{aligned} \tag{11}$$

(e.g., if $x \in (y_i, x_i)$, then as S_n is linear we have

$$|S_n'(x)| \leqq 2n\pi^{-1}|S_n(y_{i+1}) - S_n(y_i)| \leqslant 2n\pi^{-1}\omega(f, x; 4\pi/n)),$$

and moreover,

$$0 \leqslant S_n(x) - J_n(x) \leqslant \omega(f, x; 2\pi/n). \tag{12}$$

It follows from (11) that

$$\begin{aligned} \|S_n'\|_{L_p(0,2\pi)} &\leqslant 2n\pi^{-1}\tau(f; 4\pi/n)_{L_p}, \\ \|J_n'\|_{L_p(0,2\pi)} &\leqslant 2n\pi^{-1}\tau(f; 4\pi/n)_{L_p}. \end{aligned} \tag{13}$$

Moreover, (12) gives

$$\|S_n - J_n\|_{L_p} \leqslant \tau(f; 2\pi/n)_{L_p}. \tag{14}$$

Using (1) for r = 1, we obtain from (13)

$$E_n^T(S_n)_{L_p} \leqslant c(1)\tau(f; 4\pi/n)_{L_p}; \quad E_n^T(J_n)_{L_p} \leqslant c(1)\tau(f; 4\pi/n)_{L_p}. \tag{15}$$

The following inequality is obvious:

$$E_n^T(f)_{L_p} \leqslant E_n^T(S_n)_{L_p} + \|S_n - J_n\|_{L_p} + E_n^T(J_n)_{L_p}. \tag{16}$$

Using Lemma 5, from (14)-(16) we obtain

$$E_n^T(f)_{L_p} \leqslant 2c(1)\tau(f; 4\pi/n)_{L_p} + \tau(f; 2\pi/n)_{L_p} \leqslant c\tau(f; n^{-1})_{L_p}.$$

Thus, Theorem 3 is proved.

Using Lemma 9 and Theorem 3, we obtain the following.

THEOREM 4. Let f be a 2π-periodic function which has integrable bounded k-th derivative $f^{(k)}$. Then

$$E_n^T(f)_{L_p} \leqslant c^k n^{-k} \tau(f^{(k)};\ n^{-1})_{L_p}, \qquad 1 \leqslant p \leqslant \infty,$$

where c > 0 is an absolute constant.

COROLLARY. For the corresponding restrictions on the function f, we have the estimates:

a) $E_n^T(f)_C \leqslant c_3(k)\, n^{-k} \omega(f^{(k)};\ n^{-1})$,
b) $E_n^T(f)_L \leqslant c_4(k)\, \varkappa(f^{(k)};\ n)\, n^{-k-1}$,
c) $E_n^T(f)_L \leqslant c_5(k)\, n^{-k-1} V_0^{2\pi} f^{(k)}$,
d) $E_n^T(f)_{L_p} \leqslant c_6(k) n^{-k-1} \|f^{(k+1)}\|_{L_p}$.

Remark. We can define moduli $\tau_k(f;\ \delta)_{L_p}$ by analogy with the k-th continuity moduli $\omega_k(f;\ \delta)_{L_p}$. Generalizations of Theorems 1 and 3 were obtained in [16] for the moduli $\tau_k(f;\ \delta)_{L_p}$, analogous to the generalization of Jackson's theorem for $\omega_k(f;\ \delta)_{L_p}$, obtained by Stechkin [15]. We also note that using $\tau_k(f;\ \delta)_{L_p}$, in [17] inverse theorems were obtained for one-sided trigonometrical approximations in L_p, $1 \leqslant p \leqslant \infty$.

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