THEOREMS ON ESTIMATES IN THE NEIGHBORHOOD OF A SINGULAR POINT OF A MAPPING

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1. Two Statements. 1. Let X and Y be Banach spaces, $F: X \to Y$ be a mapping of X into Y, x_* be a fixed point of X, and U be a neighborhood of this point. We set $M(x_*) = \{x \in U | F(x) = F(x_*)\}$. It is required to find an upper estimate for the distance $\rho(x, M(x_*))$ in terms of the quantity $\|F(x) - F(x_*)\|$, where x is a point in U. The following linear estimate is well known in the case where the mapping F is regular at the point x_* (if F is Fréchet-differentiable at x_*, then this means that $\operatorname{Im} F'(x_*) = Y$):

$$\rho(x, M(x_*)) \leqslant K \| F(x) - F(x_*) \| \quad \forall x \in U. \tag{1.1}$$

Estimate (1.1) follows at once from the Lyusternik theorem [1, p. 41; 2]. The most complete investigation of this type of linear estimates in the regular case is offered in [2]. At the same time, in the nonregular, singular, case ($\operatorname{Im} F'(x_*) \neq Y$) estimate (1.1) is not valid in general. This is seen readily by the example of the function $f(x) = x_1^2 + x_2^2 - x_3^2$ for $x_* = 0$.

2. Let X be a topological space, Y and Z be Banach spaces, W and U respectively be neighborhoods of the point (x_*, z_*) in $X \times Z$ and of the point (x_*, y_*) in $X \times Y$, and F be a mapping of U into Z such that $F(x_*, y_*) = z_*$. It is required to prove the existence of a mapping $\varphi: U \to Y$ such that

$$F(x, \varphi(x, z)) = z \quad \forall (x, z) \in W,$$

and to obtain an upper estimate for $\|\varphi(x, z) - y_*\|$ in terms of the quantity $\|F(x, y_*) - z\|$. In the regular case [if the mapping $y \to F(x_*, y)$ is Fréchet-differentiable, then this means that $\operatorname{Im} F_y(x_*, y_*) = Z$] the implicit function theorem of [3, p. 161] gives complete solution of the above-posed problem. In the same manner as in the first case, we are interested in the nonregular case, to which the theorem of [3] is not applicable.

To investigate both the problems under consideration we prove at first a general estimation theorem that is meaningful in the nonregular case.

2. Estimation Theorem. Let V and W be Banach spaces, $\mathscr{L}(V, W)$ be the space of continuous linear mappings of V into W, and $\mathscr{L}((V, V), W)$ be the space of continuous bilinear mappings. Both these spaces are Banach spaces with respect to the following norms:

$$\| \Lambda \| = \sup_{v \in V} \{\| \Lambda v \| \mid \| v \| \leqslant 1\}, \quad \Lambda \in \mathscr{L}(V, W);$$
$$\| B \| = \sup \{ \| B(v_1, v_2) \| \mid \| v_1 \| \leqslant 1, \ \| v_2 \| \leqslant 1\},$$
$$B \in \mathscr{L}((V, V), W).$$

We fix mappings $\Lambda \in \mathscr{L}(V, W)$ and $B \in \mathscr{L}((V, V), W)$. Let us suppose that the subspace $L = \operatorname{Im} \Lambda$ is closed and complemented. Then there exist a closed subspace N and continuous projections P_1 and P_2 such that $W = L + N$, $N \cap L = \{0\}$, $\operatorname{Im} P_1 = L$, $\operatorname{Im} P_2 = N$, $\operatorname{Ker} P_1 = N$, and $\operatorname{Ker} P_2 = L$. For an arbitrary fixed $h \in V$ we introduce the linear mapping

$$G(h): V \to L \times N = Z, \tag{2.1}$$
$$G(h)x = (\Lambda x, P_2 B(h, x)). \tag{2.2}$$

Let us suppose that $\operatorname{Im} G(h) = Z$. Then, by the lemma on the right inverse operator [2], we conclude that there exist a mapping $A: Z \to V$ and a constant $C = C(h) > 0$ such that

$$G(h) \cdot A = I_Z, \tag{2.3}$$

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$$\| A(z) \| \leqslant C \| z \| , \tag{2.4}$$

where I_Z is the identity operator from Z into Z, $z = (v_1, v_2)$, $v_1 \in L$, $v_2 \in N$, and $\|z\| = \max\{\|v_1\|, \|v_2\|\}$. Let us set

$$K = \max\{32C^2 \| P_2 \| \| B \|, \ 2C, 1\}. \tag{2.5}$$

THEOREM 1. Let U be a topological space, V and W be Banach spaces, T be a neighborhood of a point (u_*, v_*) in U × V, and $\mathcal{F}$ be a mapping of T into W. Suppose that the following conditions are fulfilled:

1) The mapping $u \to \mathcal{F}(u, v)$ is continuous at the point u_* for all v from a neighborhood of v_*.

2) (The quadratic approximation condition.) There exist mappings $\Lambda \in \mathcal{L}(V, W)$ and $B \in \mathcal{L}((V, W)$, a number $\delta > 0$, and a neighborhood S of the point u_* such that the conditions $u \in S$ and $\|\bar{v}\| \le \delta$ imply that

$$\mathcal{F}(u, v_* + \bar{v}) = \mathcal{F}(u, v_*) + \Lambda\bar{v} + B(\bar{v}, \bar{v})/2 + \omega(u, \bar{v}),$$

where $\omega(u, \bar{v})$ satisfies the estimate

$$\|\omega(u, \bar{v}') - \omega(u, \bar{v}'')\| \leqslant \beta(\|\bar{v}'\| + \|\bar{v}''\|)(\|\bar{v}' - \bar{v}''\|),$$
$$\beta = \text{const} \leqslant 1/\{32(\|P_1\| + \|P_2\|)C\};$$

3) The mappings Λ and B from condition 2) and a point $h \in V$, $\|h\| = 1$, are such that $\operatorname{Im}\Lambda$ is a closed complemented subspace of W and

$$\operatorname{Im} G(h) = Z,$$

where G(h) and Z are from (2.1) and (2.2).

Then there exist neighborhoods U_0 and V_0 of the points u_* and v_* respectively and a number $\varepsilon > 0$ such that for arbitrary $u \in U_0$ and $v \in V_0 \setminus \{v_*\}$, satisfying the conditions

a) $\|(v - v_*)\|v - v_*\|^{-1} - h\| \le \varepsilon$,

b) if $\operatorname{Im}\Lambda \ne W$, then

$$\|v - v_*\| \geqslant K\left[\|P_1(\mathcal{F}(u, v) - \mathcal{F}(u_*, v_*))\| + \frac{\|P_2(\mathcal{F}(u, v) - \mathcal{F}(u_*, v_*))\|}{\|v - v_*\|}\right];$$

there exists a point $r(u, v) \in V$, for which

$$\mathcal{F}(u, v + r(u, v)) = \mathcal{F}(u_*, v_*); \tag{2.6}$$

$$\|r(u, v)\| \leqslant$$
$$\leqslant K\left[\|P_1(\mathcal{F}(u, v) - \mathcal{F}(u_* v_*))\| + \frac{\|P_2(\mathcal{F}(u, v) - \mathcal{F}(u_*, v_*))\|}{\|v - v_*\|}\right]. \tag{2.7}$$

Proof. We easily obtain the following inequality, valid for all $p \in V$, $u \in S$, and v', $v'' \in V$ such that $\|v' - v_*\| \le \delta$ and $\|v'' - v_*\| \le \delta$, from condition 3) by identity transformation:

$$\begin{aligned} &\|\mathcal{F}(u, v') - \mathcal{F}(u, v'') - \Lambda[v' - v''] - B(p, v' - v'') - \\ &- {}^1/_2 B(v' + v'' - 2(p + v_*), v' - v'')\| \leqslant \\ &\leqslant \beta[\|v' - v_*\| + \|v'' - v_*\|]\|v' - v''\|. \end{aligned} \tag{2.8}$$

We introduce the following constants, mappings, and neighborhoods:

$$\begin{aligned} &\varepsilon = 1/(16C(\|P_2\|\|B\| + 1)), \\ &R = \min\{1/(16C(\|P_1\|\|B\| + 1)), \delta/2\}, \\ &\mathcal{D}(u, v) = \mathcal{F}(u, v) - \mathcal{F}(u_*, v_*), \\ &\Delta(u, v) = \|P_1\mathcal{D}(u, v)\| + \left\|\frac{P_2\mathcal{D}(u, v)}{\|v - v_*\|}\right\|, \\ &U_0 = \{u \in S \mid \|\mathcal{F}(u, v_*) - \mathcal{F}(u_*, v_*)\| \leqslant R/4C\}, \\ &V_0 = \{v \in V \mid \|v - v_*\| \leqslant R, \ \|\mathcal{F}(u, v) - \mathcal{F}(u, v_*)\| \leqslant R/4C \ \forall u \in S\}. \end{aligned} \tag{2.9}$$

Without loss of generality we will assume that the conditions $u \in S$ and $\|v - v_*\| \le 2R$ imply that $(u, v) \in T$.

By virtue of conditions 1) and 3), the neighborhoods U_0 and V_0 are nonempty. We fix an arbitrary point $(u, v) \in U_0 \times V_0$ that satisfies conditions a) and b).

Let us set $p = \|v - v_*\|h$, $v_0 = v$, and

$$v_n = v_{n-1} - M\left(P_1(\mathscr{D}(u, v_{n-1})), \frac{P_2(\mathscr{D}(u, v_{n-1}))}{\|v - v_*\|}\right) \quad (n = 1, 2, \ldots), \tag{2.10}$$

where M is the right inverse operator from (2.3), (2.4). We apply the mapping G(h) to both sides of Eq. (2.10). By virtue of (2.3), we get

$$\Lambda(v_n - v_{n-1}) = -P_1(\mathscr{D}(u, v_{n-1})), \tag{2.11}$$

$$P_2 B(p, v_n - v_{n-1}) = -P_2(\mathscr{D}(u, v_{n-1})). \tag{2.12}$$

By induction we prove the relation

$$\Delta(u, v_n) \leqslant \Delta(u, v)/2^n. \tag{2.13}$$

Relation (2.13) is obvious for $n = 0$. Let us suppose that relation (2.13) is valid for the first m elements $v_0, v_1, \ldots, v_m$. Then, by virtue of (2.4) and (2.10), we have

$$\|v_{i+1} - v_i\| \leqslant C\Delta(u, v_i) \leqslant C\Delta(u, v)/2^i, \tag{2.14}$$

$$\|v_{i+1} - v\| \leqslant \sum_{s=0}^{i} \|v_{s+1} - v_s\| \leqslant 2C\Delta(u, v), \tag{2.15}$$

$$\|v_{i+1} - v_*\| \leqslant 2C\Delta(u, v) + \|v - v_*\| \quad (i = 0, 1, \ldots, m). \tag{2.16}$$

We show that

$$\|v_{i+1} - v_*\| \leqslant 2R \quad (i = 0, 1, \ldots, m). \tag{2.17}$$

At first, let $\operatorname{Im}\Lambda = W$. Then $P_1 = I_W$, $P_2 = 0$, and $\Delta(u, v) = \|\mathscr{F}(u, v) - \mathscr{F}(u_*, v_*)\|$. Hence, by virtue of the definitions of the sets U_0 and V_0, (2.17) follows at once from (2.16). Let us suppose that $\operatorname{Im}\Lambda \ne W$. From condition b), relation (2.16), and the definition of the constant K we conclude that

$$\|v_{i+1} - v_*\| \leqslant (2C/K + 1)\|v - v_*\| \leqslant 2\|v - v_*\|. \tag{2.18}$$

Since $\|v - v_*\| \le R$, relation (2.17) is fulfilled in this case. Further, to prove inequality (2.13) for $n = m + 1$ it is sufficient to prove the relations

$$\|P_1(\mathscr{D}(u, v_{m+1}))\| \leqslant \Delta(u, v)/2^{m+2},$$

$$\|P_2(\mathscr{D}(u, v_{m+1}))\| \leqslant \Delta(u, v)\|v - v_*\|/2^{m+2}.$$

By virtue of (2.17) and the choice of R, inequality (2.8) is valid for $v' = v_{m+1}$ and $v'' = v_m$. Using this inequality, relations (2.11) and (2.14) for $n = i + 1 = m + 1$, and the definitions of the constants β and R, we conclude that

$$\begin{aligned}
&\|P_1(\mathscr{F}(u, v_{m+1}) - \mathscr{F}(u_*, v_*))\| \leqslant \\
&\quad \leqslant \|P_1\|\|\mathscr{F}(u, v_{m+1}) - \mathscr{F}(u, v_m) - \Lambda(v_{m+1} - v_m)\| \leqslant \\
&\quad \leqslant \|P_1\|\|B\|\|v_{m+1} + v_m - 2v_*\|\|v_{m+1} - v_m\|/2 + \\
&+ \|P_1\|\beta(\|v_{m+1} - v_*\| + \|v_m - v_*\|)\|v_{m+1} - v_m\| \leqslant \Delta(u, v)/2^{m+2}.
\end{aligned}$$

If $\operatorname{Im}\Lambda = W$, then (2.13) follows from the above inequality. Now let $\operatorname{Im}\Lambda \ne W$. Then, by virtue of estimate (2.8) for $v' = v_{m+1}$, $v'' = v_m$, and $p = \|v - v_*\|h$, relations (2.12), (2.14), (2.15), and (2.17) for $n = i + 1 = m + 1$, definitions of the constants β, K, and ε, and conditions 3a) and 3b) of Theorem 1, we get

$$\begin{aligned}
&\|P_2(\mathscr{F}(u, v_{m+1}) - \mathscr{F}(u_*, v_*))\| = \|P_2(\mathscr{F}(u, v_{m+1}) - \mathscr{F}(u, v_m) - \\
&\quad - \Lambda(v_{m+1} - v_m)) + P_2(\mathscr{F}(u, v_m) - \mathscr{F}(u_*, v_*))\| \leqslant \\
&\leqslant \|P_2\|\,\|\mathscr{F}(u, v_{m+1}) - \mathscr{F}(u, v_m) - \Lambda(v_{m+1} - v_m) - B(p, v_{m+1} - v_m)\| \leqslant \\
&\quad \leqslant (1/2\|P_2\|\|B\|(\|v_{m+1} - v\| + \|v_m - v\| +
\end{aligned}$$

$$+2\|v-v_*-p\|)C\Delta(u,v_m)+4\|P_2\|\beta\|v-v_*\|C\Delta(u,v_m)\leqslant\Delta(u,v)\|v-v_*\|2^{m+2}.$$

Thus, (2.13) is proved for all $n \geq 0$. Hence we conclude that

$$\lim_{n\to\infty}\|\mathcal{F}(u,v_n)-\mathcal{F}(u_*v_*)\|=0. \tag{2.19}$$

On the other hand, by virtue of (2.4), (2.10), and (2.13), we have

$$\|v_{n+1}-v_n\|\leqslant C\Delta(u,v)2^{-n}\quad (n=0,1,\ldots), \tag{2.20}$$

Therefore, since the space V is Banach, the limit $g(u,v)=\lim_{n\to\infty}v_n\in V$ exists. We set $r(u, v) = g(u, v) - v$. By virtue of condition 3) and the definitions of the sets U_0 and V_0, the mapping $v \to \mathcal{F}(u, v)$ is continuous at each point $v \in V_0$ for all $u \in U_0$. Hence, by virtue of (2.19) and (2.20), we get assertions (2.6) and (2.7). The theorem is proved.

3. Estimation of the Distance from a Level Surface of a Mapping.

THEOREM 2. Let X and Y be Banach spaces, V' be a neighborhod of a point $x_* \in X$, and F be a mapping of V' into Y. Suppose that the following conditions are fulfilled:

1) (The quadratic approximation conditions.) There exist mappings $\Lambda \in \mathscr{L}(X, Y)$ and $B \in \mathscr{L}((X, X), Y)$ and a number $\delta > 0$ such that the condition $\|\bar{x}\| \leq \delta$ implies that

$$F(x_*+\bar{x})=F(x_*)+\Lambda\bar{x}+B(\bar{x},\bar{x})+\omega(\bar{x}),$$

where $\omega(\bar{x}')$ satisfies the estimate

$$\|\omega(\bar{x}')-\omega(\bar{x}'')\|\leqslant\beta(\|\bar{x}'\|+\|\bar{x}''\|)\|\bar{x}'-\bar{x}''\|,$$
$$\beta=\text{const}\leqslant 1/(32(\|P_1\|+\|P_2\|)C),$$

$P_1: Y \to L$, $P_2: Y \to N$ being continuous projections and C being the constant from (2.4).

2) The mappings Λ and B from condition 1) and a point $h \in V$, $\|h\| = 1$, are such that $L = \operatorname{Im}\Lambda$ is a closed complemented subspace of Y and

$$\operatorname{Im}G(h)=L\times N\overset{\text{df}}{=}Z.$$

Here G(h), L, and N are defined in the same manner as in (2.1)-(2.2).

Then there exist numbers $\varepsilon = \varepsilon(h) > 0$ and $K = K(h) > 0$ and a neighborhood $V_0 \subseteq V'$ of the point x_* such that for each point $x \in V_0$, satisfying the condition $\|(x - x_*)/\|x - x_*\| - h\| \leq \varepsilon$, there exists a point $r(x) = r(x, h)$, for which

$$F(x+r(x))=F(x_*), \tag{3.1}$$
$$\|r(x)\|\leqslant K[\|P_1(F(x)-F(x_*))\|+\|P_2(F(x)-F(x_*))\|^{1/2}]. \tag{3.2}$$

Proof. Let us set $V = X$, $W = Y$, $v = x$, and $\mathcal{F}(u, v) = F(x)$. We observe that, in the case under consideration, the mapping does not depend on the first variable. Hence conditions 1) and 2) imply that conditions 1)-3) of Theorem 1 are fulfilled. For the introduced mappings and sets we define the constants C, K, ε, and R and the neighborhood V_0 by the formulas of Sec. 1. We fix an arbitrary point $x \in V_0$ such that $\|(x - x_*)/\|x - x_*\| - h\| \leq \varepsilon$. We set $R(x) = \|P_1(F(x) - F(x_*))\| + \|P_2(F(x) - F(x_*))\|^{1/2}$. If $\|x - x_*\| \leq KR(x)$, then assertions (3.1)-(3.2) are obviously fulfilled with $r(x) = x_* - x$. Now let $\|x - x_*\| \geq KR(x)$. Then

$$\|x-x_*\|\geqslant K\left[\|P_1(F(x)-F(x_*))\|+\frac{\|P_2(F(x)-F(x_*))\|}{\|x-x_*\|}\right]. \tag{3.3}$$

Relation (3.3) means that condition b) of Theorem 1 is fulfilled for the point $v = x$. Thus, all conditions of Theorem 1 are fulfilled for the mappings and spaces under consideration. Consequently, there exists a point $r(x) = r(x, h) \in X$ such that

$$F(x+r(x))=F(x_*), \tag{3.4}$$

$$\|r(x)\|\leqslant K\left[\|P_1(F(x)-F(x_*))\|+\frac{\|P_2(F(x)-F(x_*))\|}{\|x-x_*\|}\right]. \tag{3.5}$$

Estimate (3.3) follows from the last relation and inequality (3.2). The theorem is proved.

Remark 1. Let a mapping $F: X \to Y$ be twice Fréchet-differentiable at a point x_*. Suppose that $L = \operatorname{Im} F'(x_*)$ is a closed complemented subspace of Y, N is its complement, and $P_1: Y \to L$ and $P_2: Y \to N$ are continuous projections. We introduce a linear mapping ($h \in X$)

$$G(x_*, h): X \to L \times N = Z,$$
$$G(x_*, h)\, x = (F'(x_*)\, x,\ F''(x_*)\,[h, x]).$$

Then we can set $\Lambda = F'(x_*)$, $B = F''(x_*)$, and $G(h) = G(x_*, h)$ in conditions 1) and 2) of Theorem 2. Indeed, to prove the inequality in condition 2) of Theorem 2, it is sufficient to apply the mean-value theorem to the mapping $F_0(x) = F(x) - \Lambda(x - x_*) - B(x - x_*, x - x_*)/2$.

Remark 2. If the mapping F is regular at the point x_*, then $\operatorname{Im} F'(x_*) = Y$ and, consequently, $P_2 = 0$. Then the mapping $G(x_*, h)$ is an epimorphism for each $h \in X$, since it can be identified with the mapping $F'(x_*)$. In this case, relations (3.1)-(3.2) give the assertions of the Lyusternik theorem [1, p. 41].

We introduce the sets

$$H(x_*) = \{h \in X,\ F'(x_*)\, h = 0,\ F''(x_*)\,[h, h] \in \operatorname{Im} F'(x_*)\},$$
$$H_0(x_*) = \{h \in H(x_*),\ \operatorname{Im} G(x_*, h) = L \times N\},$$

and let $TM(x_*)$ denote the cone of tangental directions to the set $M(x_*)$ at the point x_*.

COROLLARY 1. Let X and Y be Banach spaces, U_0 be a neighborhood of a point $x_0 \in X$, and F be a twice Fréchet-differentiable (at the point x_*) mapping of U_0 into Y such that $\operatorname{Im} F'(x_*)$ is a closed complemented subspace of Y. Then

$$H_0(x_*) \subseteq TM(x_*) \subseteq H(x_*). \tag{3.6}$$

Proof. Let $h \in TM(x_*)$. Then, by the definition of a tangential direction, there exist a $\delta > 0$ and a mapping $r: [-\delta, \delta] \to X$, such that $x_* + th + r(t) \in M(x_*)$, $r(t) = o(t)$. Therefore, $0 = F(x_* + th + r(t)) - F(x_*) = tF'(x_*)[h] + F'(x_*)[r(t)] + t^2F''(x_*)[h, h] + o(t^2)$ $\forall\, t \in [-\delta, \delta]$. Hence

$$F'(x_*)\,[h] = 0,$$
$$-F'(x_*)\,[r(t)/t^2] = F''(x_*)\,[h, h] + o(t^2)/t^2.$$

Passing to the limit at $t \to 0$ in the last equation, we conclude that $\lim\limits_{t\to 0} (-F'(x_*)\,[r(t)\, t^{-2}])$ exists and belongs to $\operatorname{Im} F'(x_*)$, since $\lim\limits_{t\to 0} [F''(x_*)\,[h, h] + o(t^2)\, t^{-2}] = F''(x_*)\,[h, h]$, exists, and, by the condition, the subspace $\operatorname{Im} F'(x_*)$ is closed. Thus, the right-hand part of inclusion (3.6) is proved.

Now let $h \in H_0(x_*)$. Without loss of generality we can assume that $\|h\| = 1$. Let us consider the point $x(t) = x_* + th$. Then there obviously exists a number $\delta > 0$ such that $x(t) \in V_0$ for $t \in [-\delta, \delta]$, where V_0 is the neighborhood from the conditions of Theorem 2, and, moreover, $(x(t) - x_*)\|x(t) - x_*\|^{-1} = ht|t|^{-1}$. Hence it follows from the definition of the set $H_0(x_*)$ that all conditions of Theorem 2 are fulfilled. But then, by virtue of the definitions of the sets $H(x_*)$ and $H_0(x_*)$, we conclude from relations (3.1)-(3.2) that there exists a mapping $r(t): [-\delta, \delta] \to X$ such that

$$F(x_* + th + r(t)) = F(x_*),$$
$$\|r(t)\| \leqslant$$
$$\leqslant K\,[\|P_1(F(x(t)) - F(x_*))\| + \|P_2(F(x(t)) - F(x_*))\|^{1/2} \leqslant$$
$$\leqslant \{\|P_1\|\,\|tF'(x_*)\,[h] + t^2F''(x_*)\,[h, h] + o(t^2)\|\} +$$
$$+ \|P_2(F'(x_*)\,[th]) + t^2P_2(F''(x_*)\,[h, h]) + o(t^2)\|^{1/2} = o(t).$$

The corollary is proved.

Definition 1. The mapping F is said to be 2-regular at the point x_* if it is twice Fréchet-differentiable at this point, the subspace $\operatorname{Im} F'(x_*)$ is closed and complemented, and

$$H(x_*) \setminus \{0\} = H_0(x_*) \setminus \{0\}. \tag{3.7}$$

If the mapping F is 2-regular at the point x_*, then Corollary 1 gives complete description of the cone of tangential directions $TM(x_*) = H(x_*)$.

Condition (3.7) is weaker than the condition of regularity of the mapping F at the point x_*. Indeed, if $\operatorname{Im} F'(x_*) = Y$, then, as remarked above, the mapping $G(x_*, h)$ is an epimorphism $\forall h \in X$ and, consequently, (3.7) is fulfilled. On the other hand, the regularity of the mapping F at the point x_* does not follow from (3.7). In order to show this, it it sufficient to consider the function $F(x) = x_1{}^2 + x_2{}^2 - x_3{}^2$ at the point $x_* = 0$.

We return to the investigation of the estimate of distance, in which we are interested. It follows obviously from Theorem 2 that

$$\rho(x, M(x_*)) \leqslant K[\| P_1(F(x) - F(x_*))\| + \| P_2(F(x) - F(x_*))\|^{1/2}]. \tag{3.8}$$

However, it is asserted in Theorem 2 that estimate (3.8) is valid not for each point x in a certain neighborhood V_0 of the point x_0, but only for those points x, for which $\|(x - x_*)/\|x - x_*\| - h\| \leq \varepsilon$, where h is a fixed point of X such that $\operatorname{Im} G(x_*, h) = Z$. It may turn out that if there are enough such regular points $h \in X$, then estimate (3.8) is valid for each point x in U_0. In the finite-dimensional case, as will be shown below, for this the 2-regularity of the mapping F at the point x_* is sufficient.

Let us suppose that the mapping F is twice Fréchet-differentiable at the point $x_* \in X$ and $\operatorname{Im} F'(x_*)$ is a closed complemented subspace of Y. We introduce the sets

$$H^\alpha(x_*) = \Big\{h \in X \mid \|F'(x_*)h\| \leqslant \alpha, \quad \inf_{y \in \operatorname{Im} F'(x_*)} \Big\|\frac{1}{2}F''(x_*)[h, h] - y\Big\| \leqslant \alpha\Big\},$$

$$H_0^\alpha(x_*) = \{h \in H^\alpha(x_*) \mid G(x_*, h) = L \times N\}.$$

Let $h \in H_0{}^\alpha(x_*)$. Then there exist a right inverse operator $A_h: L \times N \to X$ and a constant $C(h) > 0$ such that

$$G(x_*, h) \circ A_h = I_Z, \quad \|A_h(z)\| \leqslant C(h)\|z\|.$$

<u>Definition 2</u>. The mapping F is said to be strongly 2-regular at the point x_* if it is twice Fréchet-differentiable at this point, the subspace $\operatorname{Im} F'(x_*)$ is closed and complemented, and there exist constants $C > 0$ and $\alpha > 0$ and a family of right inverse operators $\{A_h\}$, $h \in H_0{}^\alpha(x_*)$, $\|h\| = 1$, such that

$$H^\alpha(x_*) \setminus \{0\} = H_0^\alpha(x_*) \setminus \{0\},$$
$$\|A_h(z)\| \leqslant C\|z\| \quad \forall h \in H_0^\alpha(x_*), \quad \|h\| = 1.$$

<u>THEOREM 3</u>. Let X and Y be Banach spaces, V' be a neighborhood of a point $x_* \in X$, and F be a mapping of the neighborhood V' and Y that is twice Fréchet-differentiable at the point x_*. Suppose that F is strongly 2-regular at x_*.

Then there exist a number $K > 0$, a neighborhood V_0 of the point x_*, and a mapping $x \to r(x)$ of the set V_0 into X such that

$$F(x + r(x)) = F(x_*), \tag{3.9}$$
$$\|r(x)\| \leqslant K_1[\|P_1(F(x) - F(x_*))\| + \|P_2(F(x) - F(x_*))\|^{1/2}] \tag{3.10}$$

for all $x \in V_0$.

<u>Proof</u>. Since F is strongly 2-regular at x_*, there exist constants $c > 0$, and $\alpha > 0$ such that for each element $h \in H^\alpha(x_*)$, $\|h\| = 1$, there exists a right inverse operator A_h, for which the estimate $\|A_h(z)\| \leq C\|z\|$ is valid. Therefore, by virtue of Remark 1, we conclude that for each point $h \in H^\alpha(x_*)$, $\|h\| = 1$, the conditions of Theorem 2 are fulfilled, and the constant $K > 0$ and the neighborhood V_0 of x_* in the assertion of Theorem 2 do not depend on h. Since F is twice Fréchet-differentiable at x_*, there exists a neighborhood $V_1 \subseteq V_2$ of this point such that for each $x \in V_1$

$$\|F(x) - F(x_*) - F'(x_*)(x - x_*)\| \leqslant (\alpha/2)\|x - x_*\|, \tag{3.11}$$
$$\|F(x) - F(x_*) - F'(x_*)(x - x_*) - (1/2)F''(x_*)(x - x_*, x - x_*)\| \leqslant (\alpha/2)\|x - x_*\|^2. \tag{3.12}$$

Let us set $h = (x - x_*)/\|x - x_*\|$ and $\mathscr{D}(x) = F(x) - F(x_*)$. We fix an arbitrary point $x \in U_1$, $\|x - x_*\| \le 1$. If $\|x - x_*\| \le (2/\alpha)(\|P_1(\mathscr{D}(x))\| + \|P_2(\mathscr{D}(x))\|^{1/2})$, then, taking $r(x) = x_* - x$, we get assertions (3.9) and (3.10) with the constant $K_1 = \max\{K, 2/\alpha\}$

Let

$$\| x - x_* \| \geqslant (2/\alpha) [\| P_1 (\mathscr{D} (x)) \| + \| P_2 (\mathscr{D} (x)) \|^{1/2}]. \tag{3.13}$$

Without loss of generality we can assume that $\alpha \le 2$. Then it follows from (3.13) that $\|x - x_*\| \ge (2/\alpha)\|\mathscr{D}(x)\|$. Therefore, by virtue of (3.11),

$$\| F' (x_*) h \| \leqslant \| \mathscr{D} (x) \| \| x - x_* \|^{-1} + \alpha/2 \leqslant \alpha.$$

Let us set $y = P_1\mathscr{D}(x)/\|x - x_*\|^2 - F'(x_*)h/\|x - x_*\| \in \operatorname{Im} F'(x_*)$. Since $\alpha \le 2$, from (3.12) and (3.13) we conclude that

$$\| F'' (x_*) [h, h] - y \| \leqslant \| P_2 \mathscr{D} (x) \| / \| x - x_* \|^2 + \alpha/2 \leqslant \alpha.$$

Thus, $(x - x_*)/\|x - x_*\| = h \in H^{\alpha}(x_*)$, and we can use Theorem 2, from which relations (3.9)-(3.10) follow. The theorem is proved.

Remark 3. Let $X = R^n$, $Y = R^m$, $F: R^n \to R^m$, and $F = (f, f_2, \ldots, f_m)$. Then we can show that the condition of strong 2-regularity is equivalent to the condition of 2-regularity. Thus, in the finite-dimensional case, for the validity of (3.9)-(3.10) it is sufficient that the mapping F is 2-regular at x_*.

4. Implicit Function Theorem.

THEOREM 4. Let X be a topological space, Y and Z be Banach spaces, T be a neighborhood of a point (x_*, y_*) in $X \times Y$, F be a mapping of T into Z, and $F(x_*, y_*) = z_*$. Suppose that the following conditions are fulfilled:

1) The mapping $x \to F(x, y_*)$ is continuous at x_*.

2) (The quadratic approximation condition.) There exist mappings $\Lambda \in \mathscr{L}(Y, Z)$ and $B \in \mathscr{L}((Y, Y), Z)$, a number $\delta > 0$, and a neighborhood S of the point x_* such that if $u \in S$ and $\|\bar{y}\| \le \delta$, then

$$F (x, y_* + \bar{y}) = F (x, y_*) + \Lambda\bar{y} + B (\bar{y}, \bar{y})/2 + \omega (x, \bar{y}),$$

where $\omega(x, \bar{y})$ satisfies the estimate

$$\| \omega (x, \bar{y}') - \omega (x, \bar{y}'') \| \leqslant \gamma (\| \bar{y}' \| + \| \bar{y}'' \|) \| \bar{y}' - \bar{y}'' \|,$$
$$\gamma = \text{const} \leqslant 1/(16 (\| P_1 \| + \| P_2 \|) K),$$

$P_1: Z \to L$ and $P_2: Z \to N$ being continuous projections and K being the constant from (2.5).

3) The mappings Λ and B from condition 2) are such that $L = \operatorname{Im}\Lambda$ is a closed complemented subspace of z and there exists an element $h \in Y$, $\|h\| = 1$, for which

$$G (h) h = 0, \quad \operatorname{Im} G (h) = L \times N \overset{\text{df}}{=} Z.$$

Here G(h), L, and N are defined in the same manner as in (2.1)-(2.2).

Then there exist a number $K_1 > 0$, a neighborhood U' of the point (x_*, z_*) in $X \times Z$, and a mapping $\varphi: U' \to Y$ such that

$$F (x, \varphi (x, z)) = z, \tag{4.1}$$

$$\| \varphi (x, z) - y_* \| \leqslant K_1 [\| P_1 (F (x, y_*) - z) \| + \| P_2 (F (x, y_*) - z) \|^{1/2}]. \tag{4.2}$$

Proof. Let us set $U = X \times Z$, $V = Y$, $W = Z$, $v = y$, $u = (x, y)$, and $\mathscr{F}(u, v) = F(x, y) - z$. Then conditions 1)-3) of Theorem 1 are fulfilled for the so-defined spaces U, V, and W and the mapping $\mathscr{F}$. We fix h from condition 2) of Theorem 4 and set $\mathscr{D}(x, t) = F(x, y_* + th) - F(x, y_*)$. It follows from condition 3) that there exists a number $t_0 > 0$ such that

$$\| F (x, y_* + th) - F (x, y_*) - t\Lambda h - (1/2) t^2 B (h, h) \| \leqslant \gamma t^2 \tag{4.3}$$

for all $t \in (0, t_0)$ and $x \in S$. We choose a number $t_1 \in (0, t_0)$ such that

$$t_1 \leqslant 2/K\,(\|B\| + 2\gamma), \quad y_* + th \in V_0 \quad \forall t \in (0, t_1), \tag{4.4}$$

where V_0 is the neighborhood from the assertion of Theorem 1. It follows from the equation $G(h)h = 0$ that $\Lambda h = 0$ and $P_2B(h, h) = 0$. Hence, by virtue of (4.3), (4.4), and the definition of the constant $\gamma > 0$, it is easy to show that

$$\begin{aligned} &\|P_1(\mathscr{D}(x, t))\| \leqslant t/K, \\ &\|P_2(\mathscr{D}(x, t))\| \leqslant t^2/K \quad \forall t \in (0, t_1). \end{aligned} \tag{4.5}$$

Let us set

$$\begin{aligned} &R(x, z) = F(x, y_*) - z, \\ &t(x, z) = K \max\{\|P_1(R(x, z))\|, \ \|P_2(R(x, z))\|^{1/2}\}, \\ &U' = \{(x, z) \in U_0 \mid t(x, z) \leqslant t_1\}, \\ &y(x, z) = y_* + t(x, z)\, h. \end{aligned} \tag{4.6}$$

Since $t(x, z) \in (0, t_1)$ for $(x, z) \in U'$, we can show by virtue of relations (4.6) and the definition of the point h that for the points $u = (x, z) \in U'$ and $v = y(x, z)$ conditions a) and b) of Theorem 1 are valid for $\mathscr{F}(u, v) = F(x, y) - z$. Thus, in the case under consideration, all conditions of Theorem 1 are fulfilled. Therefore, there exists a mapping $r\colon U' \to Y$ such that

$$\begin{aligned} &F(x, y(x, z) + r(x, z)) = z, \\ &\|r(x, z)\| \leqslant K[\|P_1(F(x, y(x, z)) - z)\| + \\ &\qquad + \|P_2(F(x, y(x, z)) - z)\|/t(x, z)]. \end{aligned} \tag{4.7}$$

Let us set $\varphi(x, z) = y(x, z) + r(x, z)$. Then the assertions of the theorem follow from (4.6), (4.7), and condition b) of Theorem 1 for $v = y(x, z)$.

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