FACTORIZATION OF ALMOST-PERIODIC MATRIX-VALUED FUNCTIONS

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1. By AP we shall denote the algebra of uniform, almost periodic functions, defined on the real axis $\mathbf{R}$, by M(f) the Bohr mean of the function $f \in AP$, by $\hat{f}(\lambda)$ the coefficient of $e^{i\lambda x}$ of its Fourier series, i.e., $M(fe^{-i\lambda x})$, and by AP_W the subclass of AP, consisting of functions with an absolutely convergent Fourier series. By the spectrum of $f \in AP$ we shall mean $\Omega(f) = \{\lambda \in \mathbf{R}: \hat{f}(\lambda) \neq 0\}$. We set $AP^{\pm} = \{f \in AP: \pm\lambda \geq 0 \ \forall\lambda \in \Omega(f)\}$, $AP_W^{\pm} = AP^{\pm} \cap AP_W$. The elements of $AP^{\pm}$ admit a continuous extension to functions that are analytic and bounded in $\Pi^{\pm} = \{\zeta: \pm\mathrm{Im}\,\zeta > 0\}$.

As in [2, 3], by a P-factorization of an $n \times n$ matrix-valued function G, defined on $\mathbf{R}$, we mean the representation

$$G = G_+ \Lambda G_-, \tag{1}$$

where $\Lambda(x) = \mathrm{diag}[e^{i\lambda_1 x}, \ldots, e^{i\lambda_n x}]$, $\lambda_j \in \mathbf{R}$,*

$$G_+^{\pm 1} \in AP^+, \quad G_-^{\pm 1} \in AP^-. \tag{2}$$

Replacing condition (2) by the more rigid requirement $G_+^{\pm 1} \in AP_W^+$, $G^{\pm 1} \in AP_W^-$, we obtain the definition of a P_V-factorization. The P- and P_W-factorizations have been studied in [2-5] in connection with the investigation of the solvability of singular integral equations and equations of convolution type on systems of intervals.

An obvious necessary condition for P- (P_W-) factorization is $G^{\pm 1} \in AP$ (AP_W). If the matrix-valued function G is periodic with period τ, then by the formula $X(\zeta) = G(\tau \ln \zeta / 2\pi i)$ one associates to it a matrix-valued function on the unit circumference $\mathbf{T}$. Moreover, the condition $G \in AP$ is equivalent to the continuity of X. Being continuous and nonsingular, the matrix X admits (see [6, 7]) the factorization

$$X(\zeta) = X_+(\zeta) D(\zeta) X_-(\zeta). \tag{3}$$

Here $X_+^{\pm 1}$ and $X_-^{\pm 1}$ belong to the Hardy classes H_p (inside and outside $\mathbf{T}$, respectively) for all $p < \infty$, $D(\zeta) = \mathrm{diag}[\zeta^{\kappa_1}, \ldots, \zeta^{\kappa_n}]$, while $\kappa_1, \ldots, \kappa_n$ ($\in \mathbf{Z}$) are the so-called partial indices of the matrix X. The P-factorization of a periodic matrix-valued function G is equivalent to the fact that X is factorizable in C, i.e., the factors $X_\pm$ from (3) are continuous up to $\mathbf{T}$, while the P_W-factorizability of G is equivalent to the fact that $X_\pm$ are in the Wiener algebra W of matrix-valued functions with absolutely convergent Fourier series.

Thus, the P- and P_W-factorization can be considered as some analogue of the factorization (in C and W, respectively) of matrix-valued functions defined on $\mathbf{T}$. This analogy is not complete. Thus, one has constructed examples of matrices of the form

$$Z(x) = \begin{bmatrix} e^{i\lambda x} & 0 \\ \sum_{k=1}^{m} c_k e^{i\alpha_k x} & e^{-i\lambda x} \end{bmatrix}, \quad (-\lambda < \alpha_k < \lambda), \tag{4}$$

for m = 3, which do not admit P-factorization [3, 8] (the existence of such matrices for m = 5 has been proved also in [9]), while the factorizability in C or W can be ensured by conditions on the smoothness of the matrix.

Nevertheless, the formulations of a series of results on factorization in C(W) can be carried over directly to P- (P_W-) factorization, although the proof of some of them require

*Here and in the sequel, the membership of a matrix-valued function to some functional class will be understood elementwise.

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the use of essentially different methods. We give those results which will be used in the sequel.

THEOREM A [2, 3]. The numbers λ_j (we shall call them partial P-indices) are determined from the P-factorizable matrix-valued function G to within a permutation. Their sum coincides with $\lambda(G)$, the almost-periodic index (mean motion) of the function det G. The partial P-indices are stable with respect to small perturbations of G, preserving the P-factorizability, if and only if $\lambda_1 = \ldots = \lambda_n$. In this case an invariant of the P-factorization is also the quantity $d(G) = M(G_+)M(G_-)$.

THEOREM B [10, 11]. The conditions $G \in AP_W$, $\inf\{|(G(x)h, h)|\cdot\|h\|^{-2}:\ h \in \mathbf{C}^n \setminus \{0\},\ x \in \mathbf{R}\} > 0$ are sufficient for the P_W-factorizability of G. Moreover, all of its partial P-indices coincide among themselves and with the quantity $\lambda((Gh, h))$ for any $h \in \mathbf{C}^n \setminus\{0\}$.

Here we give some new results on P-factorizations, suggested by its analogy with the factorization in C. The symbol Δ will mark the end of a proof.

2. In this section we consider the P-factorizability of matrix-valued functions of class AP_W, with partial P-indices that coincide among themselves. For $G \in AP$ with Fourier series $\Sigma\hat{G}(\lambda_j)e^{i\lambda_j x}$ we set

$$\|G\| = \sup\{|G(x)|:\ x \in \mathbf{R}\},\quad \|G\|_W = \textstyle\sum_j |\hat{G}(\lambda_j)|.$$

Thus, $G \in AP_W$ if and only if $\|G\|_W < \infty$.

Here, for a numerical matrix A, by $|A|$ we denote its largest s-number, i.e., the square root of the largest eigenvalue of A^*A.

THEOREM 1. Let F be an n × n matrix-valued function of class AP_W. The following statements are equivalent: 1) F is P_W-factorizable with zero partial P-indices; 2) F is P-factorizable with zero partial P-indices; 3) there exists a matrix-valued function X_+, belonging to the class AP^+ together with its inverse, such that

$$\|I - X_+F\| < 1; \tag{5}$$

4) there exists a matrix-valued function X_-, belonging to the class AP^- together with its inverse, such that

$$\|I - FX_-\| < 1. \tag{6}$$

Proof. The implication 1) → 2) is obvious. Assume that 2) is satisfied, i.e., $F = F_+F_-$, $F_+^{\pm 1} \in AP^+$ and $F_-^{\pm 1} \in AP^-$. We set $X_+ = \mu F_-^* F_+^{-1}$, where μ is a constant from the interval $(0, \|F_-\|^{-2})$. Then $\|I - X_+F\| = \|I - \mu F_-^*F_-\| < 1$. Thus, 2) → 3).

If, however, 3) holds, then, by a small perturbation of X_+ in the metric of AP, one can achieve that $X_+^{\pm 1} \in AP_W^+$ and that, as before, inequality (5) should hold. In this case the factor X_+ does not affect P_W-factorizability and the values of the partial P-indices. Therefore, without loss of generality we shall assume that instead of (5) we have

$$\|I - F\| < 1. \tag{7}$$

The P_W-factorizability with zero partial P-indices of a matrix $F \in AP_W$, satisfying relation (7), follows from Theorem B.

Thus, 3) ⇒ 1) and, therefore, the statements 1)-3) are equivalent. The equivalence of the statements 1), 2), 4) is established in a similar manner. Δ

Regarding the analogue of Theorem 1 for factorization in C, see [12]. It is clear that Theorem 1 remains valid if instead of the vanishing of the partial P-indices we require only that they be equal among themselves and if in the inequalities (5), (6) we replace F by $F\exp\left(-\frac{i\lambda(F)x}{n}\right)$.

THEOREM 2. Assume that the matrix-valued function $G \in AP_W$ is P-factorizable. Then: 1) For the preservation of P-factorizability under small (in the metric of AP) perturbations, not leading out from AP_W, it is necessary and sufficient that the partial P-indices of G should coincide among themselves. 2) If this condition is satisfied, then the factorization factors $G_{\pm}$ and the matrix d(G) depend continuously (in the metric of AP_W) on G.

Proof. 1) Sufficiency follows from Theorem 1 and the stability of the inequality (4) with respect to small perturbations of F.

In order to prove necessity, we consider a matrix $G \in AP_W$, represented in the form (1), having at least two distinct partial P-indices. Without loss of generality we can assume that $\lambda_1 > \lambda_2$. We set $\lambda = 1/2(\lambda_1 - \lambda_2)$ and $G_\varepsilon = G_+T_\varepsilon \operatorname{diag}[e^{i(\lambda_1+\lambda_2)x/2}Z, E^{i\lambda_3 x}, \ldots, e^{i\lambda_n x}]T_\varepsilon^{-1}G_-$. Here $T_\varepsilon = \operatorname{diag}[1, \varepsilon, 1, \ldots, 1]$, while Z is a matrix of the form (4). Then $G_\varepsilon - G = \varepsilon(G_1 - G)$, so that $\|G_\varepsilon - G\| \to 0$ for $\varepsilon \to 0$. At the same time, for any $\varepsilon \neq 0$ the matrix G does not admit P-factorization together with Z.

2) By virtue of the implication 2) $\Rightarrow$ 1) of Theorem 1, the matrix G is P_W-factorizable. But then, from the smallness of $\|F - G\|_W$, there follows the smallness of $\|I - G_+^{-1}e^{-i\nu x}FG_-^{-1}\|_W$, where $\nu = (\lambda(G)/n)$. Therefore, it is sufficient to consider the case G = I. But for

$$\| I - F \|_W < 1 \tag{8}$$

the factorization factors of the matrix F can be computed by the formulas

$$F_+^{-1} = Q \textstyle\sum_{k=0}^{\infty} \mathsf{X}^k I, \quad F_-^{-1} = \left(\sum_{k=0}^{\infty} \mathsf{Y}^k I\right) Q^{-1}. \tag{9}$$

Here Q is an arbitrary nonsingular matrix, the operators X and Y act on matrix-valued functions $f \in AP_W$ according to the rules $\mathsf{X}f = P_+(f(F - I))$, $\mathsf{Y}f = P_-((F - I)f)$, $P_\pm$ are the projections corresponding to the decomposition of AP_W into the direct sum of AP_W^+ with $A^0P_W^- = \{\varphi \in AP_W^-: M(\varphi) = 0\}$. Condition (8) ensures the absolute convergence of the series (9) in the metric AP_W. If we take Q = I, then $\max\{\|F_\pm - I\|_W, \|F_\pm^{-1} - I\|_W\} \le \text{const}\cdot\|F - I\|_W$. Taking into account the relation $M(\mathsf{Y}f) = 0$, from (9) we obtain

$$d(F) = \left(\textstyle\sum_{k=0}^{\infty} M(\mathsf{X}^k I)\right)^{-1},$$

which proves the continuous dependence of d(F) on F. Δ

Formula (9), used in the proof, follows from the known (see, for example, [6]) general lemma on the factorization of elements, close to the identity, of an abstract splitting Banach algebra. For n = 1, from the explicit formulas for the factorization factors there follows that $d(F) = \exp M(\ln Fe^{-i\lambda(F)x})$. Thus, in this case the mapping d is continuous also in the metric of AP; moreover, it can be extended by continuity to the set of all invertible elements of AP. This fact has been used at the investigation of scalar singular integral equations with semi-almost-periodic discontinuities of the coefficients [13]. The author does not know whether the corresponding refinement of Theorem 2 for n > 1 is valid or not. Nevertheless, one can obtain definite restriction on the set of the limit values of d (G_α) for $|G_\alpha - G\| \to 0$; see [3].

3. Together with each P-factorizable matrix-valued function G, also its adjoint admits a P-factorization. Moreover, the P-factorization of G* is obtained from (1) by the operation of taking adjoints:

$$G^* = G_-^*\Lambda^{-1}G_+^*. \tag{10}$$

Consequently, the partial P-indices of G and G* are obtained from each other by changing the signs. Therefore, for a Hermitian P-factorizable matrix-valued function, to each positive partial P-index λ_j there corresponds the negative $-\lambda_j$ with the same multiplicity. The corresponding assertion for factorization in C has been mentioned in [14]. It turns out that for P-factorizable matrix-valued functions, the analogue of the result from [15, [16] on a special factorization form is also valid.

THEOREM 3. If the Hermitian matrix-valued function G with signature σ is P_W-factorizable, then there exists its representation in the form

$$G = A_{00} A_0^*, \tag{11}$$

where $A_0^{\pm 1} \in AP_W^+$,

$$\Lambda_0(x) = \begin{bmatrix} & & & & & e^{i\lambda_m x} \\ & & & & \cdot^{\cdot^{\cdot}} & \\ & & & e^{i\lambda_1 x} & & \\ & & J & & & \\ & e^{-i\lambda_1 x} & & & & \\ \cdot^{\cdot^{\cdot}} & & & & & \\ e^{-i\lambda_m x} & & & & & \end{bmatrix},$$

$J = \begin{bmatrix} I_{l_+} & 0 \\ 0 & -I_{l_-} \end{bmatrix}$, the dimensions $l_{\pm}(\geqslant 0)$ of the blocks of the matrix J are determined from the conditions $l_0 = l_+ + l_- = n - 2m, l_+ - l_- = \sigma$, while $\lambda_1, \ldots, \lambda_m$ ($m \geq 0$) are all the positive partial P-indices of G.

Proof. Without loss of generality we can assume that $\lambda_1 \leq \ldots \leq \lambda_m$, while the exponents of the diagonal elements of the matrix Λ are arranged in nonincreasing order. Let $\nu_1 < \ldots < \nu_k$ be all the distinct positive partial P-indices of G, and let $l_1, \ldots, l_k$ be their multiplicity $(l_1 + \ldots + l_k = m)$. We denote also $\nu_{-s} = -\nu_s$ ($s = 1, \ldots, k$), $\nu_0 = 0$.

Comparing (1) and (10), from the Hermitian property of G we obtain

$$\Phi\Lambda = \Lambda^{-1}\Phi^*, \tag{12}$$

where $\Phi = G_-^{*-1} G_+ \in AP_W^+$. Moreover,

$$G = G_-^* \Phi \Lambda G_-. \tag{13}$$

We assume that in AP_W^+ there exists an invertible matrix-valued function Ψ:

$$\Phi\Lambda = \Psi\Lambda_0\Psi^*. \tag{14}$$

Then, setting $A_0 = G_-^*\Psi$, from (13) we obtain the representation (11).

It remains to prove the solvability of equation (14) with respect to Ψ in the indicated class. For this we represent Φ, Ψ in the block form $\Phi = (\Phi_{r,s})_{r,s=-k}^{k}$, $\Psi = (\Psi_{r,s})_{r,s=-k}^{r}$ (the blocks are numbered from left to right and from below upwards), where $\Phi_{r,r}$, $\Psi_{r,r}$ are square matrices of order $l_{|r|}$. Relation (12) is equivalent to the following one:

$$\Phi_{r,s}(x) = \Phi^*_{-s,-r}(x)\, e^{i(\nu_s - \nu_r)x}. \tag{15}$$

Therefore, $\Phi_{r,s} = 0$ and $s < r$ and

$$\Omega(\Phi_{r,s}) \subset [0, \nu_s - \nu_r] \quad \text{for } s \geqslant r. \tag{16}$$

In particular, the blocks $\Phi_{r,r}$ are constant. From the nonsingularity of $G_{\pm}$ and from the block-triangular structure of Φ, just proved, there follows the nonsingularity of these blocks.

Equation (14) is equivalent to the system

$$\sum_{\alpha=-k}^{k} \Psi_{r,-\alpha} J_\alpha \Psi^*_{-s,\alpha} e^{-i\nu_\alpha x} = \Phi_{r,s} e^{-i\nu_s x}, \tag{17}$$

where $J_0 = J$ and $J_\alpha = (\delta_{i,\, l-j+1})_{i,\, j=1}^{l}$, $l = l_{|\alpha|}$ for $\alpha \neq 0$. The relation (12), in combination with the Hermitian property of $\Psi\Lambda_0\Psi^*$, allows us to restrict ourselves to those values of the indices r, s ($= -k, \ldots, k$) for which $r + s \geq 0$.

We subject the desired matrix Ψ to the additional condition

$$\Psi_{\alpha,\beta} = 0 \quad \text{for } \alpha + \beta > 0.$$

Then for $s < r$ both parts of (17) vanish and one has to achieve the validity of (17) for $s \geq r$. We partition the remaining equations of (17) into groups, placing in the jth group those for which $s - r = j$ ($j = 0, \ldots, 2k$).

The zeroth group consists of the equations

$$\Psi_{r,-r}J_r\Psi^*_{-r,r} = \Phi_{r,r} \quad (r = 1, \ldots, k),$$
$$\Psi_{00}J\Psi^*_{00} = \Phi_{00}.$$

All of them are solvable in the class of constant nonsingular matrices (for the first k equations this is obvious, while for the last one has to make use of the Hermitian property of Φ_{00}, resulting from (15), and of the equality of the signatures of the matrices G and $\Phi\Lambda$).

Now we assume that all the equations of the jth groups have been solved for $j \le j_0$ and the spectrum of the obtained $\Psi_{\alpha,\beta}$ $(\alpha + \beta \ge -j_0)$ is concentrated in $[0, \nu_{-\beta} - \nu_\alpha]$. Each equation of the $(j_0 + 1)$th group can be rewritten in the form

$$\Psi_{r,-r}J_r\Psi^*_{-s,r}(x)\, e^{i(\nu_s-\nu_r)x} + \Psi_{r,-s}(x)\, J_s\Psi^*_{-s,s} = \Phi_{r,s}(x) - \sum_{\alpha=r+1}^{s-1} \Psi_{r,-\alpha}(x)\, J_\alpha\Psi^*_{-s,\alpha}(x)\, e^{i(\nu_s-\nu_\alpha)x}. \tag{18}$$

By virtue of the assumption and of (16), the right-hand side of the equality (18) represents a matrix-valued function of class AP_W with spectrum in $[0, \nu_s - \nu_r]$. The same is true for the matrix $\Psi_{r,-s}$, determined from (18), if for $r \ne -s$ one sets $\Psi_{-s,r} = 0$.

If, however, $r = -s$ (which is possible only for odd j_0), then Eq. (18) can be rewritten in the following manner:

$$2\mathrm{Re}\,(\Psi_{r,r}(x)\, J_r\Psi^*_{r,-r}e^{i\nu_r x}) = \Phi_{r,-r}(x)\, e^{i\nu_r x} - \sum_{\alpha=r+1}^{-r-1} \Psi_{r,-\alpha}(x)\, J_\alpha\Psi^*_{r,\alpha}(x)\, e^{-i\nu_\alpha x}.$$

The right-hand side of this equation is Hermitian, which can be easily seen on the basis of (15) and by replacing the summation index α by $-\alpha$. Since the spectrum of all the terms of the right-hand side is concentrated in $[-\nu_s, \nu_s]$, it can be represented in the form 2Re X, where $X \in AP_W$ and $\Omega(X) \subset [0, \nu_s]$. It remains to set $\Psi_{r,r}(x) = e^{i\nu_s x}X(x)\,\Psi_{r,-r}^{*-1}J_r$.

Thus, under the made assumption, all the equations of the $(j_0 + 1)$th group are solvable and the obtained solutions $\Psi_{r,s}(s - r = j_0 + 1)$ are matrix-valued functions of the class AP_W with spectrum in $[0, \nu_{-s} - \nu_r]$. Thus, we have proved the existence of a matrix $\Psi \in AP_W^+$, satisfying (14). This matrix is invertible in AP_W^+ since $\det \Psi = \Pi \det \Psi_{r,-r}$ is a nonzero constant.

<u>COROLLARY 1.</u> Let N be a Hermitian-positive matrix-valued function of class AP_W. Then, under the condition $N^{-1} \in L_\infty$, it can be represented in the form

$$N = N_+N_+^*, \tag{19}$$

where $N_+^{\pm 1} \in AP_W^+$.

Indeed, in the Hermitian-positive case, the requirement $N^{-1} \in L_\infty$ guarantees the P_W-factorizability of N by virtue of Theorem B. However, the special form (19) of the P_W-factorization is ensured by Theorem 3 by taking into account that for $\sigma = n$ we necessarily have $m = l_- = 0,\ l_+ = n$, i.e., $\Lambda_0 = I$.

Thus, for each matrix-valued function G, invertible in AP_W, one can determine matrix-valued functions A_1, A_2, invertible in AP_W^+, such that

$$A_1A_1^* = GG^*,\quad A_2A_2^* = G^*G. \tag{20}$$

The matrices A_j are determined by the conditions (20) to within a right constant unitary factor. The matrix-valued functions

$$U_1 = A_1^{-1}G,\quad U_2 = GA_2^{*-1} \tag{21}$$

are unitary (see [17]), P- or P_W-factorizable only simultaneously with G, and have the same collection of partial P-indices. In connection with this we mention that, for unitary-valued matrices, Theorem 1 has a simpler form:

<u>COROLLARY 2.</u> Let F be a unitary matrix-valued function of class AP_W. For its P-(equivalently: P_W-) factorizability with zero partial P-indices it is necessary and sufficient that there exist a matrix-valued function X, belonging together with its inverse to the class AP^+ or AP^-, such that $\|F - X\| < 1$.

Besides (20), (21), at the investigation of P-factorizability, there exist other methods for passing from arbitrary invertible matrices from AP_W to unitary ones. One can, for example, determine the matrix-valued functions B_1, B_2, invertible in AP_W^+, from the relations $B_1B_1^* = (GG^*)^{1/2}$, $B_2B_2^* = (G^*G)^{1/2}$ and to set $U = B_1^{-1}GB_2^{*-1}$. The matrix-valued function U is unitary-valued since $U^*U = B_2^{-1}G^*(B_1B_1^*)^{-1}GB_2^{*-1} = B_2^{-1}G^*(GG^*)^{-1/2}GB_2^{*-1} = B_2^{-1}(G^*G)^{1/2}B_2^{*-1} = I$.

If the values of G are normal, then $GG^* = G^*G$ and, without loss of generality, we can assume that $B_1 = B_2$. In this case, the quadratic forms, determined by the matrices G and U, are congruent. In particular, for a Hermitian G, the matrix U will be simultaneously Hermitian and unitary. We give a criterion for the P_W-factorizability of such matrices.

THEOREM 4. Assume that the matrix-valued function $G \in AP_W$ is simultaneously Hermitian and unitary and that for all $x \in \mathbf{R}$, the eigenspace $\mathfrak{L}(x)$, corresponding to the eigenvalue 1, of the matrix G(x) forms with some fixed subspace $\mathfrak{L} \subset \mathbf{C}^n$ a maximal angle, not exceeding $\pi/4 - \varepsilon$. Then G is P_W-factorizable with zero partial P-indices.

Proof. We select in $\mathbf{C}^n$ an orthonormal basis so that the first $k = \dim \mathfrak{L}$ of its elements should form a basis in $\mathfrak{L}$. The matrix in this basis of the operator of multiplication by G(x) will be denoted by A(x) and we partition it into blocks in accordance with the decomposition $\mathbf{C}^n = \mathfrak{L} \oplus \mathfrak{L}^\perp$:

$$A(x) = \begin{bmatrix} A_{11}(x) & A_{21}^*(x) \\ A_{21}(x) & A_{22}(x) \end{bmatrix}.$$

The square of the operator cosine of the angle between $\mathfrak{L}$ and $\mathfrak{L}(x)$ is given by the matrix $1/2(I + A_{11})$, and, by assumption, its spectrum is separated from 1/2. In other words, the matrix $A_{11}(x)$ is uniformly positive. By considering the angle between $\mathfrak{L}^\perp$ and $\mathfrak{L}(x)^\perp$, we establish the uniform negativity of $A_{22}(x)$. But then $\begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix} A$ has a uniformly positive real part and, by virtue of Theorem B, it is P_W-factorizable. At the same time, also the matrix G is P_W-factorizable, differing only by constant factors. Δ

An analogue of Theorem 4 for factorizations of type (3) is formulated in [8].

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WEAKLY EXTREMAL PROPERTIES OF BANACH SPACES

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In this article we will be interested in theorems of the form:

THEOREM A. Let X be a Banach space and let some condition P = P(f) be satisfied for all linear functionals $f \in X^*$. Then assertion Q holds. Or, briefly

$$(\forall f \in X^* P(f)) \Rightarrow Q.$$

The classical Banach–Steinhaus theorem is of that form: if a sequence $\{x_n^*\}$ of elements of a Banach space X satisfy the condition $\forall f \in X^* \sup_n |f(x_n)| < \infty$, then the sequence $\{x_n\}$ is bounded in norm.

The principal goal of this article, arising under the influence of [1] (although Theorem 1, basic for all applications, was proved earlier by the author [see 2, Theorem 2.2.3]), is to clear up under which restrictions on the Banach space X one can substitute for the condition $\forall f \in X^*$, P(f) the weaker

$$\forall f \in \operatorname{ext} U(X^*)\, P(f),$$

where ext $U(X^*)$ is the set of extreme points of the unit ball $U(X^*)$ of the space X^*.

It is interesting that for all types of theorems of the form A, such a sufficient restriction on the space X (and for separable X it is also necessary) always turns out to be $X \not\supset c_0$, i.e., the space X does not contain subspaces isomorphic to c_0 (it is understood, except for cases when no restrictions are required, Choquet's theorem "works").

Before passing to a presentation of the results, we recall that a subset B of the unit ball $U(X^*)$ of the dual Banach space X^* is said to be total if $\forall x \in X \setminus 0$, $\sup\{|f(x)|: f \in B\} > 0$; a subset $B \subset U(X^*)$ is said to be normalizing if

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