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Nonexistence of Brownian models for certain multiclass queueing networks*

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We present two multiclass queueing networks where the Brownian models proposed by Harrison and Nguyen [3,4] do not exist. If self-feedback is allowed, we can construct such an example with a two-station network. For a three-station network, we can construct such an example without self-feedback.

Keywords: Brownian model, multiclass network, heavy traffic, diffusion approximation, performance analysis.

1. Introduction

It has been demonstrated that Brownian models are effective for approximate analysis of queueing networks, particularly when no explicit analytical methods are available [3,4,1]. In 1988 Harrison [2] proposed Brownian models as approximations for *multiclass* open queueing networks. That work was expanded by Harrison and Nguyen in two recent papers [3,4] with the aim of developing a systematic method for performance analysis of open networks. The network models they consider have general routing and general service requirements. The corresponding Brownian model for a d-station network is a reflected Brownian motion in the d-dimensional non-negative orthant $\mathbb{R}^d_+$. Their work provides explicit formulas for calculating data of the reflected Brownian motion in terms of primitive network data.

In this note we present two network examples where the Brownian model (reflected Brownian motion) proposed by Harrison and Nguyen does not exist. In

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our examples, all customers follow one deterministic route. Customers may visit a particular station more than once, and a customer's service requirements at successive visits to a station may differ. Such a network is an example of a *multiclass* network. If self-feedback (i.e., a customer leaving a station may immediately visit the same station) is allowed, we can construct such an example with $d = 2$. For $d = 3$, we can construct an example without self-feedback. At this point, we are not able to offer an explanation as to why Harrison and Nguyen's models fail for these particular examples.

For an open single-class network, Reiman [8] proved a heavy traffic limit theorem which says that under *heavy traffic conditions* Brownian models are good approximations of queueing networks. The existence of the limiting processes (reflected Brownian motions) was resolved by Harrison and Reiman [5]. Peterson [7] proved an analogous heavy traffic limit theorem for a network populated by several types of customers where each type follows a deterministic, feedforward path. The existence of Peterson's limiting process depends heavily on the special feedforward structure of the network. Reiman [9] proved a heavy traffic limit theorem for self-feedback one-station networks, and his proof was simplified by Dai and Kurtz [6]. However, there are no limit theorems for general multiclass networks with more than one station. Our examples show that it is difficult to *formulate* a correct Brownian model for general multiclass networks, let alone to prove a general heavy traffic limit theorem.

2. Reflected Brownian motion

Let θ be a d-dimensional vector, Γ be a $d \times d$ positive definite matrix and R be a $d \times d$ matrix.

DEFINITION 1

The matrix R is said to be an $\mathcal{S}$ *matrix* if there exists a $u > 0$ such that $Ru > 0$ and a *completely*-$\mathcal{S}$ *matrix* if each principal submatrix of R is an $\mathcal{S}$ matrix.

Here and later, vectors are envisioned as column vectors and vector inequalities are interpreted componentwise.

DEFINITION 2

A *semimartingale reflected Brownian motion* (SRBM) associated with data (θ, Γ, R) is a continuous, $\{\mathcal{F}_t\}$-adapted, d-dimensional process W together with a family of probability measures $\{P_x, x \in \mathbb{R}^d_+\}$ defined on some filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$ such that for each $x \in \mathbb{R}^d_+$ under P_x,

$$W(t) = X(t) + RY(t) \geq 0, \quad \text{for all} \quad t \geq 0,$$

where

(i) X is a d-dimensional Brownian motion with drift vector θ and covariance matrix Γ such that $\{X(t) - \theta t, \mathcal{F}_t, t \geqslant 0\}$ is a martingale and $X(0) = x$ P_x-a.s.,
(ii) Y is a continuous, $\{\mathcal{F}_t\}$-adapted, d-dimensional process such that P_x-a.s. for each $i \in \{1, \ldots, d\}$, the ith component Y_i of Y satisfies
(a) $Y_i(0) = 0$ and Y_i is non-decreasing,
(b) Y_i can increase only when $W_i = 0$, i.e., $\int_0^\infty 1_{\{W_i(s) \neq 0\}} \mathrm{d}Y_i(s) = 0$.

Remark

The vector θ, the matrix Γ and the matrix R are called the *drift* vector, the *covariance* matrix and the *reflection* matrix of the SRBM, respectively.

In [3,4] Harrison and Nguyen do not give a formal definition of a reflected Brownian motion, which was used as their Brownian model. It is obvious that the definition of SRBM given above is very weak. In particular, the reflected Brownian motion alluded to in Harrison and Nguyen [3,4] is a semimartingale reflected Brownian motion as defined here. The following proposition was proved by Reiman and Williams [10].

PROPOSITION 1

If there exists an SRBM associated with data (θ, Γ, R), then R must be a completely-$\mathcal{S}$ matrix. In particular, the diagonal elements of R are positive.

Remark

Taylor and Williams [11] recently proved that R being completely-$\mathcal{S}$ is also sufficient for the existence and uniqueness (in law) of an SRBM.

3. A two-station network with self-feedback

Consider a two-station network pictured in fig. 1. Customers arrive at station one according to a Poisson process with rate 1. Each customer follows a deterministic route whose sequence of visitation is 1, 1, 2, 2, 1 and then the customer departs. Hence each customer makes 5 stops before exiting the network. We designate those customers in their kth stop as class k customers. The service times for class k customers are assumed to be exponentially distributed with mean m_k

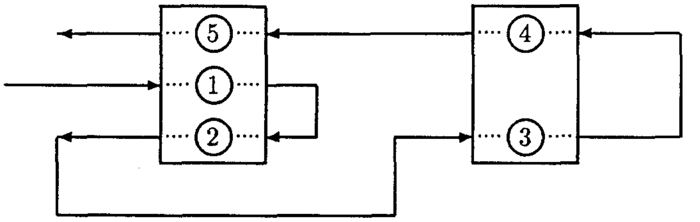

Fig. 1. A two-station network with self-feedback.

$(k = 1, \ldots, 5)$. Using the notation in Harrison and Nguyen [4], we have the number of stations $d = 2$, the number of classes $c = 5$, and the constituency matrix C and the routing matrix P given by

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.1}$$

Put $M = \text{diag}(m_1, \ldots, m_5)$ and $Q = (I - P')^{-1}$, where prime denotes transpose. It is easy to check that the vector of arrival rates for each class is

$$\lambda = Q(1,0,0,0,0)' = (1,1,1,1,1)'.$$

We assume that servers are perfectly reliable, hence the vector μ of availability rates defined in (2.2) of [4] is $(1, 1)'$ and the vector ρ of traffic intensities defined in (3.4) of [4] is $(m_1 + m_2 + m_5, m_3 + m_4)'$. Also, the matrix Δ defined in (4.16) of [4] is simply C'.

In section 4 of [4] the authors develop an approximating Brownian system model, eventually defined by (4.29)–(4.33); in that Brownian approximation the server workload process is modeled as a reflected Brownian motion W^* with reflection matrix $(I + G)^{-1}$, where

$$G = CMQP'\Delta = CMQP'C'.$$

Later, in section 6 of [4], the authors describe a "refined" Brownian approximation which is exactly the same except that G is replaced by

$$\hat{G} = G[\text{diag}(\rho)]^{-1},$$

where $\text{diag}(\rho)$ denotes the diagonal matrix with diagonal elements $\rho_1, \ldots, \rho_d$. Thus, as explained in section 6 of [4], the reflection matrix in the refined Brownian approximation is

$$R = (I + \hat{G})^{-1} = \text{diag}(\rho)[CMQC']^{-1}, \tag{3.2}$$

and the refined Brownian approximation is precisely the so-called QNET approximation proposed earlier in [3]. One can easily check that

$$CMQC' = \begin{pmatrix} m_1 + 2m_2 + 3m_5 & 2m_5 \\ 2m_3 + 2m_4 & m_3 + 2m_4 \end{pmatrix}.$$

Hence, the determinant of $(I + \hat{G})$ is

$$\det(I + \hat{G}) = \frac{1}{\rho_1 \rho_2}[(m_1 + 2m_2)(m_3 + 2m_4) + 2m_4 m_5 - m_3 m_5].$$

Choose $m = (0.1, 0.05, 0.9, 0.05, 0.8)'$, so that $\rho = (0.95, 0.95)'$. Then, $\det(I + \hat{G}) = -0.4875$ and

$$R = \begin{pmatrix} -2.1591 & 3.4545 \\ 4.1023 & -5.6136 \end{pmatrix},$$

which is *not* completely-$\mathcal{S}$ because the diagonal elements of R are negative. Therefore, by proposition 1, the corresponding SRBM does not exist.

For the "unrefined" Brownian approximation developed in section 4 of [4] one has the reflection matrix $R = (I + G)^{-1}$, and with the specific data above that works out to be

$$R = \begin{pmatrix} -4.0777 & 6.2136 \\ 7.3786 & -10.2913 \end{pmatrix},$$

so again the proposed Brownian approximation does not exist. Incidentally, if one takes $m = (0.1, 0.05, 1/6, 0.05, 0.8)'$, then $I + \hat{G}$ is not invertible. Similarly, if one takes $m = (0.1, 0.05, 0.81953125, 0.05, 0.8)'$, then $I + G$ is not invertible.

Remark

If we do not allow self-feedback, one can prove that R is always completely-$\mathcal{S}$ when $d = 2$.

4. A three-station network example

Consider a three-station network. Customers arrive at station 1 from outside according to a Poisson process with rate 1. The route for each customer is deterministic with the sequence of visiting stations given by 1, 2, 1, 3, 2, 3, 1 and then the customer departs. Thus each customer makes 7 stops before exiting the system. As in section 3, customers in their kth stop in the route are called class k customers. The service time for a class k customer is exponentially distributed with mean $m_k (k = 1, \dots, 7)$. We again assume that the servers are reliable. Note that if customers at station 2 have zero service times, then the subnetwork consisting of stations 1 and 3 is exactly the one discussed in section 3. Using the notation in [4], we have $d = 3$, $c = 7$ and the routing matrix P given by: $P_{k,k+1} = 1$ for $k = 1, \dots, 6$ and $P_{kl} = 0$ otherwise. The constituency matrix is

$$C = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Put $M = \mathrm{diag}(m_1, \dots, m_7)$ and $Q = (I - P')^{-1}$. The vector ρ of traffic intensities is $(m_1 + m_3 + m_7, m_2 + m_5, m_4 + m_6)'$. The reflection matrix R for the "refined" Brownian approximation is given by $R = (I + \hat{G})^{-1} = [\mathrm{diag}(\rho)](CMQC')^{-1}$, where

$$CMQC' = \begin{pmatrix} m_1 + 2m_3 + 3m_7 & m_3 + 2m_7 & 2m_7 \\ m_2 + 2m_5 & m_2 + 2m_5 & m_5 \\ 2m_4 + 2m_6 & m_4 + 2m_6 & m_4 + 2m_6 \end{pmatrix}.$$

Set $m = (0.1, 0.5, 0.05, 0.9, 0.45, 0.05, 0.8)'$. Then

$$R = \begin{pmatrix} -2.0270 & 0.1067 & 3.1951 \\ 1.1628 & 0.9388 & -2.2830 \\ 2.6884 & -1.1415 & -2.8377 \end{pmatrix},$$

which is not a completely-$\mathcal{S}$ matrix because R_{11} and R_{33} are negative. Furthermore, if we use the "unrefined" Brownian approximation then

$$R = (I + G)^{-1} = \begin{pmatrix} -3.1486 & 0.3890 & 4.6312 \\ 1.8055 & 0.7560 & -3.0755 \\ 3.9780 & -1.4239 & -4.4991 \end{pmatrix},$$

which is not completely-$\mathcal{S}$ either. For this three-station network, the Brownian models proposed in [3] and [4] do not exist.

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