ASYMPTOTIC PROPERTIES OF KRAWTCHOUK POLYNOMIALS

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1. The classical Krawtchouk orthogonal polynomials of a discrete variable can be defined with the help of the following equation:

$$K_n(x) = K_n(x; N, p) = \frac{(-1)^n}{n!\,\rho(x)} \Delta^n \left[\frac{p^x q^{N-x+n} N!}{\Gamma(x-n+1)\,\Gamma(N-x+1)} \right], \tag{1}$$

where $0 < p < 1$, $q = 1 - p$, $\Gamma(z)$ is the Euler gamma-function,

$$\rho(x) = \rho(x; N, p) = \frac{N!\, p^x q^{N-x}}{\Gamma(x+1)\,\Gamma(N-x+1)},$$

$$\Delta f(x) = f(x+1) - f(x),$$

$$\Delta^n f(x) = \Delta(\Delta^{n-1} f(x)) \quad (n > 1).$$

The difference properties of these polynomials have been rather well-studied (see, for example, [1]). Recurrence relations of a different kind and connections with the other classical orthogonal polynomials have been established. Krawtchouk polynomials in a definite sense are the discrete analog of Hermite polynomials. Let us make note of [2], where the properties of Krawtchouk polynomials are considered in connection with their applications to coding theory. In particular, the asymptotic properties of the zeros of $K_n(x; N, p)$ were considered for fixed n and $N \to \infty$.

On the other hand, the problem of the asymptotic properties of the Krawtchouk polynomials $K_n(x; N, p)$, for which the degree n grows along with N, has remained uninvestigated. In connection with this let us observe that the corresponding problem for the classical orthogonal polynomials of a continuous variable (Hermite, Laguerre and Jacobi polynomials) was exhaustively investigated in the words of Darboux, Stieltjes, S. N. Bernshtein, Szëgo et al. (see [3]). The basic instrument of analysis of asymptotic properties in these works was the differential equation which the corresponding classical orthogonal polynomials satisfy. The differential equation imposes a very harsh condition on the behavior of its solutions and in the final account it plays a decisive role in the analysis of the asymptotic properties of Hermite, Laguerre and Jacobi polynomials. As to the classical orthogonal polynomials of a discrete variable, they then are the solutions of the corresponding difference equations which leave for their solutions incomparably "greater freedom" on the intervals contained between the points of the discontinuity of the corresponding weight function, and this circumstance materially hardens the analysis of the asymptotic properties of the orthogonal polynomials of a discrete variable. It turned out that the asymptotic properties of the Krawtchouk polynomials $K_n(x; N, p)$ essentially depends on the restrictions imposed on the growth of n (depending on N). In this paper we study the asymptotic properties of the Krawtchouk polynomials $K_n(x; N, p)$ for $n = O(N^{1/3})$ and their zeros for $n = o(N^{1/4})$.

2. Let $\tilde{Q} = Np + (2Npq)^{1/2} Q$,

$$I_{n,N}(x) = (2Npq\pi n!)^{1/2} \left(\frac{1}{Npq}\right)^{n/2} \rho(\tilde{x})\, e^{x^2/2} K_n(\tilde{x}), \tag{2}$$

$$H_m(x) = (-1)^m e^{x^2} \frac{d^m}{dx^m} \{e^{-x^2}\}.$$

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The basic result of this paper is the following

THEOREM. Let p be a fixed number, $0 < p < 1$, A_1 and A_2 two positive constants. Then uniformly with respect to

$$-A_1 n^{1/2} \leqslant x \leqslant A_1 n^{1/2}, \quad 0 \leqslant n \leqslant A_2 N^{1/3} \ (N = 1, 2, \ldots) \tag{3}$$

we have $(N \to \infty)$

$$I_{n,N}(x) = e^{-x^2/2} H_n(x)/(2^n n!)^{1/2} + O\left(\left(\frac{n^{3+1/2}}{N}\right)^{1/2}\right). \tag{4}$$

Proof. Let us take advantage of the following discrete analogue of the Leibnitz formula:

$$\Delta^n f(x) g(x) = \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k} f(x) \Delta^k g(x+n-k), \tag{5}$$

which can be easily established by induction. Letting $x^{[0]} = 1$, $x^{[n]} = x(x-1), \ldots (x - n + 1)$ $(n \geq 1)$, we find from (1) and (5)

$$\begin{aligned} \rho(x) K_n(x) &= \frac{(-q)^n}{n!} \Delta^n [\rho(x) x^{[n]}] \\ &= \frac{(-q)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k} \rho(x) \Delta^k (x+n-k)^{[n]} = \\ &= \frac{(-q)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} n^{[k]} (x+n-k)^{[n-k]} \Delta^{n-k} \rho(x). \end{aligned} \tag{6}$$

(Here we used the fact that $\Delta^{\ell} x^{[m]} = m^{[\ell]} x^{[m-\ell]}$.) Let $\Delta_h f(x) = f(x+h) - f(x)$, $\Delta_h^n f(x) = \Delta_h (\Delta_h^{n-1} f(x))$ $(n \geqslant 2)$. Letting $h = 1/(2Npq)^{1/2}$ and using the known formula $\Delta_h^n f(x) = h^n f^{(n)}(x + nh\theta)$ $(0 < \theta < 1)$ we have due to (2) and (6)

$$\begin{aligned} I_{n,N}(x) &= (2\pi Npq)^{1/2} e^{x^2/2} \left(\frac{q}{Np}\right)^{n/2} (-1)^n \cdot \\ &\cdot \frac{1}{(n!)^{1/2}} \sum_{k=0}^{n} \binom{n}{k} n^{[k]} (\tilde{x}+n-k)^{[n-k]} \Delta_h^{n-k} \rho(\tilde{x}) = \\ = e^{x^2/2} (-1)^n \sum_{k=0}^{n} & \frac{(n^{[k]})^{3/2} (q/p)^{k/2}}{N^{k/2} k! \, (Np)^{n-k}} (\tilde{x}+n-k)^{[n-k]} \frac{(2\pi Npq)^{1/2}}{2^{(n-k)/2} ((n-k)!)^{1/2}} \frac{d^{n-k}}{dt_k^{n-k}} \rho(\tilde{t}_k) \\ & (t_k = x + h(n-k)\theta_k, \ 0 < \theta_k < 1). \end{aligned} \tag{7}$$

Let $S_N(\xi)$ be such that

$$\rho(\tilde{\xi}) = \frac{1}{(2\pi Npq)^{1/2}} \exp(-\xi^2 + S_N(\xi)). \tag{8}$$

Using the integral theorem of Cauchy and the representation (8), we find

$$\begin{aligned} \frac{(2\pi Npq)^{1/2}}{2^{(n-k)/2} ((n-k)!)^{1/2}} \frac{d^{n-k}}{dt_k^{n-k}} \rho(\tilde{t}_k) &= \frac{((n-k)!)^{1/2} (2\pi Npq)^{1/2}}{2^{(n-k)/2} 2\pi i} \int_{c_k} \frac{\rho(\tilde{\xi}_k) \, d\xi_k}{(t_k - \xi_k)^{n-k+1}} = \\ &= \frac{(2\pi Npq)^{1/2} ((n-k)!)^{1/2}}{2^{(n-k)/2} 2\pi i} \left[\frac{1}{(2\pi Npq)^{1/2}} \int_{c_k} \frac{e^{-\xi_k^2} d\xi_k}{(t_k - \xi_k)^{n-k+1}} + \right. \\ &\left. + \int_{c_k} \frac{(\rho(\tilde{\xi}_k) - e^{-\xi_k^2}/(2\pi Npq)^{1/2}) \, d\xi_k}{(t_k - \xi_k)^{n-k+1}} \right] = \\ &= \frac{(-1)^{n-k} e^{-t_k^2} H_{n-k}(t_k)}{2^{(n-k)/2} ((n-k)!)^{1/2}} + \frac{((n-k)!)^{1/2}}{2^{(n-k)/2} 2\pi i} \int_{c_k} \frac{e^{-\xi_k^2} (e^{S_N(\xi_k)} - 1) \, d\xi_k}{(t_k - \xi_k)^{n-k+1}} \\ &= Q_{n-k}(t_k) + r_{n-k}(t_k) \quad (0 \leqslant k \leqslant n), \end{aligned}$$

where c_k is a closed contour, containing t_k. From (7) and (8) we deduce

$$I_{n,N}(x) = (-1)^n e^{x^2/2} \Big[Q_n(t_0) L_n +$$

$$+\sum_{k=1}^{n}\frac{(n^{[k]})^{3/2}}{k!\,N^{k/2}}\left(\frac{q}{p}\right)^{k/2}Q_{n-k}L_{n-k}+$$
$$+\sum_{k=0}^{n}\frac{(n^{[k]})^{3/2}}{k!\,N^{k/2}}\left(\frac{q}{p}\right)^{k/2}r_{n-k}(t_k)\,L_{n-k}\Big]$$
$$=(-1)^n e^{x^2/2}\,[Q_n(t_0)\,L_n+B_1+B_2], \tag{9}$$

where

$$L_{n-k}=(\tilde{x}+n-k)^{[n-k]}/(Np)^{n-k}.$$

Let us evaluate $|r_{n-k}(t_k)|, |L_{n-k}|$ and $|Q_{n-k}(t_k)|$. We need the asymptotic formula for the Euler gamma-function [4, p. 83]

$$\ln\Gamma(z)=\left(z-\frac{1}{2}\right)\ln z-z+\frac{1}{2}\ln(2\pi)+R_0(z), \tag{10}$$

where $|\arg z|<\pi/2$,

$$|R_0(z)|\leqslant\frac{k(z)}{12|z|},\qquad k(z)=\sup_{u\geqslant 0}\left|\frac{z^2}{u^2+z^2}\right|. \tag{11}$$

We have from (10) for $\tau=z/N$, $\rho(z)=N!p^zq^{N-z}/(\Gamma(z+1)\Gamma(N-z+1))$

$$\ln\rho(z)=-N\left[\tau\ln\frac{\tau}{p}+(1-\tau)\ln\frac{1-\tau}{q}\right]-$$
$$-\frac{1}{2}\ln(2\pi N\tau(1-\tau))+R_0(N)-R_0(z)-R_0(N-z). \tag{12}$$

Letting

$$H(\tau)=\tau\ln\frac{\tau}{p}+(1-\tau)\ln\frac{1-\tau}{q},$$

we have $H(p)=H'(p)=0$, $H''(p)=1/(pq)$. Therefore [5, p. 99]

$$H(\tau)=\frac{1}{2pq}(\tau-p)^2+r(\tau), \tag{13}$$

where

$$|r(\tau)|\leqslant\frac{|\tau-p|^3}{3(pq)^2},\quad \text{if}\quad |\tau-p|\leqslant\frac{1}{2}\min\{p,q\}. \tag{14}$$

On the other hand, since $\tau(1-\tau)-pq=(p-\tau)(\tau-q)$, then

$$\ln(\tau(1-\tau))=\ln(pq)+\ln\left(1+\frac{\tau(1-\tau)-pq}{pq}\right)=\ln(pq)+\ln\left(1+\frac{(p-\tau)(\tau-q)}{pq}\right).$$

Hence, taking into account

$$|\ln(1+\gamma)|=\left|\int_1^{1+\gamma}\frac{dz}{z}\right|<2|\gamma|\qquad\left(|\gamma|<\frac{1}{2}\right),$$

we find for $|(p-\tau)(\tau-q)|\leqslant pq/2$

$$\ln(\tau(1-\tau))=\ln(pq)+D(\tau), \tag{15}$$

where

$$|D(\tau)|\leqslant\frac{2}{pq}|(p-\tau)(\tau-q)|. \tag{16}$$

Composing (12), (13), and (15), we have ($\tau = z/N$)

$$\ln\rho(z) = -\frac{N}{2pq}(\tau - p)^2 - \frac{1}{2}\ln(2\pi Npq) + R_0(N) - R_0(z) - R_0(N-z) - Nr(\tau) - D(\tau)/2.$$

Hence, letting $z = \tilde{\xi} = Np + (2Npq)^{1/2}\xi$, we arrive at the representation (8), in which

$$S_N(\xi) = R_0(N) - R_0(\tilde{\xi}) - R_0(N - \tilde{\xi}) - Nr\left(p + \left(\frac{2pq}{N}\right)^{1/2}\xi\right) - D\left(p + \left(\frac{2pq}{N}\right)^{1/2}\xi\right)/2. \tag{17}$$

Let us evaluate $|S_N(\xi_k)|$ on the circle c_k with center at t_k and radius $d_k = (n-k)/2)^{1/2}$ $(k < n)$. From condition (3) of the theorem, it follows that for sufficiently large N

$$|x| \leqslant \frac{1}{4}(2Npq)^{1/2} - \left(\frac{n}{2}\right)^{1/2} - \frac{n}{(2Npq)^{1/2}}. \tag{18}$$

For $\xi_k = t_k + d_k e^{i\varphi}$ $(0 \leqslant \varphi \leqslant 2\pi)$, $t_k = x + (n-k)h\theta_k$, $\tau = \tilde{\xi}_k/N$ we have from (18)

$$|\tau - p| = \left(\frac{2pq}{N}\right)^{1/2}|t_k + d_k e^{i\varphi}| \leqslant \frac{pq}{2} \leqslant \frac{1}{2}\min\{p, q\}, \tag{19}$$

$$|(p-\tau)(\tau-q)| \leqslant \frac{pq}{2}\left(|p-q| + \frac{pq}{2}\right) \leqslant \frac{pq}{2}, \quad |\tau - q| < 1, \tag{20}$$

It follows from (19) and (20) that for $\tau = \tilde{\xi}_k/N$, the conditions, for the observance of which the bounds (14) and (16) were proved, are satisfied. Further, from (18) we obtain

$$\mathrm{Re}(\tilde{\xi}_k) \geqslant Np - (2Npq)^{1/2}|\xi_k| \geqslant Np - \frac{1}{2}Npq = Np(1 - q/2), \tag{21}$$

$$\mathrm{Re}(N - \tilde{\xi}_k) \geqslant Nq(1 - p/2), \tag{22}$$

$$|\mathrm{Im}(\tilde{\xi}_k)| \leqslant (Nnpq)^{1/2}, \quad |\mathrm{Im}(N - \tilde{\xi}_k)| \leqslant (Nnpq)^{1/2}. \tag{23}$$

From (19)-(23) and from the condition of the theorem, it follows as well for large enough N that

$$\mathrm{Re}(\tilde{\xi}_k^2) \geqslant \left(\frac{Np}{2}\right)^2 - Nnpq = Np\left(\frac{Np}{4} - nq\right) > 0, \tag{24}$$

$$\mathrm{Re}((N - \tilde{\xi}_k)^2) \geqslant \left(\frac{Nq}{2}\right)^2 - Nnpq = Nq\left(\frac{Nq}{4} - np\right) > 0. \tag{25}$$

Observing that for $\mathrm{Re}(z^2) > 0$

$$\sup_{u \geqslant 0}\left|\frac{z^2}{u^2 + z^2}\right| = 1 \tag{26}$$

and composing (11), (14), (16), (17), (19)-(26), we obtain for $0 \leqslant k \leqslant n$, $t_k = x + (n-k)h\theta_k$, $\xi_k = t_k + d_k e^{i\varphi}$, $\tau = \tilde{\xi}_k/N$

$$|S_N(\xi_k)| \leqslant \frac{1}{12N} + \frac{1}{12Np(1-q/2)} + \frac{1}{12Nq(1-p/2)} + \frac{N}{3(pq)^2}\left(\frac{2pq}{N}\right)^{3/2}\left(|x| + \left(\frac{n}{2}\right)^{1/2} + \frac{n}{(2Npq)^{1/2}}\right)^3 + \frac{1}{pq}\left(\frac{2pq}{N}\right)^{1/2}\left(|x| + \left(\frac{n}{2}\right)^{1/2} + n/(2pqN)^{1/2}\right) = G_{N,n}(x). \tag{27}$$

From (27), in turn, we find

$$|e^{S_N(\xi_k)} - 1| \leqslant G_{N,x}(x)\, e^{G_{N,n}(x)}. \tag{28}$$

Then, using the Stieltjes formula, we have ($k < n$)

$$\left(\frac{(n-k)!}{2^{n-k}}\right)^{1/2}\frac{1}{2\pi}\int_{c_k}\left|\frac{\exp(-\xi_k^2)}{(t_k-\xi_k)^{n-k+1}}\right||d\xi_k|=\frac{((n-k)!)^{1/2}}{2\pi(n-k)^{(n-k)/2}}\int_0^{2\pi}e^{-(t_k+d_k\cos\varphi)^2+d_k^2\sin^2\varphi}\,d\varphi=$$

$$=\frac{((n-k)!)^{1/2}e^{d_k^2-t_k^2/2}}{2\pi(n-k)^{(n-k)/2}}\int_0^{2\pi}e^{-(t_k/2^{1/2}+2^{1/2}d_k\cos\varphi)^2}d\varphi<\frac{(n-k)!^{1/2}e^{d_k^2-t_k^2/2}}{(n-k)^{(n-k)/2}}<(2\pi(n-k))^{1/4}e^{-t_k^2/2+1/(24(n-k))}. \tag{29}$$

We find from (27)-(29) for sufficiently large N $(t_k=x+h(n-k)\theta_k,\ 0<\theta_k<1)$

$$|r_{n-k}(t_k)|\leqslant G_{N,n}(x)e^{G_{N,n}(x)}(2\pi(n-k))^{1/4}e^{-t_k^2/2+1/(24(n-k))}, \tag{30}$$

where x and n satisfy (3).

For L_{n-k} we have from (9)

$$1+\left(\left(\frac{2q}{Np}\right)^{1/2}x\right)^{n-k}\quad\left(\leqslant L_{n-k}=\exp\left(\sum_{\nu=1}^{n-k}\ln\left(1+\left(\frac{2q}{Np}\right)^{1/2}x+\frac{1+n-k-\nu}{Np}\right)\right)\right.$$
$$\left.\leqslant\exp\left(\left(\frac{2q}{Np}\right)^{1/2}x(n-k)+\frac{(n-k)^2}{Np}\right)\right).$$

Hence, it follows that uniformly with respect to $0\le k\le n$ we have the estimate $(n=O(N^{1/3}))$

$$L_{n-k}=1+O\left(\left(\frac{n^2x^2}{Np}\right)^{1/2}+\frac{n^2}{N}\right)\qquad(x=O(n^{1/2})). \tag{31}$$

To estimate $|Q_{n-k}(t_k)|$ let us use the following known [3, p. 250] result $(m\ge1)$:

$$\max_{-\infty<z<\infty}e^{-z^2/2}|H_m(z)|=O((2^m m!)^{1/2}m^{-1/12}). \tag{32}$$

Then for $|Q_{n-k}(t_k)|$ we obtain $(0\le k<n)$

$$|Q_{n-k}(t_k)|=O((n-k)^{-1/12})e^{-t_k^2/2}. \tag{33}$$

Also, for k = n

$$|Q_0(t_n)|=e^{-t_n^2}. \tag{34}$$

Further, since

$$\frac{d}{dz}\{e^{-z^2}H_n(z)\}=-e^{-z^2}H_{n+1}(z),$$

then by Lagrange's theorem we find $(x<\eta<t_0)$

$$e^{-t_0^2}H_n(t_0)=e^{-x^2}H_n(x)-e^{-\eta^2}H_{n+1}(\eta)nh\theta_0, \tag{35}$$

and for $x=O(n^{1/3})$ and $x\le t\le x+nh$

$$\exp((x^2-t^2)/2)=1+\left(\frac{n^3}{N}\right)^{1/2}\exp\left(O\left(\frac{n^3}{N}\right)^{1/2}\right). \tag{36}$$

From (9), (31), (32), (35) and (36) we also have

$$e^{x^2/2}L_nQ_n(t_0)=(-1)^n\frac{e^{-x^2/2}}{(2^n n!)^{1/2}}H_n(x)+O\left(\left(\frac{n^{3-1/6}}{N}\right)^{1/2}\right) \tag{37}$$
$$(x=O(n^{1/2}),\ n=O(N^{1/3})).$$

In order to estimate the sum B_1, figuring in (9), we represent it in the following manner

$$B_1=\sum_{1\leqslant k\leqslant n/2}+\sum_{n/2<k\leqslant n}=B_1'+B_1''. \quad (38)$$

Then from (31), (33), and (36) we have

$$e^{x^2/2}|B_1'|=O\left(\left(\frac{n^{3-1/6}}{N}\right)^{1/2}\right)$$
$$(x=O(n^{1/2}),\ n=O(N^{1/3})). \quad (39)$$

For B_1'' we have from (31), (33), (34), and (36)

$$e^{x^2/2}|B_1''|=O\left(\sum_{n/2\leqslant k\leqslant n}\frac{1}{k!}\left(\frac{qn^3}{pN}\right)^{k/2}\right)=$$
$$=O\left(\frac{1}{[n/2]!}\left(\frac{qn^3}{pN}\right)^{[n/2]/2}\exp\left(\left(\frac{qn^3}{pN}\right)^{1/2}\right)\right), \quad (40)$$

where [n/2] is the integer part of n/2.

Finally, for the sum B_2 [see (9)] from (27), (30), (31), and (36) we find

$$e^{x^2/2}|B_2|=O(n^{1/4}G_{N,n}(x))=O\left(\left(\frac{n^{3+1/2}}{N}\right)^{1/2}\right) \quad (41)$$
$$(x=O(n^{1/2}),\ n=O(N^{1/3})).$$

Juxtaposing (9) and (37)-(41), we arrive at (4). The theorem is proved.

Let $x_{j,N}=x_{j,N}(n,p)$ $(j=1,2,\ldots,n)$ be the zeros of $\tilde{I}_{n,N}(x)$ situated in decreasing order: $x_{1,N}>x_{2,N}>\ldots>x_{n,N}$. Then by (2) $\tilde{x}_{1,N}>\tilde{x}_{2,N}>\ldots>\tilde{x}_{n,N}$ are the zeros of the Krawtchouk polynomial $K_n(t)$. Then, let $z_j=z_j(n)$ $(1\leqslant j\leqslant n+1)$ be the points of the extrema of the function $Z_n(x)=\exp(-x^2/2)H_n(x)$ such that $z_j>z_{j+1}$ $(1\leqslant j\leqslant n)$.

<u>COROLLARY 1</u>. If $\eta=\{\eta_N\}_1^\infty,\eta_N>0,\eta_N\to 0(N\to\infty)$, then for all sufficiently large N

$$z_j>x_{j,N}>z_{j+1}\ (1\leqslant j\leqslant n,\ 1\leqslant n\leqslant\eta_N N^{1/4}). \quad (42)$$

The proof of this corollary immediately stems form the asymptotic formula (4) and from the fact that the least maximum of $|Z_n(x)|$ in succession is not less (see [3, Theorems 7.6.3 and 8.22.9]) than $(2^n n!)^{1/2}n^{-1/4}$ and [see (45)] $z_{n+1},z_1=O(n^{1/2})$.

Let $x_j=x_j(n)$ $(1\leqslant j\leqslant n)$ be the zeros of the Hermite polynomial $H_n(x)$, arranged in decreasing order: $x_1>x_2>\ldots>x_n$, $0<i_1<i_2<\ldots$ the zeros of the Airy function (see [3, p, 32]). Below we need the following (see [3, p. 141]):

$$x_\nu(n)=(2n+1)^{1/2}-6^{-1/3}(2n+1)^{-1/6}(i_\nu+\varepsilon_n)$$
$$(\varepsilon_n\to 0\ (n\to\infty)). \quad (43)$$

<u>COROLLARY 2.</u> Let $\eta=\{\eta_N\}_1^\infty$, $\eta_N>0$, $\eta_N\to 0\ (N\to\infty)$. Then uniformly with respect to $1\leq n\leq\eta_N N^{1/4}$ (N = 1, 2, ...) for the least zero of the Krawtchouk polynomial, we have

$$\tilde{x}_{n,N}=Np\,(1-(2q/(Np))^{1/2}x_1(n))+O(n^{7/4}).$$

<u>Proof.</u> Since $\tilde{x}_{n,N}=N_p+(2Npq)^{1/2}x_{n,N}$, then for the proof of Corollary 2 it is sufficient to obtain for $1\leqslant n\leqslant\eta_N N^{1/4}\ (N\to\infty)$ the estimate

$$|x_{n,N}+x_1(n)|=O\left(\left(\frac{n^{3+1/2}}{N}\right)^{1/2}\right). \quad (44)$$

The function $Z_n(x)$ is even (odd) if n is even (odd) and satisfies $Z_n''(x)+(2n+1-x^2)Z_n(x)=0$. Therefore, we have

$$-(2n+1)^{1/2} \leqslant z_{n+1} < x_n = -x_1 < z_n < x_{n-1} = -x_2. \tag{45}$$

On the other hand, using the asymptotic formula (see [3, p. 208])

$$Z_n(x) = 3^{1/3}\pi^{-3/4}2^{n/2+1/4}(n!)^{1/2}n^{-1/12}\cdot \{A(t) + O(n^{-2/3})\}, \quad x = (2n+1)^{1/2} - 2^{-1/2}3^{-1/3}n^{-1/6}t, \tag{46}$$

where $t = O(1)$, $A(t)$ is the Airy function, we conclude that $|Z_n(z_n)|$ and $Z_n(z_{n+1})|$ in succession are no less than $(2^n n!)^{1/2}n^{-1/12}$. Then, since $Z_n''(x) = -(2n+1-x^2)Z_n(x)$, the function $|Z_n(x)|$ is convex (upwards) for $x_n \leq x \leq x_{n-1}$ and for $z_{n+1} \leq x \leq x_n$. Since due to (43) and (45) $|z_k - x_n| = O(n^{-1/6})$ $(k = n,\ n+1)$, then from the indicated properties of $Z_n(x)$ we find

$$|Z_n(x)/((x-x_n)(2^n n!)^{1/2})| \geqslant A_3 n^{1/12},\ x \in [z_{n+1}, z_n], \tag{47}$$

where $0 < A_3$ is an absolute constant. It follows from (47) that if $n = o(N^{1/4})$, then by (4) in the neighborhood $(x_n - \delta_n,\ x_n + \delta_n)$, where $\delta_n = \delta_n(N) = (n^{3+1/2}/N)^{1/2}$, the function $I_{n,N}(x)$ for sufficiently large N changes sign, that is, in this neighborhood there is contained at least one zero of $I_{n,N}(x)$. The estimate (44) will be established if we show that this zero coincides with $x_{n,N}$. With this object, let us consider $Z_n'(x) = -xH_n(x)e^{-x^2/2} + H_n'(x)e^{-x^2/2}$. Using the fact that $H_n'(x) = 2nH_{n-1}(x)$, $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$, we, hence, have

$$Z_n'(x) = nZ_{n-1}(x) - \frac{1}{2}Z_{n+1}(x). \tag{48}$$

Let us use (46) twice, replacing at first n by n − 1, and then by n + 1. Correspondingly, for x we have the representation, $x = (2k+1)^{1/2} - 2^{-1/2}3^{-1/3}k^{-1/6}t_k$ $(k = n-1,\ n+1)$, where $|t_{n-1} - t_{n+1}| = O(n^{-1/3})$. Hence, in turn, we find $n^{5/12}|A(t_{n-1}) - A(t_{n+1})| = O(n^{1/12})$. Therefore, observing that $n^{1/2}/(n-1)^{1/12} - (n+1)^{5/12} = O(n^{-7/12})$, we deduce from (46) and (48) for $x \in [z_{n+1}, z_n]$

$$|Z_n'(x)| \leqslant A_4(2^n n!)^{1/2}n^{1/12}, \tag{49}$$

where $0 < A_4$ is a constant.

Due to the theorem on the mean we have (k = n, n + 1)

$$\left|\frac{Z_n(z_k)}{z_k - x_n}\right| = \left|\frac{Z_n(z_k) - Z_n(x_n)}{z_k - x_n}\right| = |Z_n'(\eta)| \quad (\eta \in (z_{n+1}, z_n)). \tag{50}$$

Juxtaposing (46), (49), and (50), we conclude that there is an absolute constant $A_5 > 0$, for which $|z_k - x_n| \geq A_5 n^{-1/6}$. This means that for sufficinetly large N there will be $(x_n - \delta_n,\ x_n + \delta_n) \in (z_{n+1}, z_n)$. Indeed, this follows from the fact that $\delta_n = \delta_n(N) = o(n^{-1/4}) = o(n^{-1/6}) = o(|z_k - z_n|)$ $(n = o(N^{1/4}))$. Since, in accord with Corollary 1 $x_{n,N} \in (z_{n+1}, z_n)$ and $x_{k,N} \notin (z_{n+1}, z_n)$ $(k \neq n)$, all the more $x_{k,N} \notin (x_n - \delta_n,\ x_n + \delta_n)$ $(k \neq n)$. So, $x_{n,N} \in (x_n - \delta_n,\ x_n + \delta_n)$. In this way, (44) and along with it Corollary 2 are proved.

3. Let us note one application of Corollary 2 to coding theory. Following [2], let us introduce some necessary notation. Let E_m^N be the space of sequences of length N from the elements {0, 1, ..., m − 1} with the Hamming metric, $M(E_m^N, D)$ the maximal cardinality of a D-code $W \subset E_m^N$, $D(E_m^N, M)$ the magnitude, inverse for $M(E_m^N, D)$, equal to the maximal number D such that there exists a D-code $W \subset E_m^N$ of cardinality M. In the notation adopted in this paper, let us formulate one result established in [2]: that is, the bound $(m \geq 2)$

$$M(E_m^N, D) \leqslant \sum_{j=0}^{n}(m-1)^j\binom{N}{j} \quad (D \geqslant \bar{x}_{n,N-1} + 1), \tag{51}$$

where $\tilde{x}_{n,N-1} = \tilde{x}_{n,N-1}(n, (m-1)/m)$ is the smallest zero of $K_n(x; N-1, (m-1)/m)$. For fixed n and $N \to \infty$, it was proved in [2] that

$$D\left(E_m^N, \sum_{j=0}^{n} (m-1)^j \binom{N}{j}\right) \leqslant \frac{m-1}{m} N \left(1 - \left(\frac{2}{N(m-1)}\right)^{1/2} x_1(n)\right) + O(1). \tag{52}$$

COROLLARY 3. Let $m \geqslant 2$, $\eta = \{\eta_N\}_1^\infty$, $\eta_N > 0$, $\eta_N \to 0$ $(N \to \infty)$. Then uniformly with respect to $1 \leqslant n \leqslant \eta_N N^{1/4}$ $(N = 1, 2, \ldots)$ we have

$$D\left(E_m^N, \sum_{j=0}^{n} (m-1)^j \binom{N}{j}\right) \leqslant \frac{m-1}{m} N \left(1 - \left(\frac{2}{N(m-1)}\right)^{1/2} x_1(n)\right) + O(n^{7/4}). \tag{53}$$

If we let $p = (m-1)/m$, $q = 1/m$, then the assertion of Corollary 3 immediately follows from Corollary 2 and (51). It is clear that (53) is a generalization of (52) to the case when n grows along with N, remaining no more than $o(N^{1/4})$.

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