INVERSE PROBLEM FOR INTEGRAL OPERATORS

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In this note, we investigate the inverse problem for integral operators of the form

$$Af=\int_0^x M(x,t)f(t)\,dt+g(x)\int_0^\pi f(t)v(t)\,dt,\quad 0\leqslant x\leqslant\pi. \tag{1}$$

Let $M(x, t, \lambda)$ denote the kernel of the integral operators $M_\lambda = (E - \lambda M)^{-1}M$, where E is the identity operator and

$$Mf=\int_0^x M(x,t)f(t)\,dt.$$

Let us set

$$g(x,\lambda)=g(x)+\lambda\int_0^x M(x,t,\lambda)g(t)\,dt. \tag{2}$$

Then (see, e.g., [1]) the eigenvalues λ_k of A coincide with the zeros of the function

$$\mathscr{L}(\lambda)=1-\lambda\int_0^\pi v(x)g(x,\lambda)\,dx, \tag{3}$$

which is called the characteristic function (c.f.) of A. Here the eigen- and the associated functions $g_k(x)$ of the operator have the form

$$g_{k+\nu}(x)=\frac{\partial^\nu}{\partial\lambda^\nu}g(x,\lambda)\Big|_{\lambda=\lambda_k},\quad \nu=0,1,\ldots,r_k-1,$$

if r_k is the multiplicity of λ_k ($\lambda_k=\lambda_{k+1}=\ldots=\lambda_{k+r_k-1}$). Let us set $\beta_k = g_k(\pi)$. We will call the totality of the numbers $\{\lambda_k, \beta_k\}$ the spectral data of A.

In this note, we consider the following problems.

Problem 1. To find the function $v(x)$ with respect to preassigned functions $M(x, t)$ and $g(x)$ and the spectrum $\{\lambda_k\}$ of an operator $A = A(M, g, v)$ of the form (1).

Problem 2. To find the functions $v(x)$ and $g(x)$ with respect to the preassigned function $M(x, t)$ and the spectral data $\{\lambda_k, \beta_k\}$ of the operator $A(M, g, v)$.

1. Let the function $M(x, t)$ satisfy the following condition (condition M_1): The functions

$$\frac{\partial^{\nu+j}}{\partial x^\nu\partial t^j}M(x,t),\quad \nu,j=0,1,$$

are continuous for $0 \le t \le x \le \pi$, and $M(x, x) = -i$, $\frac{\partial}{\partial x}M(x,t)\Big|_{t=x}=0$.

Then the operator $D = M^{-1}$ has the form

$$Dy=iy'(x)+\int_0^x H(x,t)y(t)\,dt,\quad y(0)=0,$$

where $H(x, t)$ is a continuous function for $0 \le t \le x \le \pi$.

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Definition. The operator A is said to belong to $\Lambda^{(1)}{}_{00}$ if the function $M(x, t)$ satisfies the condition M_1, the functions $g(x)$ and $v(x)$ are absolutely continuous for $0 \le x \le \pi$; $g'(x), v'(x) \in \mathcal{L}_2(0, \pi)$, and $a_0 \cdot b_0 \neq 0$, where

$$a_0 = 1 + ig(0)v(0) + \int_0^\pi v(\tau)\Big(ig'(\tau) + \int_0^\tau H(\tau, s)\, g(s)\, ds\Big) d\tau, \qquad b_0 = ig(0)v(\pi). \tag{4}$$

For simplicity, we solve Problems 1 and 2 for operators of the class $\Lambda^{(1)}{}_{00}$.

THEOREM 1. Let $A \in \Lambda_{00}^{(1)}$. Then the spectral data $\{\lambda_k, \beta_k\}$, $k = 0, \pm 1, \pm 2, \ldots$, of the operator A have the form

$$\lambda_k = 2k + \alpha + \varkappa_k, \quad \beta_k = \alpha_1 + \varkappa_{k_1}, \qquad \lambda_k \neq 0, \quad \alpha_1 \neq 0, \quad \varkappa_k, \varkappa_{k1} \in l_2. \tag{5}$$

THEOREM 2. Let there be given functions $M(x, t)$ and $g(x)$, $0 \le t \le x \le \pi$, such that $M(x, t)$ satisfies the condition M_1, $g(x)$ is absolutely continuous, $g'(x) \in \mathcal{L}_2(0, \pi)$, $g(0) \neq 0$. Further, let there be given numbers λ_k, $k = 0, \pm 1, \pm 2, \ldots$, of the form $\lambda_k = 2k + \alpha + \varkappa_k$, $\lambda_k \neq 0$, $\varkappa_k \in l_2$. Then there exists a unique operator $A(M, g, v) \in \Lambda_{00}^{(1)}$, for which λ_k are the eigenvalues.

THEOREM 3. If a function $M(x, t)$ that satisfies the condition M_1 and numbers λ_k and β_k, $k = 0, \pm 1, \pm 2, \ldots$, of the form (5) are given, then there exists a unique operator $A(M, g, v) \in \Lambda_{00}^{(1)}$, for which $\{\lambda_k, \beta_k\}$ are the spectral data.

We prove some lemmas.

LEMMA 1. Let there be given numbers λ_k, $k = 0, \pm 1, \pm 2, \ldots$, of the form $\lambda_k = 2k + \alpha + \varkappa_k$, $\lambda_k \neq 0$, $\varkappa_k \in l_2$. Set

$$\mathcal{L}(\lambda) = \exp(p\lambda) \cdot \prod_{k=-\infty}^{\infty} \Big(1 - \frac{\lambda}{\lambda_k}\Big) \exp\Big(\frac{\lambda}{\lambda_k}\Big), \tag{6}$$

where

$$p = p_0 + \sum_{k=-\infty}^{\infty} \Big(\frac{1}{\lambda_k} - \frac{1}{\lambda_k^0}\Big),$$

$$p_0 = i\pi \exp(i\alpha\pi), \qquad \lambda_k^0 = 2k + \alpha$$

(the case where α is an even integer brings in insignificant changes). Then the following representation is valid for $\mathcal{L}(\lambda)$:

$$\mathcal{L}(\lambda) = \gamma \cdot (1 - \exp(i(\alpha - \lambda)\pi)) + \int_0^\pi W(t) \exp(-i\lambda t)\, dt, \qquad \gamma = \prod_{k=-\infty}^{\infty} \frac{\lambda_k^0}{\lambda_k}, \quad w(t) \in \mathcal{L}_2(0, \pi). \tag{7}$$

Proof. The function $\mathcal{L}_0(\lambda) = 1 - \exp(i(\alpha - \lambda)\pi)$ has zeros λ^0_k and admits the representation

$$\mathcal{L}_0(\lambda) = \exp(p_0\lambda) \cdot \prod_{k=-\infty}^{\infty} \Big(1 - \frac{\lambda}{\lambda_k^0}\Big) \cdot \exp\Big(\frac{\lambda}{\lambda_k^0}\Big).$$

Therefore,

$$\mathcal{L}(\lambda) = \gamma \mathcal{L}_0(\lambda) F(\lambda), \quad F(\lambda) = \prod_{k=-\infty}^{\infty} \Big(1 + \frac{\varkappa_k}{\lambda_k^0 - \lambda}\Big). \tag{8}$$

We show that $|F(\lambda)| < C_\delta$ in the domain $G_\delta = \{\lambda : |\lambda - \lambda_k^0| \geqslant \delta\}$ for a fixed $\delta > 0$. We choose a natural number N such that $|\kappa_k \leqslant \delta/2$ for $|k| \geqslant N$. Then, for $\lambda \in G_\delta$

$$F(\lambda)=\exp(H_N(\lambda))\prod_{|k|<N}\left(1+\frac{\varkappa_k}{\lambda_k^0-\lambda}\right), \tag{9}$$

where

$$H_N(\lambda)=\sum_{|k|\geqslant N}\ln\left(1+\frac{\varkappa_k}{\lambda_k^0-\lambda}\right)=\sum_{|k|\geqslant N}\frac{\varkappa_k}{\lambda_k^0-\lambda}\cdot\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu+1}\cdot\left(\frac{\varkappa_k}{\lambda_k^0-\lambda}\right)^\nu.$$

Since

$$|H_N(\lambda)|\leqslant\sum_{|k|\geqslant N}\frac{|\varkappa_k|}{|\lambda_k^0-\lambda|}\cdot\sum_{\nu=0}^{\infty}\frac{1}{2^\nu}\leqslant C\Big(\sum_{|k|\geqslant N}\frac{1}{|2k+\alpha-\lambda|^2}\Big)^{1/2},$$

it follows from (9) that $|F(\lambda)| < C_\delta$ for $\lambda\in G_\delta$.

Further, it follows from (8) that

$$\mathscr{L}(\lambda_n^0)=-\gamma\cdot\varkappa_n b_n\left(\frac{d}{d\lambda}\mathscr{L}_0(\lambda)\right)\Big|_{\lambda=\lambda_n^0},$$

$$b_n=\prod_{k=-\infty,k\neq n}^{\infty}\left(1+\frac{\varkappa_k}{\lambda_k^0-\lambda_n^0}\right),$$

i.e., $\mathscr{L}(\lambda_n^0)\in l_2$. Let us consider the function

$$\Delta(\lambda)=\mathscr{L}(\lambda)-\gamma\mathscr{L}_0(\lambda). \tag{10}$$

Let us set $\theta_n = \Delta(\lambda^0{}_n)$. It is obvious that $\theta_n\in l_2$. Let us construct a function $w(t)\in\mathscr{L}_2$ • (0, π) such that

$$\theta_n=\int_0^\pi w(t)\exp(-i\lambda_n^0 t)\,dt.$$

Let us consider the function

$$\theta(\lambda)=\int_0^\pi w(t)\exp(-i\lambda t)\,dt$$

and set $S(\lambda)=(\mathscr{L}_0(\lambda))^{-1}(\theta(\lambda)-\Delta(\lambda))$. The function $S(\lambda)$ is an analytic integral function of λ. We have

$$|\mathscr{L}_0(\lambda)|>C\cdot(1+\exp(\operatorname{Im}\lambda\pi))$$

in the domain G_δ. It follows from (8) and (10) that $\Delta(\lambda)=\gamma\mathscr{L}_0(\lambda)\cdot(F(\lambda)-1)$. Using the maximum modulus principle for analytic functions, we see that the function $S(\lambda)$ is bounded and, consequently, $S(\lambda)\equiv C$ by the Liouville theorem (see [2, p. 209]). Since x, $x\to-\infty$ lim S• (ix) = 0 (x real), it follows that C = 0, and we arrive at relation (7). The lemma is proved.

LEMMA 2. The integral equation

$$P(x,t,\alpha)=i\int_\alpha^{x-t+\alpha}H(t+\xi,\xi)\,d\xi+i\int_\alpha^{x-t+\alpha}ds\int_0^t H(s+t,s+\xi)\cdot P(s+\xi-\alpha,\xi,\alpha)\,d\xi,$$
$$0\leqslant t\leqslant x\leqslant\pi-\alpha,\quad 0\leqslant\alpha\leqslant\pi, \tag{11}$$

has a unique solution $P(x, t, \alpha)$, and the functions $P(x, t, \alpha)$, $(\partial/\partial x)P(x, t, \alpha)$, $(\partial/\partial\alpha)P(x, t, \alpha)$ are continuous with respect to all the variables.

Proof. We solve Eq. (11) by the method of successive approximations:

$$P_1(x,t,\alpha)=i\int_\alpha^{x-t+\alpha}H(t+\xi,\xi)\,d\xi,$$

$$P_{k+1}(x,t,\alpha) = i\int_{\alpha}^{x-t+\alpha} ds \int_0^t H(s+t, s+\xi)\cdot P_k(s+\xi-\alpha, \xi, \alpha)\, d\xi.$$

Let us set $C_0 = \max |H(x,t)|,\ 0 \leqslant t \leqslant x \leqslant \pi$. Then

$$|P_1(x,t,\alpha)| \leqslant C_0\cdot\pi,$$

$$|P_{k+1}(x,t,\alpha)| \leqslant C_0 \int_0^{\pi} ds \int_0^t |P_k(s+\xi, \xi, \alpha)|\, d\xi.$$

Hence, by induction, we obtain the estimates

$$|P_k(x,t,\alpha)| \leqslant (C_0\pi)^k \frac{t^{k-1}}{(k-1)!}, \quad k = 1, 2, \ldots$$

Thus, the continuous function

$$P(x,t,\alpha) = \sum_{k=1}^{\infty} P_k(x,t,\alpha)$$

is a solution of Eq. (11). The existence and continuity of the derivatives with respect to x and α follow from the term-by-term differentiability of the series. The lemma is proved.

LEMMA 3. Let $A \in \Lambda_{00}^{(1)}$. Then the following statements are valid:

1) The c.f. $\mathscr{L}(\lambda)$ of the operator A has the form

$$\mathscr{L}(\lambda) = 1 - \lambda \int_0^{\pi} m(t) \exp(-i\lambda(\pi - t))\, dt, \tag{12}$$

where

$$m(t) = g(0)u(t) + \int_0^t u(\tau) Q(t,\tau)\, d\tau,$$
$$u(t) = v(\pi - t), \tag{13}$$
$$Q(t,\tau) = \frac{d}{dt}\Big(g(t-\tau) + \int_0^{t-\tau} P(\pi - t + s, s, t - \tau - s) g(t - \tau - s)\, ds\Big),$$

The function $P(x, t, \alpha)$ is a solution of Eq. (11). In addition, the function m(t) is continuous, $m'(t) \in \mathscr{L}_2(0, \pi)$, $1 + im(\pi) = a_0$, $m(0) = -ib_0$, where a_0 and b_0 have the form (4).

2) The following representation is valid for the function $g(\pi, \lambda)$:

$$g(\pi, \lambda) = g(0)\exp(-i\lambda\pi) + \int_0^{\pi} \gamma(t) \exp(-i\lambda t)\, dt, \tag{14}$$

where $\gamma(t) = \mu'(t) \in \mathscr{L}_2(0, \pi)$,

$$\mu(\pi - t) = -g(t) - \int_0^t P(\pi - \tau, t - \tau, \tau) g(\tau)\, d\tau. \tag{15}$$

Proof. It is clear that $M_\lambda^{-1} y = Dy - \lambda y$, $y(0) = 0$, and the function $z(x) = M(x + \alpha, \alpha, \lambda)$ is a solution of the Cauchy problem

$$iz'(x) + \int_0^x H(x+\alpha, t+\alpha) z(t)\, dt = \lambda z(x), \quad z(0) = -i,$$
$$0 \leqslant x \leqslant \pi - \alpha. \tag{16}$$

for fixed $\alpha \in [0, \pi]$. Consequently,

$$M(x+\alpha, \alpha, \lambda) = -i\Big(\exp(-i\lambda x) + \int_0^x P(x, t, \alpha) \exp(-i\lambda(x - t))\, dt\Big), \tag{17}$$

since the right-hand side of Eq. (17) is also a solution of the Cauchy problem (16). We substitute the obtained representation (17) in (2) and get

$$g(x,\lambda)=g(x)-i\lambda\int_0^x \exp(-i\lambda t)\cdot\left(g(x-t)+\int_0^{x-t} P(t+\tau,\tau,x-t-\tau)\,g(x-t-\tau)\,d\tau\right)dt, \qquad (18)$$

Hence, in particular, the representation (14) follows. Further, we substitute (18) in (3) and get (12), where the function m(t) is defined by Eq. (13) and is a continuous function. We show that $m'(t)\in\mathscr{L}_2(0,\pi)$. To this end, we write m(t) in the form

$$m(t)=g(0)u(t)+\int_0^t u(t-\tau)R(t,\tau)\,d\tau,$$

$$R(t,\tau)=g'(\tau)+P(\pi-t+\tau,\tau,0)g(0)+$$

$$+\int_0^\tau g'(\tau-s)P(\pi-t+s,s,\tau-s)\,ds+\int_0^\tau g(\tau-s)\tilde{P}(\pi-t+s,s,\tau-s)\,ds,$$

$$\tilde{P}(x,t,\alpha)=\left(\frac{\partial}{\partial\alpha}-\frac{\partial}{\partial x}\right)P(x,t,\alpha).$$

By virtue of (11), we get

$$\tilde{P}(\pi-t+s,s,\tau-s)=$$

$$=-iH(\tau,\tau-s)-i\int_{\tau-s}^{\pi-t+\tau-s}d\eta\int_0^s H(\eta+s,\eta+\xi)\,\tilde{P}(\eta+\xi-\tau+s,\xi,\tau-s)\,d\xi.$$

Thus, the function R(t, τ) is continuously differentiable with respect to t and, consequently, $m'(t)\in\mathscr{L}_2(0,\pi)$. The lemma is proved.

Proof of Theorem 1. By virtue of Lemma 3, the c.f. $\mathscr{L}(\lambda)$ of the operator A has the form

$$\mathscr{L}(\lambda)=a_0-b_0\exp(-i\lambda\pi)+\int_0^\pi w(t)\exp(-i\lambda t)\,dt=a_0\mathscr{L}_0(\lambda)+\mathscr{L}_1(\lambda), \qquad (19)$$

where

$$\mathscr{L}_0(\lambda)=1-\exp(i(\alpha-\lambda)\pi),$$

$$\mathscr{L}_1(\lambda)=\int_0^\pi w(t)\exp(-i\lambda t)\,dt,$$

$$\exp(-i\alpha\pi)=a_0\cdot b_0^{-1},\quad w(t)=i\frac{d}{dt}m(\pi-t)\in\mathscr{L}_2(0,\pi).$$

The estimate $|\mathscr{L}_0(\lambda)|>C(1+\exp(\operatorname{Im}\lambda\pi))$, is valid in the domain $G_\delta=\{\lambda: |\lambda-\lambda_k^0|\geqslant\delta\}$, where λ°_k = 2k + α and, consequently,

$$|a_0|\cdot|\mathscr{L}_0(\lambda)|>|\mathscr{L}_1(\lambda)|$$

for sufficiently large $|\lambda|$. Therefore, by the Rouche theorem (see [2, p. 246]), 2N + 1 zeros λ_k, $k=0,\pm1,\ldots,\pm N$, of the function $\mathscr{L}(\lambda)$, lie inside the contour $\Gamma_N=\{\lambda: |\lambda-\alpha|=2N+1$ for sufficiently large N and exactly one zero λ_k of the function $\mathscr{L}(\lambda)$, lie inside the contour $\gamma_k(\delta)=\{\lambda: |\lambda-\lambda_k^0|=\delta\}$ for sufficiently large λ_k, i.e., $\lambda_k = 2k+\alpha+\kappa_k$, $\kappa_k = o(1)$. Substituting this expression in (19), we make the more precise assertion that $\varkappa_k\in l_2$. Using (14), we now easily obtain the desired asymptotic formula for the numbers β_k. The theorem is proved.

Proof of Theorem 2. We construct the function $\mathscr{L}(\lambda)$ by Eq. (6) with respect to the given numbers λ_k and, by Lemma 1, the representation (7) is valid for $\mathscr{L}(\lambda)$. Let us set

$$m(t)=-ib_0+i\int_0^t w(\pi-\tau)\,d\tau,\qquad b_0=\gamma\exp(i\alpha\pi).$$

Further, let a function u(t) be a solution of Eq. (13). It is clear that u(t) is continuous, $u'(t)\in\mathscr{L}_2(0,\pi)$, and u(0) ≠ 0. Let us set v(t) = u(π − t) and consider an operator A(M, g, v) of the form (1). Let $\mathscr{L}^*(\lambda)$ be the c.f. of A. Then, as in the proof of Lemma 3, we get

$$\mathscr{L}^*(\lambda)=1-\lambda\int_0^\pi m(t)\exp(-i\lambda(\pi-t))\,dt$$

or, after integrating by parts,

$$\mathscr{L}^*(\lambda)=1+im(\pi)-im(0)\exp(-i\lambda\pi)+\int_0^{\pi}w(t)\exp(-i\lambda t)\,dt.$$

Comparing this equation with relation (7) and taking into account the relations $\mathscr{L}(0)=\mathscr{L}^*(0)=1$, im (0) = γ exp ($i\alpha\pi$), we get $\mathscr{L}^*(\lambda)\equiv\mathscr{L}(\lambda)$, and $1+im(\pi)=\gamma$, and, consequently, the operator $A\in\Lambda_{00}^{(1)}$ and has spectrum λ_k. If it is assumed that there also exists an operator $A(M,g,\tilde{v})\in\Lambda_{00}^{(1)}$ with the same spectrum λ_k, then it would follow from Lemma 3 and the uniqueness of solution of the integral equation (13) that $v(t)=\tilde{v}(t)$, $t\in[0,\pi]$. The theorem is proved.

Proof of Theorem 3. For simplicity, we restrict ourselves to the case where all λ_k are different. As in the proof of Theorem 2, let us construct the functions $\mathscr{L}(\lambda)$, m(t), and P(x, t, α) with respect to the preassigned numbers λ_k and the function M(x, t). We set $\mu_k=\lambda_k-\alpha$, $g=\alpha_1\exp(i\alpha\pi)$, $\tilde{\beta}_k=\beta_k-g\exp(-i\lambda_k\pi)$. It is clear that $\tilde{\beta}_k\in l_2$. The system of the functions $\exp(-i\mu_k t)$ forms a Riesz basis in $\mathscr{L}_2(0,\pi)$, since it is complete and quadratically close (see [3]) to the orthogonal basis exp (−2kit). Let $h(t)\in\mathscr{L}_2(0,\pi)$ be such that

$$\tilde{\beta}_k=\int_0^{\pi}h(t)\exp(-i\mu_k t)\,dt,$$

and set

$$\mu(t)=-g-\int_t^{\pi}h(\tau)\exp(i\alpha\tau)\,d\tau.$$

Let the function g(t) be a solution of Eq. (15). It is clear that g(t) is continuous, $g'(t)\in\mathscr{L}_2(0,\pi)$, and $g(0)=g\neq 0$. As in Theorem 2, we now find the function v(t). By the same token, an operator A(M, g, v) of the form (1) has been constructed, and the numbers λ_k and β_k are the spectral data of A. As in Theorem 2, the uniqueness follows obviously from Lemma 3. In the case of multiple λ_k, the system of the functions $t^{\nu}\exp(-i\mu_k t)$, $\nu=0,1,\ldots,r_k-1$, where r_k is the multiplicity of λ_k, is a Riesz basis. The theorem is proved.

Remark. Results, analogous to the above ones, hold also for other classes of operators, e.g., for the operators $A\in\Lambda_{\nu\mu}^{(m)}$, $\max(\nu,\mu)<m$, whose c.f. have the form

$$\mathscr{L}(\lambda)=\sum_{k=0}^{m-1}\frac{1}{\lambda^k}(a_k-b_k\exp(-i\lambda\pi))+\frac{1}{\lambda^{m-1}}\int_0^{\pi}w_m(t)\exp(-i\lambda t)\,dt,$$

$$w_m(t)\in\mathscr{L}_2(0,\pi),\ a_\nu\cdot b_\mu\neq 0,$$

$$a_k=b_j=0,\ k=0,1,\ldots,\nu-1,\ j=0,1,\ldots,\mu-1.$$

Let us also observe that similar results are valid also for the case where M^{-1} is an integro-differential operator of second order.

2. In spite of the qualitative difference of the above-considered problems from the inverse problems for ordinary differential operators, there is a connection between them. In this section, by the example of Borg's theorem [4] we show how the inverse problem for ordinary differential operators can be reduced to Problem 1. To this end, we give here a general uniqueness theorem for the solution of Problem 1.

Let us consider an operator A of the form (1) under the assumption that the function M(x, t) is the Hilbert–Schmidt kernel and $g(x),\ v(x)\in\mathscr{L}_2(0,\pi)$.

THEOREM 4. Let the system of the eigen- and associated functions $g_k(x)$ of the operator A(M, g, v) be complete in $\mathscr{L}_2(0,\pi)$ and let λ_k and $\tilde{\lambda}_k$ be the spectra of the operators A = A(M, g, v) and $\tilde{A}$ = A(M, g, $\tilde{v}$), respectively. If $\lambda_k=\tilde{\lambda}_k$ for all k, then $v(x)=\tilde{v}(x)$ a.e. on the segment $[0,\pi]$.

Indeed, under the conditions of the theorem, it follows from (3) that

$$\int_0^{\pi}(v(x)-\tilde{v}(x))\,g(x,\lambda)\,dx=\lambda^{-1}(\tilde{\mathscr{L}}(\lambda)-\mathscr{L}(\lambda)),$$

where $\mathscr{L}(\lambda)$, and $\tilde{\mathscr{L}}(\lambda)$ are the cf. of the operators A and $\tilde{A}$, respectively. Therefore,

$$\int_0^{\pi}(v(x)-\tilde{v}(x))\,g_k(x)\,dx=0$$

and, consequently, $v(x) = \tilde{v}(x)$ a.e. on the segment $[0, \pi]$. The theorem is proved.

Let us consider the boundary-value problems $L_i = L(q(x), h, H_i)$, $i = 1, 2$,

$$\begin{gathered}-y''+q(x)\,y=\lambda y,\ q'(x)\in\mathcal{L}_2(0,\pi),\\ y'(0)-hy(0)=y'(\pi)+H_i y(\pi)=0,\ H_1\neq H_2.\end{gathered}\tag{20}$$

Let the functions $\varphi(x,\lambda)$ and $\psi_i(x,\lambda)$ be the solutions of Eq. (20) under the initial conditions

$$\begin{gathered}\varphi(0,\lambda)=\psi_i(\pi,\lambda)=1,\ \varphi'(0,\lambda)=h,\\ \psi_i'(\pi,\lambda)=-H_i\end{gathered}$$

and let $M(x, t, \lambda)$ be the Green function of the operator $y'' - q(x)y = \lambda y$, $y(0) = y'(0) = 0$. Then the eigenvalues μ_{ni}, $n = 0, 1, 2, \ldots$ of the problems L_i are the zeros of the functions $\Delta_i(\lambda) = \psi'_i(0, \lambda) - h\psi_i(0, \lambda)$, and the functions $\Delta_i(\lambda)$ are determined uniquely by their zeros. Let $\tilde{\mu}_{ni}$ be the eigenvalues of the problems $\tilde{L}_i = L(\tilde{q}(x), \tilde{h}, \tilde{H}_i)$ and let the functions $\tilde{\varphi}(x,\lambda)$, $\tilde{\psi}_i(x, \lambda)$, $\tilde{M}(x, t, \lambda)$, $\tilde{\Delta}_i(\lambda)$ be constructed analogously for the problems $\tilde{L}_i$.

We know from the theory of transformation operators (see, e.g., [5]), that if a function $G(x, t)$ satisfies the conditions

$$\begin{gathered}\frac{\partial^2 G(x,t)}{\partial x^2}-q(x)G(x,t)=\frac{\partial^2 G(x,t)}{\partial t^2}-\tilde{q}(t)G(x,t),\quad 0\leqslant t\leqslant x\leqslant\pi,\\ G(x,x)=h+\frac{1}{2}\int_0^x(q(t)-\tilde{q}(t))\,dt,\\ \left(\frac{\partial G(x,t)}{\partial t}-\tilde{h}G(x,t)\right)\Big|_{t=0}=0,\end{gathered}\tag{21}$$

then

$$\varphi(x,\lambda)=(E+G)\,\tilde{\varphi}(x,\lambda),\ M_\lambda(E+G)=(E+G)\,\tilde{M}_\lambda,\tag{22}$$

where

$$\begin{gathered}(E+G)f=f(x)+\int_0^x G(x,t)f(t)\,dt,\\ M_\lambda f=\int_0^x M(x,t,\lambda)f(t)\,dt.\end{gathered}$$

Let us consider the family of the operators

$$\begin{gathered}L_{\alpha,i}(q(x),h,H_1,H_2),\ -\infty<\alpha<\infty,\\ L_{\alpha,i}y=y''-q(x)\,y+\alpha y,\\ y'(0)-hy(0)=y'(\pi)+H_i y(\pi)=0.\end{gathered}$$

The inverse operators $A_{\alpha,i} = L^{-1}_{\alpha,i}$ have the form

$$A_{\alpha,i}f=\int_0^x M(x,t,\alpha)f(t)\,dt+\frac{\varphi(x,\alpha)}{\Delta_i(\alpha)}\int_0^{\pi}\psi_i(t,\alpha)f(t)\,dt,$$

and $\mu_{ni} - \alpha$ is the spectrum of $A_{\alpha,i}$. Analogously, the operators

$$\tilde{A}_{\alpha,i}f=\int_0^x\tilde{M}(x,t,\alpha)f(t)\,dt+\frac{\tilde{\varphi}(x,\alpha)}{\tilde{\Delta}_i(\alpha)}\int_0^{\pi}\tilde{\psi}_i(t,\alpha)f(t)\,dt$$

are inverse to the operators $L_{\alpha,i}(\tilde{q}(x),\tilde{h},\tilde{H}_1,\tilde{H}_2)$ and have the spectrum $\tilde{\mu}_{ni} - \alpha$. Now, we show that Borg's theorem [4] can be obtained as a corollary of Theorem 4.

Borg's Theorem. If $\mu_{ni} = \tilde{\mu}_{ni}$ for $i = 1, 2$, then

$$q(x)=\tilde{q}(x),\ h=\tilde{h},\ H_i=\tilde{H}_i.$$

Proof. Let a function G(x, t) satisfy the conditions (21). Let us set $B_{\alpha,i} = (E + G)^{-1}A_{\alpha,i}(E + G)$. Then, using (22), we get

$$B_{\alpha,i}f = \int_0^x \tilde{M}(x,t,\alpha)f(t)\,dt + \frac{\tilde{\varphi}(x,\alpha)}{\tilde{\Delta}_i(\alpha)}\int_0^\pi v_i(t,\alpha)f(t)\,dt,$$

where

$$v_i(x,\alpha) = (E + G^*)\psi_i(x,\alpha),$$

$$(E + G^*)f = f(x) + \int_x^\pi G(t,x)f(t)\,dt.$$

Under the conditions of the theorem, the operators $\tilde{A}_{\alpha,i}$ and $B_{\alpha,i}$ have identical spectra and, consequently, by Theorem 4 we have

$$\tilde{\psi}_i(x,\alpha) = (E + G^*)\psi_i(x,\alpha).$$

Since

$$\varphi(x,\alpha) = (H_1 - H_2)^{-1}(\Delta_2(\alpha)\psi_1(x,\alpha) - \Delta_1(\alpha)\psi_2(x,\alpha)),$$

we have $\tilde{\varphi}(x,\alpha) = (E + G^*)\varphi(x,\alpha)$, which, together with (22), gives $(E + G^*) = (E + G)^{-1}$. This is possible only in the case where $G(x, t) \equiv 0$. Consequently, $q(x) \equiv \tilde{q}(x)$, $h = \tilde{h}$, $H_i = \tilde{H}_i$. The theorem is proved.

In analogous manner we can obtain a uniqueness theorem for the reconstruction of a differential operator with semidecomposable boundary conditions with respect to two spectra (see [6, Theorem 3]).

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