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TOPOLOGICAL SPACE OBJECTS IN A TOPOS II: $\mathcal{E}$-COMPLETENESS AND $\mathcal{E}$-COCOMPLETENESS

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It is well known that topoi satisfy strong internal completeness and cocompleteness conditions: Lawvere [4] announced the existence of internal Kan extensions; proofs may be found in Kock and Wraith [3] and Diaconescu [2]. In this paper I give an explicit construction of the limit of an internal functor and lift the completeness and cocompleteness of $\mathcal{E}$ to the category of topological space objects in $\mathcal{E}$ defined by internalizing the definition in terms of open sets (as in [7] and [8]).

I: Notational Conventions and Definitions

We adopt the internal language of a topos exposed in Osius [6] in which predicates are used to name the subobjects whose construction they describe. For instance the subobject $\forall_{a\in A}\exists_{b\in A}(a\in C \Rightarrow b\in C) \rightarrow PA$ is the result of applying the functor $\forall_{pr\cdot PA\times A\rightarrow A}$ to the result of applying the functor $\exists_{pr_3 : PA\times A\times A\rightarrow PA\times A}$ to the subobject $\epsilon_A \times A \Rightarrow (twist_{2,3})^{-1}\epsilon_A \times A$. The existence of elements as such is not intended to be implied.

The union map $\bigcup : P^2A \rightarrow PA$ is the exponential adjoint of the characteristic morphism of the subobject of $P^2A\times A$ consisting of those pairs (S,a) such that $\exists_{B\in PA}(a\in B \cap B\in S)$. The intersection map $\underline{\bigcap}$: $P^2A\rightarrow PA$ is defined similarly using $\forall_{B\in PA}(B\in S \Rightarrow a\in B)$.

Internal forms of the functors $\exists_f$ and $\forall_f$ are the morphisms from PA to PB taking a subobject S to $\exists_f S$ and $\forall_f S$ respectively. To distinguish the usage we underline internal functors: $\underline{\exists_f}$, $\underline{\forall_f}$, $\underline{f^{-1}}$. A construction of these maps may be found in Osius [5].

DEFINITION: A topological space object in a topos is a pair (A, T_A),

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where $T_A \subseteq PA$ satisfies

1. $\phi \in T_A$ and $A \in T_A$
2. $\forall_{B, B' \in PA}((B \in T_A \cap B' \in T_A) \Rightarrow (B \cap B') \in T_A)$
3. $\forall_{S \in P^2 A}(S \subseteq T_A \Rightarrow \bigcup S \in T_A)$.

A morphism $f:(A, T_A) \to (B, T_B)$ is called continuous iff $\exists_{f^{-1}} T_B \subseteq T_A$ and open iff $\exists_{\exists_f} T_A \subseteq T_B$. Top($\mathcal{E}$) is the category of topological space objects and continuous morphisms in $\mathcal{E}$.

In Sets this is the definition of a topological space. In Sh(X) for X a topological space, a topological space object is an étal space $(Y, T_Y) \xrightarrow{p} (X, T_X)$ together with a second topology T_Y' on Y making p continuous with $T_Y' \subseteq T_Y$. A proof of this appears in Stout [8].

II: Initial Topologies and Finite Limits

In order to construct limit topologies in Sets-based topology, we first construct the limit in Sets and then give it the topology induced by the projections. All that is really needed is to transport this construction to a topos is to show that all of the inferences in the proof that the construction works are intuitionistically valid. This is essentially what I did in Chapter 3 of [7].

There is a less tedious way which provides a technique easily applicable in other situations. All formations of closures under operations can be done using the concept of a tractible predicate and an argument similar to Mikkelsen's (unpublished) topos proof of the Tarski fixed point theorem.

DEFINITION: Let L be a complete lattice in $\mathcal{E}$. Then a predicate $\gamma \xrightarrow{i} L$ is called tractible iff

$$\begin{array}{ccccc} P\gamma & & \longrightarrow & & 1 \\ \downarrow{\scriptstyle \exists i} & & & & \downarrow{\scriptstyle tr} \\ PL & \xrightarrow{inf} & L & \xrightarrow{\chi_\gamma} & \Omega \end{array}$$

commutes.

REMARK: Mikkelsen's version of the Tarski fixed point theorem has as its key poiny the tractibility of the predicate obtained by pulling $\leq$

back along $(h, 1_L)$, where h is an order preserving endomorphism.

LEMMA 1: *Let* $f:C \to A$ *be any morphism,* $D \subseteq PA \times C$ *be such that* $(\underline{\bigcap} \times C)^{-1} D \subseteq \{(S, c) \in P^2 A \times C \mid \forall_{B \in PA}(B \in S \Rightarrow (B, c) \in D)\}$, *then the predicate* γ *specifying those* $A' \in PA$ *such that* $\forall_{c \in C}((A', c) \in D \Rightarrow f(c) \in A')$ *is tractible.*

Proof: We calculate $(\underline{\Xi}_i)^{-1}(\underline{\bigcap})^{-1}\{A' \mid \forall_{c \in C}((A', c) \in D \Rightarrow f(c) \in A')\}$ and show that it is all of $P\gamma$. The Beck condition for $\forall$ allows us to interchange the second pullback with the quantification to get

$$(\underline{\Xi}_i)^{-1}\{S \in P^2 A \mid \forall_{c \in C}((\underline{\bigcap} \times C)^{-1}\{(A', c) \mid (A', c) \in D \Rightarrow f(c) \in A'\} = S)\}.$$

Calculation of the inside pullback gives

$$(\underline{\Xi}_i)^{-1}\{S \in P^2 A \mid \forall_{c \in C}((\underline{\bigcap} \times C)^{-1} D \Rightarrow \forall_{A' \in PA}(A' \in S \Rightarrow f(c) \in A'))\}.$$

By hypothesis this is larger than

$$(\underline{\Xi}_i)^{-1}\{S \in P^2 A \mid \forall_{c \in C}(\forall_{A'' \in PA}(A'' \in S \Rightarrow (A'', c) \in D) \Rightarrow \forall_{A' \in PA}(A' \in S \Rightarrow f(c) \in A'))\}.$$

Using the intuitionistic rules $\forall_g A \Rightarrow \forall_g B \geq \forall_g (A \Rightarrow B)$ and $A \Rightarrow (B \Rightarrow C) \leq (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$ we obtain a smaller subobject

$$(\underline{\Xi}_i)^{-1}\{S \in P^2 A \mid \forall_{c \in C}\forall_{A' \in PA}(A' \in S \Rightarrow ((A', c) \in D \Rightarrow f(c) \in A'))\}.$$

We may now apply the Beck condition to move the other pullback inside the quantifier and then use the definition of the internalized quantifier to calculate the result. This has the effect of replacing the condition $A' \in S$ with $A' \in S \cap \forall_{c \in C}((A', c) \in D \Rightarrow f(c) \in A)$ giving

$$\{S \in P\gamma \mid \forall_{c \in C}\forall_{A' \in PA}((A' \in S \cap \forall_{c \in C}((A', c) \in D \Rightarrow f(c) \in A)$$

$$\Rightarrow ((A', c) \in D \Rightarrow f(c) \in A'))\}.$$

The expression inside the quantifiers is always true by the $\forall$-elimination rule, proving the lemma.

COROLLARY: *If* $f:A^n \to A$ *is any morphism, then*

$$\{A' \mid \forall_{a_1 \ldots a_n \in A}((a_1 \in A' \cap a_2 \in A' \cap \ldots \cap a_n \in A') \Rightarrow f(a_1 \ldots a_n) \in A')\}$$

is tractible.

Proof: $(\underline{\cap}\times A^n)^{-1}\{(A', a_1, \ldots, a_n) \mid a_1 \in A' \cap \ldots \cap a_n \in A'\} = \{(S, a_1, \ldots, a_n) \mid \forall_{A' \in PA}(A' \in S \Rightarrow a_1 \in A' \cap \ldots \cap a_n \in A'\}$ so the lemma applies.

<u>COROLLARY</u>: *If* $f:(PA)^n \to A$ *is any morphism, then*

$$\{A' \mid \forall_{A_1, \ldots, A_n \in PA}((A_1 \leq A' \cap \ldots \cap A_n \leq A') \Rightarrow f(A_1, \ldots, A_n) \in A')\}$$

is tractable.

Proof: The proof for arbitrary n is a direct generalization of the proof for $n=1$ which follows from the fact that $A_1 \leq A'$ iff $\forall_{a \in A}(a \in A_1 \Rightarrow a \in A')$ and the following calculation:

$$(\underline{\cap}\times PA)^{-1}\{(A', A_1) \mid \forall_{a \in A}(a \in A_1 (a \in A_1 \Rightarrow a \in A')\}$$

$$= \{(S, A_1) \mid \forall_{a \in A}(a \in A_1 \Rightarrow \forall_{B \in PA}(B \in S \Rightarrow a \in B))\}$$

$$= \{(S, A_1) \mid \forall_{a \in A} \forall_{B \in PA}(a \in A_1 \Rightarrow (B \in S \Rightarrow a \in B))\}$$

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so the lemma applies.

A direct proof shows that for any $A' \subset A$ the predicate $\{B \mid B < A'\}$ is tractable.

<u>LEMMA 2</u>: *The intersection of two tractable predicates is tractable.*

Proof: Let Y and S be tractable. Then $P\gamma \subseteq \underline{\cap}^{-1}\gamma$ and $P\delta \subseteq \underline{\cap}^{-1}\delta$ so $P\gamma \cap P\delta \subseteq \underline{\cap}^{-1}(\gamma \cap \delta)$. But the power object functor preserves order so $P(\gamma \cap \delta) \leq P\gamma \cap P\delta$, and thus $\gamma \cap \delta$ is tractable.

As a result of these lemmata we can observe that the internal forms of the predicates "T is a topology" and "T is a topology larger than S" are tractable.

<u>THEOREM 1</u>: *If* γ *is a tractable predicate then the smallest global section of* γ *is* $\inf \underline{\Xi}_1[\gamma]$.

Proof: Tractability tells us that $\inf \underline{\exists}_i[\gamma]$ is a global section of γ. If δ is any other global section of γ then $[\delta] \leq [\gamma]$, so since inf reverses order $\inf \underline{\exists}_i[\delta] \geq \inf \underline{\exists}_i[\gamma]$. For global sections δ, $\inf \underline{\exists}_i[\delta] = \delta$ so this proves the theorem.

COROLLARY: For any $S \leq PA$ there is a smallest topology larger than S.

COROLLARY: Top(E) is finitely complete.

Proof: Let F be a functor from a finite category D to Top(E) and L be its limit in E. The limit cone gives a finite family of morphisms $l_i : L \to F(D_i)$. The limit topology on L is the smallest topology containing the union of all of the $\exists \underline{l}_i{}^{-1}(T_{FD_i})$. Any other cone induces a unique morphism by the universal mapping property in E. It is continuous because the inverse image of a subbase is in the topology.

A subobject $B \leq PA$ is called a basis for T_A if T_A is its closure under internal unions. Precisely as in set-based topology we can produce a basis of open rectangles in the product topology on the product of two spaces by taking the image of $T_A \times T_B$ along the morphism $r_{A,B}: PA \times PB \to P(A \times B)$ defined as the exponential adjoint of the characteristic morphism of the subobject $\{(A', B', a, b) | a \in A' \cap b \in B'\}$. In fact it is not necessary to use the whole topology in the formation of a basis of rectangles. It will suffice to replace T_A and T_B with bases.

III: Final Topologies and Finite Colimits

In order to produce final topologies it suffices to consider only the case of final topologies induced by one map and then use the tractability of the topology predicate to generalize the result to any finite collection of maps.

THEOREM 2: Given a morphism $f:(A, T_A) \to B$ in $\mathcal{E}$, the largest topology on B making f continuous is $(f^{-1})^{-1} T_A$.

Proof: Adjointness and the definition of continuity tell us that this is the largest possible candidate. That it is a topology is the result of the preservation of all logical operations by pullback, exactly as in Sets.

COROLLARY: For any finite set of functions f_i from A_i to B, there is a largest topology on B making all of the f_i continuous.

Proof: Apply the theorem to find the largest topologies making each of the functions continuous and then form the subobject of the object of topologies consisting of that collection of topologies and take its intersection.

COROLLARY: Top($\mathcal{E}$) has finite colimits.

IV: $\mathcal{E}$-Limits and $\mathcal{E}$-Colimits in $\mathcal{E}$

In [3], Lawvere observed that for a category object C in a topos, the category of internal functors from C to $\mathcal{E}$ is again a topos, and furthermore that any functor between category objects induces a functor between the resulting functor categories which has both a right and a left adjoint. The constructions of the adjoints (which are a sort of internal Kan extention) may be found in Diaconescu [2] or Kock and Wraith [3]. The object of this section is to give an explicit construction, using the data in the functor, of the limit. The construction follows a suggestion of John Gray.

DEFINITION: (Benabou [1]) A category object in a category with finite limits is a 6 tuple (V, E, c, d, u, o), with V and E objects of the category (the object of objects and the object of morphisms, respectively), c and d morphisms giving the domain and codomain, u a morphism giving the inclusion of the unit maps, and o a morphism from $E\ {}_c\!\times_d E$ to E (composition) such that the following conditions hold

1. $cu = du = 1_V$
2. $d\,pr_1 = do$ and $c\,pr_2 = co$
3. $o(E\times u)(E, c) = 1_E$ and $o(u\times E)(d, E) = 1_E$
4. $o(E\ {}_d\!\times_d\ o) = o(o\ {}_d\!\times_d E)$.

An internal functor from a category object $(V, E, \ldots)$ *to* C *(the base category) is a triple* (F, p, σ) *with* F *an object of* C, $p: F \to V$ *giving the action of* F *on objects as a* V*-indexed set, and* $\sigma: F\, {}_p{\times}_d E \to F$ *(the structure map) giving the action on morphisms.*

The constant functor with value K *from* $(V, E, \ldots)$ *to* C *is the functor* $(K \times V, pr_2, pr_2 \times_d c)$.

A natural transformation from (F, p, σ) *to* (F', p', σ') is a mor*phism from* F *to* F' *such that* $p'n = p$ *and* $ns = s'(n \times_d E)$.

Diaconescu [2] identified the colimit as the coequalizer of the structure map and the projection. It is easy to show that the natural transformation to the constant functor so constructed satisfies the universal mapping property for cocones.

The construction of the limit L as a subobject of F^V picks out the collection of all sections of p which are consistent with the action of σ.

PROPOSITION: If (F, p, σ) *is an internal functor from* $(V, E, \ldots)$ *to* E, *then the limit of* F *can be constructed as a subobject of* F^V.

Proof: The condition that $a: A \times V \to F$ be a natural transformation from a constant functor is the same as the commutativity of the following two diagrams:

$$\begin{array}{ccccc} (A \times V)\, {}_{pr}{\times}_d E & \longrightarrow & A \times V \times E & \xrightarrow{(pr_1,\, cpr_2)} & A \times V \\ \downarrow {\scriptstyle a \times_d E} & & & & \downarrow {\scriptstyle a} \\ F\, {}_p{\times}_d E & & \xrightarrow{\quad\sigma\quad} & & F \end{array}$$

$$\begin{array}{ccc} A \times V & \xrightarrow{\quad a \quad} & F \\ & {\scriptstyle pr} \searrow \quad \swarrow {\scriptstyle p} & \\ & V & \end{array}$$

The first of these is equivalent to the commutativity of

$$\begin{array}{ccccc}
(A\times V)_{pr}\times_d E & \xrightarrow{(a\times E\times V)} & (F^V\times V)_{pr}\times_d E & & \\
 & \swarrow & & \searrow{\scriptstyle ev\times E} & \\
F^V\times V\times E & & & & F_p\times_d E \\
\Big\downarrow{\scriptstyle (pr_1,\ cpr_2)} & & & & \Big\downarrow{\scriptstyle \sigma} \\
F^V\times V & & \xrightarrow{\quad ev \quad} & & F
\end{array}$$

which, using the representability of partial maps and exponential adjointness is equivalent to the commutativity of

$$A \xrightarrow{a} F^V \underset{((\sigma(ev\times_d E))^{\sim})^{\wedge}}{\overset{((ev(pr_1,\ cpr_2))^{\sim})^{\wedge}}{\rightrightarrows}} F^{(V\times E)}.$$

The commutativity of the second diagram is equivalent to the commutativity of the square

$$\begin{array}{ccc}
A & \xrightarrow{a} & F^V \\
{\scriptstyle a}\Big\downarrow & & \Big\downarrow{\scriptstyle p^V} \\
F^V & \xrightarrow{pr_2^{\,V}} & V^V
\end{array} .$$

Thus a is a natural transformation to F from a constant functor iff its exponential adjoint equalizes both of the pairs of maps produced above. The limit L is thus the equalizer of both pairs of maps.

V: <u>$\mathcal{E}$-Limit and $\mathcal{E}$-Colimit Topologies</u>

To extend the completeness and cocompleteness of $\mathcal{E}$ to Top($\mathcal{E}$), we first observe that by giving both V and E the discrete topology any category object in $\mathcal{E}$ may be considered to be a category object in Top($\mathcal{E}$). The other definitions remain the same but take on new meaning because "morphism" and "product" are interpreted in Top($\mathcal{E}$). In particular, constant functors are required to be given the product topology.

For the topos <u>Sets</u> these results show that the category of topological spaces is (small) complete and cocomplete.

THEOREM 3: Top($\mathcal{E}$) is $\mathcal{E}$-cocomplete.

Proof: There is a topology induced on the colimit by the construction of a final topology from the injection and the topology on F. This makes the colimit natural transformation from F to $C \times V$ continuous and makes the induced map to any other constant functor continuous as well.

THEOREM 4: Top($\mathcal{E}$) is $\mathcal{E}$-complete.

Proof: There is a topology induced on the object $L \times V$ by the two maps $L \times V \to F^V \times V \xrightarrow{ev} F$ and pr_2 by the construction of initial topologies. This satisfies the universal mapping property of the limit topology in that for any other natural transformation a from a constant functor $A \times V$ to F, there is a continuous natural transformation from $A \times V$ to $F \times V$ of the form $f \times V$. The only catch is that the topology so constructed on $L \times V$ is not a product topology. Therefore, the problem is to find the product topology best approximating the induced topology $T_{L \times V}$.

LEMMA 3: The projection from a product to one of its factors is open: i.e., $\exists_{\underline{\exists}_{pr}} T_{A \times V} \leq T_A$.

Proof: Since existential quantification preserves unions and $\exists_{r_{A,V}} T_A \times T_V$ is a basis for the product topology, it will suffice to show that

$$\exists_{\underline{\exists}_{pr}} \exists_{r_{A,V}} T_A \times T_V \leq T_A.$$

The argument is essentially that the projection of a rectangle $U \times V$ is the open subobject obtained by truncating U to the support of V. This is obtained formally by the commutativity of several diagrams.

First, truncation and union commute, that is, $\underline{\bigcup} r_{PA,1} = r_{A,1}(\underline{\bigcup} \times P1)$. Both maps are the exponential adjoint of the characteristic morphism of

$$\{(S, U, a, i) \in P^2A \times P1 \times A \times 1 \mid \exists_{A' \in PA}(i \in U \cap A' \tilde{\in} S \cap a \in A')\}.$$

Since $\underline{\bigcup}$ is split by $\{\ \}$, this shows $r_{A,1} = \underline{\bigcup} r_{PA,1}(\{\ \} \times P1)$.

Second, the rectangle map commutes with internal existentiation:

$\exists_{A\times f} r_{A,V} = r_{A,B} PA\times\exists_f$. Both are the exponential adjoint of the characteristic morphism of $\{(A', V', a, b) \in PA\times PV\times A\times B \mid a\in A' \cap f(b)\in V'\}$.

Combining these results gives the commutativity of

$$\begin{array}{ccccccc}
T_A\times P1 & \xrightarrow{\{\ \}\times P1} & PT_A\times P1 & \longrightarrow & PT_A & \longrightarrow & T_A \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \exists_i\times P1} & & \downarrow{\scriptstyle \exists_i} & & \downarrow \\
PA\times P1 & \xrightarrow{\{\ \}\times P1} & P^2A\times P1 & \xrightarrow{r_{PA,1}} & P^2A & \xrightarrow{\underline{U}} & PA
\end{array}$$

$$PA\times P1 \xrightarrow{r_{A,1}} PA$$

where the right hand square commutes by the union axiom.

Thus $\exists_{r_{A,1} PA\times\exists_{terminal}}(T_A\times T_V) = \exists_{r_{A,1}}(T_A\times P1) \le T_A$. But $r_{A,1} PA\times\exists_{terminal}$ is $\exists_{pr_1} r_{A,V}$, so this tells us that

$$\exists_{\exists_{pr}} \exists_{r_{A,V}}(T_A\times T_V) \le T_A$$

as needed.

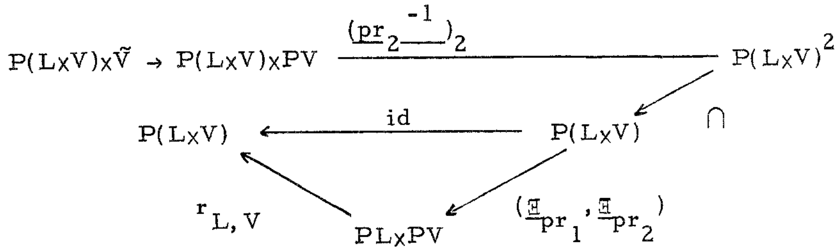
This lemma tells us that the topology used for L must contain $\exists_{\exists_{pr}} T_{L\times V}$. The next two lemmas show that the smallest such topology is the one we need. Even in Sets this depends heavily on the fact that V is discrete.

LEMMA 4: The partial map representor $\tilde{V}$ is a basis for the discrete topology on $\tilde{V}$.

Proof: The construction of $\tilde{V}$ (as in Kock and Wraith [3]) induces a factorization of the singleton through $\tilde{V}$. The unit law for the power-object triple tells us that $\underline{\cup}\,\exists_{\{\}_V} = id_{PV}$, so the singleton map, thought of as a subobject, is a basis. Thus $\tilde{V}$ is a basis since it contains the singleton.

The next lemma tells us that the basis $\tilde{V}$ gives a basis for the product topology on $V\times L$ which consists of slices rather than thick rectangles. In Sets this follows by taking the points as basic open sets.

LEMMA 5: The following diagram commutes:



Proof: Taking exponential adjoints in this diagram gives two maps into the subobject representor which we need to show are equal. This is customarily done by showing that they represent the same subobject. This involves the computation of an involved but straightforward pullback which is carried out in diagrams 1 and 2 up to the determination of the subobjects of P(LxV)xPVxLxV. The Beck conditions and definitions of $\cap$ and internal functors are used extensively in these calculations.

In the result of the calculation in diagram 1 the two terms can be combined under one pair of quantifiers to get

$$\{(X', V'', l, v) \mid \exists_{l' \in L} \exists_{v' \in V}((l', v) \in X' \cap v \in V'' \cap (l, v') \in X' \cap v' \in V'')\}.$$

The requirement in the next stage of the pullback that V'' be in $\tilde{V}$ tells us that since $v \in V''$ and $v' \in V''$, $v = v'$. Thus in P(LxV)x$\tilde{V}$ we obtain the subobject $\{(X', V'', l, v) \mid \exists_{l' \in L} \exists_{v' \in V}((l', v) \in X' \cap v \in V'' \cap (l, v'') \in X' \cap v' \in V'' \cap v = v'\}$ which reduces to $\{(X', V'', l, v) \mid (l, v) \in X' \cap v \in V''\}$ as obtained in Diagram 2. Diagrams 1 and 2 are shown on page 12.

COROLLARY: The product topology obtained by using the topology generated by $\exists_{\exists_{pr}} T_{L \times V}$ contains $T_{L \times V}$.

Proof: Preceding the diagram in the lemma by the subobject $T_{L \times V} \times V$ and taking the image along the top map gives a basis for the topology $T_{L \times V}$, since the projection is continuous and V is a basis for the topology on V. Taking the image along the bottom map gives a subobject of the rectangle basis for the product topology. This shows that a basis for the topology $T_{L \times V}$ is contained in the product topology so

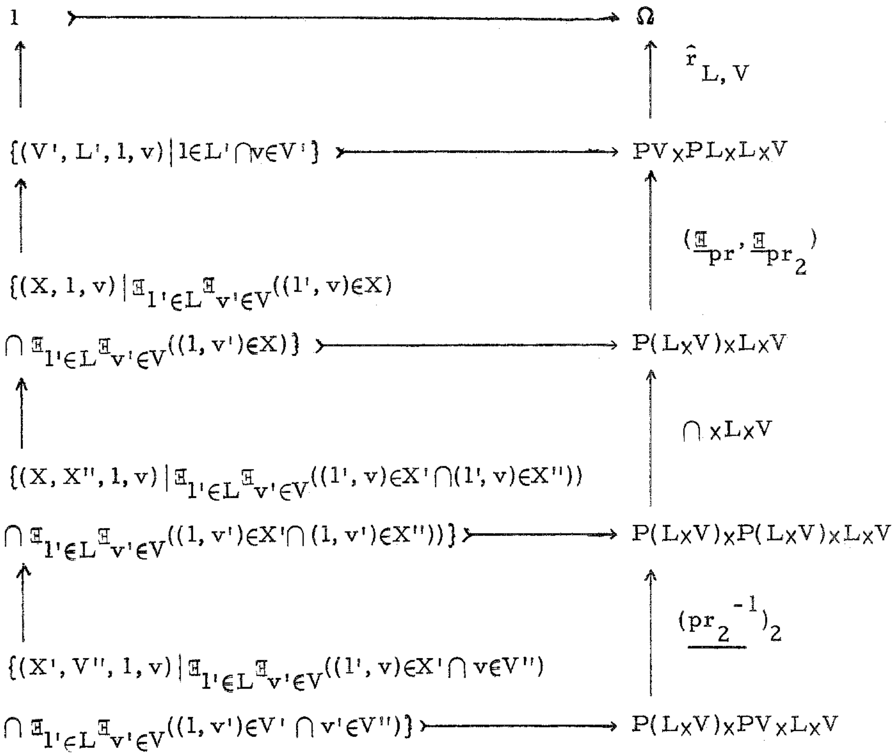


Diagram 1

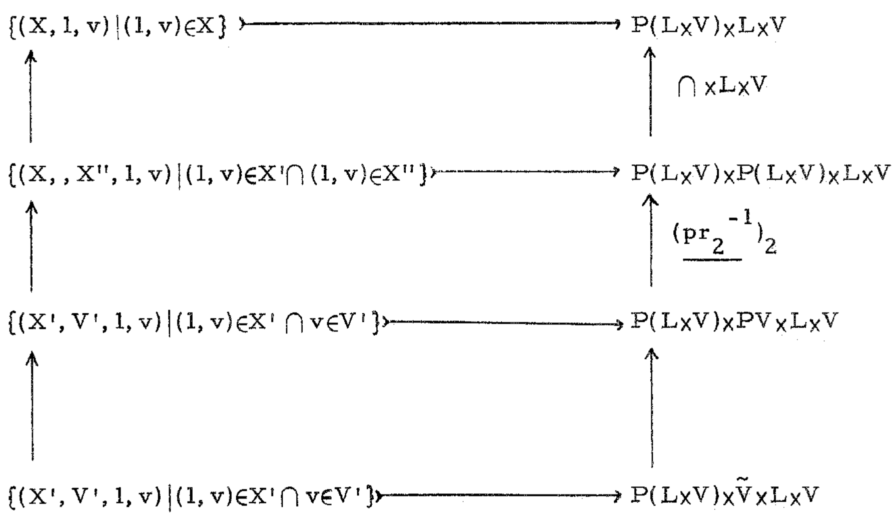


Diagram 2

the topology itself must be.

This tells us that the product topology makes $(L\times V, T_{\text{product}}) \to (F, T_F)$ a natural transformation from a constant functor to F. It remains to show that the topology on L makes the map $f: A \to L$, induced by the universal mapping property in $\underline{E}$, continuous. For this, we need the following internal form of the Beck conditions:

<u>LEMMA 6</u>: <u>If</u>

$$\begin{array}{ccc} A & \xrightarrow{g'} & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{g} & D \end{array}$$

<u>is a pullback, then</u> $(\underline{f^{-1}})\underline{\exists}_g = \underline{\exists}_{g'}\underline{(f')^{-1}}$.

Proof: This is a direct consequence of the external Beck conditions for the pullback

$$\begin{array}{ccc} PC\times A & \xrightarrow{\quad PC\times g' \quad} & PC\times B \\ {\scriptstyle PC\times f'}\downarrow & & \downarrow{\scriptstyle PC\times f} \\ PC\times C & \xrightarrow{\quad PC\times g \quad} & PC\times D \end{array} .$$

To show that f is continuous when $f\times V$ is continuous when $L\times V$ is given the topology $T_{L\times V}$ and L is given the topology with subbase $\exists_{\underline{\exists}_{pr}} T_{L\times V}$, it will suffice to show that $\exists_{\underline{f^{-1}}} \exists_{\underline{\exists}_{pr}} T_{L\times V} \le T_A$.

Now, the square

$$\begin{array}{ccc} L & \xrightarrow{pr} & L\times V \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\times V} \\ A & \xrightarrow{pr} & A\times V \end{array}$$

is a pullback, so $\underline{f^{-1}}\,\underline{\exists}_{pr} = \underline{\exists}_{pr}\underline{(f\times V)^{-1}}$. Thus,

$$\exists_{\underline{f^{-1}}} \exists_{\underline{\exists}_{pr}} T_{L\times V} = \exists_{\underline{\exists}_{pr}\underline{(f\times V)^{-1}}} T_{L\times V}.$$

Since $f\times V$ is continuous $\exists_{\underline{(f\times V)^{-1}}} T_{L\times V} \le T_{A\times V}$. Therefore, applying $\exists_{\underline{\exists}_{pr}}$ gives

$$\exists_{\underline{f^{-1}}} \exists_{\underline{\exists}_{pr}} T_{L\times V} \le \exists_{\underline{\exists}_{pr}} T_{A\times V} \le T_A.$$

The last inequality follows from Lemma 3.

I would like to thank Gerhard Osius for his helpful comments.

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