Contributed paper

A MARKOV-MODULATED M/G/1 QUEUE I: STATIONARY DISTRIBUTION

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Received 20 August 1986
(Revised 20 October 1986)

Abstract

We consider an M/G/1 queueing system in which the arrival rate and service time density are functions of a two-state stochastic process. We describe the system by the total unfinished work present and allow the arrival and service rate processes to depend on the current value of the unfinished work. We employ singular perturbation methods to compute asymptotic approximations to the stationary distribution of unfinished work and in particular, compute the stationary probability of an empty queue.

Keywords

State dependent M/G/1 queue, Markov modulated queues, singular perturbations.

This research was supported in part by NSF Grants DMS-84-06110, DMS-85-01535 and DMS-86-20267, and grants from the U.S. Israel Binational Science Foundation and the Israel Academy of Sciences.

1. Introduction

We consider an M/G/1 queueing system where the arrival rate and service time density are modulated by a two-state stochastic process. An example of such a system arises when there are two independent arrival streams regulated by a switch that allows only one of the two streams to enter the queueing system. Here the switch is the modulating process, which is Markov if the switching times are exponentially distributed. Other examples of queueing systems which are modulated by another stochastic process are examined in [1]–[8].

In [4], Regterschot and de Smit treat an M/G/1 queue with arrivals and service modulated by an N-state Markov chain and obtain exact expressions for several performance measures using a matrix factorization method. Here we treat the case where $N = 2$, but with the arrivals, service and modulating processes depending on $U(t)$, the unfinished work in the system. This state-dependent model is no longer susceptible to analysis by the methods used in [4]. Therefore we employ singular perturbation methods which allow us to compute the asymptotic expansion of the stationary distribution of the unfinished work and in particular the stationary probability of finding an empty queue.

We denote the modulating process as $Z(t)$ with the state space $\{0, 1\}$. Thus the state space of the Markov process $\{U(t), Z(t)\}$ is $[0, \infty) \times \{0, 1\}$. We define the conditional switching rates α and β by

$$\begin{aligned} &\Pr[Z(t+\Delta t) = 1 | Z(t) = 0, \quad U(t) = w] = \alpha(w)\Delta t + o(\Delta t) \\ &\Pr[Z(t+\Delta t) = 0 | Z(t) = 1, \quad U(t) = w] = \beta(w)\Delta t + o(\Delta t) \end{aligned} \tag{1.1}$$

and the conditional arrival rates and service time distributions by

$$\begin{aligned} &\Pr[\text{arrival in } (t, t+\Delta t) | Z(t) = i, \quad U(t) = w] = \lambda_i(w)\Delta t + o(\Delta t) \\ &\Pr[\text{service time} < z | Z(t^*) = i, \quad U(t^*) = w] = \int_0^z b_i(s, w)\,\mathrm{d}s \qquad (i = 0, 1) \end{aligned} \tag{1.2}$$

where t^* denotes the instant at which the customer enters the queueing system. We assume that the rate at which service is provided is unity and thus is independent of the state of the process $(U(t), Z(t))$.

The state dependence given by (1.1)–(1.2) describes a variety of effects. Having the switching rates sensitive to the buffer content creates a feedback mechanism which can be used to control the system's work backlog. The dependence of the arrival rates λ_i on w, indicates possibly discouraged arrivals. The dependence of the service time on the values of $Z(t^*)$ allows for the two classes of customers to have different service requirements, while the dependence of the service time on $U(t^*)$ allows for the fact that a customer may be more likely to submit a longer (shorter) job if the system's buffer content is small (large). The latter would be an important consideration if shorter jobs are given priority over longer jobs.

We make the following assumptions:

(i) The switching rates α, β, the arrival rates λ_i and the densities $b_i(z, w)$ are smooth functions of w.

(ii) $\lambda_i(w) \geqq 0, i = 0, 1$; $\alpha(w)$, $\beta(w)$ and $\lambda_0(w) + \lambda_1(w)$ are strictly positive for all $w \geqq 0$.

(iii) the system's capacity of unfinished work is infinite, so that the state space of the process $U(t)$ is $[0, \infty)$.

(iv) The moment generating functions $M(\theta, w, i)$ of the service time densities

$$M(\theta, w, i) = \int_0^\infty e^{\theta z} b_i(z, w) \mathrm{d}z \qquad (i = 0, 1)$$

exist in a neighborhood of $\theta = 0$ for all w, so that all the moments of $b_i(z, w)$ exist. We remark here that the basic asymptotic approach developed in sect. 2 can also be used to treat the cases where the switching and arrival rates vanish either at a point or over an entire interval, or the functions are non-smooth in their dependence on w; this approach can also be used for finite capacity models in which the unfinished work process is confined to the interval $[0, K]$ for some $K > 0$.

An important quantity associated with our model is the state-dependent traffic intensity $r(w)$ defined by

$$r(w) = \frac{\lambda_0(w)\ \beta(w)}{\mu_0(w)\ [\alpha(w) + \beta(w)]} + \frac{\lambda_1(w)\,\alpha(w)}{\mu_1(w)\ [\alpha(w) + \beta(w)]} \tag{1.3}$$

where

$$\frac{1}{\mu_i(w)} = \int_0^\infty z b_i(z, w) \mathrm{d}z \qquad (i = 0, 1). \tag{1.4}$$

Our final assumption is (v) $r(w) \leqq 1 - \delta < 1$ for some $\delta > 0$, for all w sufficiently large. This guarantees the stability of the system and hence the existence of a stationary distribution for the Markov process $\{U(t), Z(t)\}$.

We show in sect. 2 that the structure of the state-dependent model is much richer than the corresponding state-independent model discussed in [4]. The present model is indeed very sensitive to the behavior of the function $r(w)$. We consider separately three cases distinguished by the structure of $r(w)$. These are

1. $r(w) < 1$ for all $w \geqq 0$
2. $r(0) = 1$ $r'(0) < 0$ $r(w) < 1$ for $w > 0$
3. $r(w) > 1$ for $0 \leqq w < w_0$ $r(w_0) = 1$ $r(w) < 1$ for $w > w_0$.

While these cases are by no means exhaustive, their consideration will be sufficient to demonstrate the rich structure of the state-dependent model. We refer to a point w^* where $r(w^*) = 1$, as an equilibrium point. Hence, $w^* = 0$ is an equilibrium point for case 2 while case 3 has an equilibrium point at w_0 in the interior of the interval $[0, \infty)$.

Due to the complexity of the state-dependent model, it is unlikely that exact expressions for performance measures such as the stationary density of unfinished work, can be obtained for arbitrary, smooth functions $\alpha(w)$, $\beta(w)$, and $\lambda_i(w)$. Therefore we obtain approximate expressions for the stationary distribution. We introduce a parameter ϵ and write the rates as

$$\alpha = \frac{1}{\epsilon}\,\alpha(w), \quad \beta = \frac{1}{\epsilon}\,\beta(w), \quad \lambda_i = \frac{1}{\epsilon}\,\lambda_i(w), \tag{1.5}$$

and the service time densities as

$$b_i = \frac{1}{\epsilon}\, b_i\left(\frac{z}{\epsilon}, w\right). \tag{1.6}$$

We analyze the system for small ϵ, i.e. $0 < \epsilon \ll 1$. To clarify the specific choice of ϵ, suppose we had a state-dependent arrival rate model with $\lambda_0 = 10/(1 + w)$, $\lambda_1 = 15/(1 + w)$, $\alpha = 30$, $\beta = 40$ and $b_0 = b_1 = 20\mathrm{e}^{-20z}$. Then clearly we may write

$$\lambda_0 = \frac{1}{\epsilon} \cdot \frac{1}{1+w}, \quad \lambda_1 = \frac{1}{\epsilon} \cdot \frac{3}{2(1+w)}, \quad \alpha = \frac{3}{\epsilon}, \quad \beta = \frac{4}{\epsilon},$$

$$b_0 = b_1 = \frac{2}{\epsilon}\,\mathrm{e}^{-2z/\epsilon} \text{ with } \epsilon = 0.1.$$

Alternatively, we could write

$$\lambda_0 = \frac{2}{\epsilon(1+w)}, \quad \lambda_1 = \frac{3}{\epsilon(1+w)}, \quad \alpha = \frac{6}{\epsilon}, \quad \beta = \frac{8}{\epsilon} \quad \text{and}$$

$$b_0 = b_1 = \frac{4}{\epsilon}\mathrm{e}^{-4z/\epsilon} \text{ with } \epsilon = 0.2.$$

The choice of ϵ is not unique, but our final asymptotic results are independent of the particular choice of ϵ.

We observe from (1.5) and (1.6) that the mean requested service times are

$$\int_0^\infty z b_i\left(\frac{z}{\epsilon}, w\right) \frac{dz}{\epsilon} = \epsilon \int_0^\infty u b_i(u, w) du = O(\epsilon) \tag{1.7}$$

and are thus small, and that the system has rapid arrival and switching rates ($O(1/\epsilon)$).

The scaling introduced above is similar to the scaling used to obtain diffusion approximations to more complex Markov processes (see e.g. Burman [10] and Iglehart and Whitt [11] – [13]). Unlike diffusion approximations, however, the approximate methods developed in this paper are not restricted to the heavy traffic region (where $r \cong 1$) and the results we obtain involve all the moments of the service time distributions, rather than merely the first two.

Our approximation techniques are based on singular perturbation methods such as the WKB method, boundary layer theory, and the method of matched asymptotic expansions ([14] – [17]). We suitably modify existing methods, developed in the context of differential equations to deal with integro-differential-difference equations that arise in queueing models. We have previously employed similar methods in other systems ([18] – [23]).

This paper is organized as follows. In sect. 2 we formulate the stationary forward Kolmogorov equation associated with the Markov process $\{U(t), Z(t)\}$ and construct asymptotic approximations to the stationary distribution of the process, treating each of the cases 1–3 separately. In particular, the probability of finding an empty system (i.e. $U(t) = 0$) is computed.

2. Stationary distribution

We compute the stationary distribution for the joint process $\{U(t), Z(t)\}$ by solving the appropriate forward Kolmogorov equation. For $i = 0, 1$ let

$$\begin{aligned} p(w,i) &= \lim_{t\to\infty} \frac{d}{dw} \Pr[U(t) < w, \quad Z(t) = i], \qquad w > 0 \\ A_i &= \lim_{t\to\infty} \Pr[U(t) = 0, \quad Z(t) = i] \end{aligned} \tag{2.1}$$

where the existence of the limits follows from our assumption (v) that the state-dependent traffic intensity $r(w)$ is less than and remains bounded away from unity for sufficiently large w. With the scaling (1.5)–(1.6), the functions $p(w, i)$ satisfy

$$\frac{\mathrm{d}}{\mathrm{d}w}\begin{pmatrix} p(w,0) \\ p(w,1) \end{pmatrix} + \frac{1}{\epsilon}\begin{pmatrix} -\alpha(w) - \lambda_0(w) & \beta(w) \\ \alpha(w) & -\beta(w) - \lambda_1(w) \end{pmatrix}\begin{pmatrix} p(w,0) \\ p(w,1) \end{pmatrix}$$

$$+ \frac{1}{\epsilon}\int_0^{w/\epsilon} \begin{pmatrix} \lambda_0(w-\epsilon z)\, p(w-\epsilon z,0)\, b_0(z, w-\epsilon z) \\ \lambda_1(w-\epsilon z)\, p(w-\epsilon z,1)\, b_1(z, w-\epsilon z) \end{pmatrix} \mathrm{d}z$$

$$+ \frac{1}{\epsilon^2}\begin{pmatrix} \lambda_0(0)\, b_0\left(\frac{w}{\epsilon},0\right) A_0 \\ \lambda_1(0)\, b_1\left(\frac{w}{\epsilon},0\right) A_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.2}$$

$$\begin{pmatrix} \alpha(0) + \lambda_0(0) & -\beta(0) \\ -\alpha(0) & \beta(0) + \lambda_1(0) \end{pmatrix}\begin{pmatrix} A_0 \\ A_1 \end{pmatrix} = \epsilon\begin{pmatrix} p(0,0) \\ p(0,1) \end{pmatrix}, \tag{2.3}$$

and

$$\int_0^\infty [p(w,0) + p(w,1)]\,\mathrm{d}w + A_0 + A_1 = 1.$$

Our goal is to solve (2.2) and (2.3) asymptotically, for $\epsilon \ll 1$. We first construct an asymptotic approximation to $p(w, i)$ for $w = O(1)$ and $w \gg \epsilon$, by employing a WKB technique. Next we construct an approximation to $p(w, i)$ which is valid for $w = O(\epsilon)$ and connect this approximation to the WKB approximation by the principle of asymptotic matching [14].

We now assume $w \gg \epsilon$. We drop the last term on the left side of (2.2) and extend the limit on the integral in equation (2.2) from w/ϵ to ∞. This leads to errors that are exponentially small for $w = O(1)$, and hence smaller than any power of ϵ. With these two simplifications we seek solutions of (2.2) for $w = O(1)$ in the WKB form

$$\begin{pmatrix} p(w,0) \\ p(w,1) \end{pmatrix} \sim \mathrm{e}^{-\frac{1}{\epsilon}\psi(w)} \left[\begin{pmatrix} K_0(w) \\ K_1(w) \end{pmatrix} + \epsilon\begin{pmatrix} K_0^{(1)}(w) \\ K_1^{(1)}(w) \end{pmatrix} + \epsilon^2\begin{pmatrix} K_0^{(2)}(w) \\ K_1^{(2)}(w) \end{pmatrix} + \ldots\right]. \tag{2.4}$$

Substituting (2.4) in (2.2), expanding the functions λ_i and b_i for $\epsilon \ll 1$, and equating the coefficient of each power of ϵ separately to zero, we obtain a recursive sequence of equations to determine the functions $K_i^{(j)}(w)$ and $\psi(w)$. The first of these equations is

$$\begin{pmatrix} \theta_0(w) & -\beta(w) \\ -\alpha(w) & \theta_1(w) \end{pmatrix} \begin{pmatrix} K_0(w) \\ K_1(w) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.5}$$

where

$$\begin{aligned} \theta_0(w) &= \psi'(w) + \alpha(w) + \lambda_0(w) - \lambda_0(w) \int_0^\infty b_0(z,w)\, e^{z\psi'(w)} dz \\ \theta_1(w) &= \psi'(w) + \beta(w) + \lambda_1(w) - \lambda_1(w) \int_0^\infty b_1(z,w)\, e^{z\psi'(w)} dz. \end{aligned} \tag{2.6}$$

In order for (2.5) to have a non-trivial solution, the matrix in (2.5) must be singular, which implies that

$$\theta_0(w)\ \theta_1(w) = \beta(w)\ \alpha(w). \tag{2.7}$$

Equations (2.6) and (2.7) determine the function $\psi'(w)$. Clearly $\psi' = 0$ is a solution of (2.7) but this solution is unacceptable by normalization considerations. A routine convexity argument establishes the existence of a unique non-zero solution of (2.7) which satisfies $\psi'(w) > 0$ if $r(w) < 1$, $\psi'(w) = 0$ if $r(w) = 1$, and $\psi'(w) < 0$ if $r(w) > 1$. Thus the solution ψ' passes through the origin as the traffic intensity goes through the critical value one. Since ψ' is uniquely determined, ψ is known up to an additive constant which must be chosen so that $\psi(w)$ vanishes at the point where the probability density peaks, in order to satisfy the normalization condition. Hence for cases 1 and 2 we choose $\psi(0) = 0$, while for case 3 we choose $\psi(w_0) = 0$ where w_0 satisfies $r(w_0) = 1$.

With (2.7) satisfied, (2.5) has the solution which we write as

$$K_0(w) = \frac{\beta(w)}{\beta(w) + \theta_0(w)} K(w), \qquad K_1(w) = \frac{\alpha(w)}{\alpha(w) + \theta_1(w)} K(w) \tag{2.8}$$

where $K(w)$ is as yet undetermined.

At the next order in ϵ in the expansion of (2.2), we obtain

$$\begin{pmatrix} \theta_0(w) & -\beta(w) \\ -\alpha(w) & \theta_1(w) \end{pmatrix} \begin{pmatrix} K_0^{(1)}(w) \\ K_1^{(1)}(w) \end{pmatrix} = \begin{pmatrix} [1-\lambda_0(w)\, I_0(w)]\, K_0'(w) \\ [1-\lambda_1(w)\, I_1(w)]\, K_1'(w) \end{pmatrix}$$

$$- \begin{pmatrix} (\tfrac{1}{2}\lambda_0(w)\, \psi''(w)\, J_0(w) + \int_0^\infty (\lambda_0 b_0)_w\, z\, e^{z\psi'(w)} dz)\, K_0(w) \\ (\tfrac{1}{2}\lambda_1(w)\, \psi''(w)\, J_1(w) + \int_0^\infty (\lambda_1 b_1)_w\, z\, e^{z\psi'(w)} dz)\, K_1(w) \end{pmatrix} \tag{2.9}$$

where for $j = 0, 1$

$$I_j(w) = \int_0^\infty z b_j(z,w)\, e^{z\psi'(w)} dz, \quad J_j(w) = \int_0^\infty z^2 b_j(z,w)\, e^{z\psi'(w)} dz, \tag{2.10}$$

with $(\lambda_i b_i)_w = \frac{\partial}{\partial w}(\lambda_i(w)\; b_i(z, w))$. The solvability condition for (2.9) is that the right side be orthogonal to the solution $(L_0(w), L_1(w))^T$ (T denotes transpose), of

$$\begin{pmatrix} \theta_0(w) & -\alpha(w) \\ -\beta(w) & \theta_1(w) \end{pmatrix} \begin{pmatrix} L_0(w) \\ L_1(w) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This yields

$$L_0(w) = \alpha(w)\; K_0(w), \qquad L_1(w) = \beta(w)\; K_1(w). \tag{2.11}$$

Taking the inner product of (2.9) with the 2-vector $(\alpha(w)\; K_0(w),\; \beta(w)\; K_1(w))$ and using (2.8) and (2.10), we obtain the following first order linear differential equation to determine the function $K(w)$:

$$\frac{K'(w)}{K(w)} = \frac{\xi_1(w) + \xi_2(w)}{\xi_3(w)}, \tag{2.12}$$

where

$$\xi_1(w) = -\tfrac{1}{2}\psi'' \left[\frac{\beta\lambda_0 J_0}{(\beta+\theta_0)^2} + \frac{\alpha\lambda_1 J_1}{(\alpha+\theta_1)^2}\right]$$

$$\xi_2(w) = -\frac{1}{\beta+\theta_0}\left(\frac{\beta(1-\lambda_0 I_0)}{\beta+\theta_0}\right)' - \frac{1}{\alpha+\theta_1}\left(\frac{\alpha(1-\lambda_1 I_1)}{\alpha+\theta_1}\right)'$$

$$\xi_3(w) = \frac{\beta(1-\lambda_0 I_0)}{(\beta+\theta_0)^2} + \frac{\alpha(1-\lambda_1 I_1)}{(\alpha+\theta_1)^2}.$$

We observe that the right side of (2.12) is known in terms of the function $\psi(w)$ and hence $K(w)$ is known up to a multiplicative constant. We have thus obtained the leading term in the asymptotic expansion of $p(w, i)$, valid for $w \gg \epsilon$, as

$$\begin{pmatrix} p(w,0) \\ p(w,1) \end{pmatrix} \sim C\frac{K(w)}{K(w^*)}\, \mathrm{e}^{-\frac{1}{\epsilon}\psi(w)} \begin{pmatrix} \dfrac{\beta(w)}{\beta(w)+\theta_0(w)} \\[2ex] \dfrac{\alpha(w)}{\alpha(w)+\theta_1(w)} \end{pmatrix} \tag{2.13}$$

with $\psi(w)$ satisfying (2.7) and $K(w)$ is the solution of (2.12). The point w^* is chosen so that $\psi(w^*) = 0$, thus $w^* = 0$ for cases 1 and 2 and $w^* = w_0$ for case 3. The constant C appearing in (2.13) is as yet undetermined. Note that the higher order corrections, $K_i^{(j)}(w)$ for $j \geqq 1$, can be obtained in a straightforward manner.

We now determine the constant C and construct the asymptotic expansion of $p(w, i)$ that is valid in the "inner" region, where $w = O(\epsilon)$. Thus we introduce the stretched variable

$$\eta = w/\epsilon \tag{2.14}$$

and the local functions

$$Q_i(\eta; \epsilon) = p(w, i; \epsilon). \tag{2.15}$$

Rewriting (2.2) in terms of η, we expand the functions $Q_i(\eta; \epsilon)$ as

$$\begin{pmatrix} Q_0(\eta;\epsilon) \\ Q_1(\eta;\epsilon) \end{pmatrix} \sim \begin{pmatrix} Q_0(\eta) \\ Q_1(\eta) \end{pmatrix} + \epsilon \begin{pmatrix} Q_0^{(1)}(\eta) \\ Q_1^{(1)}(\eta) \end{pmatrix} + \dots \tag{2.16}$$

and expand the coefficient functions in (2.2) as

$$\begin{aligned} \lambda_i(\epsilon\eta) &\sim \lambda_i(0) + \epsilon\eta\lambda_i'(0) + \dots \\ b_i(z, \epsilon\eta) &\sim b_i(z,0) + \epsilon\eta b_{i,w}(z,0) + \dots \\ \alpha(\epsilon\eta) &\sim \alpha(0) + \epsilon\eta\alpha'(0) + \dots \\ \beta(\epsilon\eta) &\sim \beta(0) + \epsilon\eta\beta'(0) + \dots . \end{aligned} \tag{2.17}$$

To leading order in ϵ, we obtain the following set of equations for $Q_i(\eta)$

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\begin{pmatrix} Q_0(\eta) \\ Q_1(\eta) \end{pmatrix} + \begin{pmatrix} -\alpha(0)-\lambda_0(0) & \beta(0) \\ \alpha(0) & -\beta(0)-\lambda_1(0) \end{pmatrix} \begin{pmatrix} Q_0(\eta) \\ Q_1(\eta) \end{pmatrix}$$

$$+ \int_0^\eta \begin{pmatrix} \lambda_0(0)\; Q_0(\eta - z)\; b_0(z,0) \\ \lambda_1(0)\; Q_1(\eta - z)\; b_1(z,0) \end{pmatrix} \mathrm{d}z$$

$$+ \frac{1}{\epsilon}\begin{pmatrix} \lambda_0(0)\; b_0(\eta,0)\; A_0 \\ \lambda_1(0)\; b_1(\eta,0)\; A_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.18}$$

$$\begin{pmatrix} \alpha(0)+\lambda_0(0) & -\beta(0) \\ -\alpha(0) & \beta(0)+\lambda_1(0) \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} = \epsilon \begin{pmatrix} Q_0(0) \\ Q_1(0) \end{pmatrix}.$$

In addition to satisfying (2.18), we demand that $Q_i(\eta)$ for $\eta \gg 1$, agree with or "match" the WKB expansion (2.13) for $w \ll 1$. We write this matching condition as

$$Q_i(\eta)\big|_{\eta \gg 1} \sim p(w,i)\big|_{w \ll 1}, \qquad \epsilon\eta = w. \tag{2.19}$$

Since the coefficient functions in (2.18) are frozen at $w = 0$, (2.18) is easily solved by introducing the Laplace transforms

$$\hat{b}_i(s,0) = \int_0^\infty \mathrm{e}^{-sz}\; b_i(z,0)\mathrm{d}z,$$

$$\hat{Q}_i(s) = \int_0^\infty \mathrm{e}^{-sz}\; Q_i(z)\mathrm{d}z \qquad (i = 0,1) \tag{2.20}$$

to obtain

$$\hat{Q}_0(s) = \frac{\Gamma_1(s,0)\left[Q_0(0) - \lambda_0(0)\frac{A_0}{\epsilon}\hat{b}_0(s,0)\right] - \beta(0)\left[Q_1(0) - \lambda_1(0)\frac{A_1}{\epsilon}\hat{b}_1(s,0)\right]}{D(s,0)}$$

(2.21)

$$\hat{Q}_1(s) = \frac{\Gamma_0(s,0)\left[Q_1(0) - \lambda_1(0)\frac{A_1}{\epsilon}\hat{b}_1(s,0)\right] - \alpha(0)\left[Q_0(0) - \lambda_0(0)\frac{A_0}{\epsilon}\hat{b}_0(s,0)\right]}{D(s,0)}$$

where

$$\Gamma_0(s,0) = s - \alpha(0) - \lambda_0(0) + \lambda_0(0)\,\hat{b}_0(s,0)$$

$$\Gamma_1(s,0) = s - \beta(0) - \lambda_1(0) + \lambda_1(0)\,\hat{b}_1(s,0)$$

$$D(s,0) = \Gamma_0(s,0)\,\Gamma_1(s,0) - \alpha(0)\,\beta(0).$$

It can be easily shown [24] that the transcendental equation $D(s, 0) = 0$ has a positive root $c = c(0)$ which satisfies $\alpha(0) + \beta(0) < c(0) < \alpha(0) + \beta(0) + \max\{\lambda_0(0), \lambda_1(0)\}$. This root corresponds to a simple pole of $\hat{Q}_i(s)$ unless the numerators in the expressions (2.21) also vanish when $s = c(0)$. The matching condition (2.19) precludes this exponential growth of the local functions $Q_i(\eta)$, so the apparent pole at $s = c(0)$ must be eliminated so that

$$\Delta_1(0)\left[Q_0(0) - \lambda_0(0)\frac{A_0}{\epsilon}\hat{b}_0(c(0),0)\right] = \beta(0)\left[Q_1(0) - \lambda_1(0)\frac{A_1}{\epsilon}\hat{b}_1(c(0),0)\right]$$

(2.22)

$$\Delta_0(0)\left[Q_1(0) - \lambda_1(0)\frac{A_1}{\epsilon}\hat{b}_1(c(0),0)\right] = \alpha(0)\left[Q_0(0) - \lambda_0(0)\frac{A_0}{\epsilon}\hat{b}_0(c(0),0)\right]$$

(2.23)

where $\Delta_i(0) = \Gamma_i(c(0), 0)$, $i = 0, 1$. Equations (2.22)–(2.23), together with (2.3) yield four equations that relate the four constants A_0, A_1, $Q_0(0)$ and $Q_1(0)$, but only three of these equations are independent so that any three of the constants may be expressed in terms of the fourth. Thus, the local solutions $\hat{Q}_i(s)$ are determined up to a single multiplicative constant. This constant, as well as the constant C appearing in (2.13) will be determined by imposing the matching condition (2.19) and the normalization requirement. At this point it is necessary to distinguish among the three cases 1–3.

Case 1. Here $r(w) < 1$ for all $w \geqq 0$. The stationary density will be peaked near $w = 0$ (where $w = O(\epsilon)$) so that it is appropriate, to leading order in ϵ, to impose the normalization condition on the local solutions $Q_i(\eta)$. This yields

$$A = A_0 + A_1 = 1 - r(0) \tag{2.24}$$

$$A_0 = \frac{\beta(0)\ (1 - r(0))}{\beta(0) + \Delta_1(0)}, \quad A_1 = \frac{\alpha(0)\ (1 - r(0))}{\alpha(0) + \Delta_0(0)} \tag{2.25}$$

$$Q_0(0) = A_0(\alpha(0) + \lambda_0(0) - \Delta_1(0))/\epsilon,$$

$$Q_1(0) = A_1(\beta(0) + \lambda_1(0) - \Delta_0(0))/\epsilon \tag{2.26}$$

and hence the local solutions $Q_i(\eta)$ are now totally determined. It remains to fix the constant C in the WKB expansion by using the asymptotic matching condition (2.19). For $\eta \gg 1$, the local solutions have the expansions

$$Q_0(\eta) \sim \frac{(1 - r(0))\ (s^*(0))\ [\Delta_0(0)\ \Gamma_1(s^*(0), 0) - \alpha(0)\ \beta(0)]\ e^{s^*(0)\eta}}{\epsilon[\alpha(0) + \Delta_0(0)]\ D_s(s^*(0), 0)}$$

$$\tag{2.27}$$

$$Q_1(\eta) \sim \frac{(1 - r(0))\ (s^*(0))\ [\Delta_1(0)\ \Gamma_0(s^*(0), 0) - \alpha(0)\ \beta(0)]\ e^{s^*(0)\eta}}{\epsilon[\beta(0) + \Delta_1(0)]\ D_s(s^*(0), 0)}$$

where

$$D_s(s^*(0), 0) = \frac{\mathrm{d}}{\mathrm{d}s} D(s, 0)\big|_{s = s^*(0)}$$

and $s^*(0)$ satisfies

$$\left(s - \alpha(0) - \lambda_0(0) + \lambda_0(0) \int_0^\infty e^{-sw}\, b_0(w,0)\mathrm{d}w\right) \cdot \left(s - \beta(0) - \lambda_1(0) + \lambda_1(0) \int_0^\infty e^{-sw}\, b_1(w,0)\mathrm{d}w\right) = \alpha(0)\,\beta(0) \tag{2.28}$$

with $s^*(0) = \max\{s \mid s < 0 \text{ and (2.28) is satisfied}\}$. The existence of a negative solution of (2.28) is an immediate consequence of the fact that $r(0) < 1$ for case 1. Note that $s^*(0)$ corresponds to simple poles of $\hat{Q}_i(s)$.

For $w \ll 1$, the WKB expansions look like

$$\begin{aligned} p(w,0) &\sim C\, e^{-\frac{1}{\epsilon}\psi'(0)\, w} \left(\frac{\beta(0)}{\beta(0) + \theta_0(0)}\right) \\ & \qquad\qquad w \ll 1 \\ p(w,1) &\sim C\, e^{-\frac{1}{\epsilon}\psi'(0)\, w} \left(\frac{\alpha(0)}{\alpha(0) + \theta_1(0)}\right). \end{aligned} \tag{2.29}$$

Noting that $\eta = w/\epsilon$ and $s^*(0) = -\psi'(0)$ from (2.28) and (2.7), the matching condition (2.19) fixes the constant C as

$$\begin{aligned} C &= \frac{(1 - r(0))\, s^*(0)}{\epsilon D_s(s^*(0), 0)}\, \frac{[\Delta_0(0)\, (\Gamma_1(s^*(0), 0) - \alpha(0)) - \alpha(0)\, (\beta(0) - \Gamma_0(s^*(0), 0))]}{\alpha(0) + \Delta_0(0)} \\ &= \frac{(1 - r(0))\, s^*(0)}{\epsilon D_s(s^*(0), 0)}\, \frac{[\Delta_1(0)\, (\Gamma_0(s^*(0), 0) - \beta(0)) - \beta(0)\, (\alpha(0) - \Gamma_1(s^*(0), 0))]}{\beta(0) + \Delta_1(0)} \end{aligned} \tag{2.30}$$

where the last equality follows from the identities

$$\Delta_0(0)\, \Delta_1(0) = \alpha(0)\, \beta(0) = \Gamma_0(s^*(0), 0)\, \Gamma_1(s^*(0), 0).$$

Thus, both the WKB expansion and the local solution are now completely determined.

Case 2. Here $r(0) = 1$ and $r(w) < 1$ for $w > 0$. The stationary density will now be concentrated in an $O(\sqrt{\epsilon})$ region about $w = 0$, rather than the thinner $O(\epsilon)$ region as in case 1. Since the WKB expansion is valid for all $w \gg \epsilon$ and in particular for $w = O(\sqrt{\epsilon})$, we may impose the normalization condition directly on the WKB expansion. As we shall show, $A_i = O(\sqrt{\epsilon})$ for $i = 0, 1$ so that the probabilities of finding the system empty may be, to leading order in ϵ, excluded from the normalization requirement. Using Laplace's method for the asymptotic evaluation of integrals, we determine the constant C in (2.13) as

$$C = \left(\frac{2\psi''(0)}{\pi\epsilon}\right)^{1/2} \tag{2.31}$$

where

$$\psi''(0) = \frac{-2[\alpha(0) + \beta(0)]\, r'(0)}{\alpha(0)\,\lambda_1(0)\, m_1^{(2)}(0) + \beta(0)\,\lambda_0(0)\, m_0^{(2)}(0) - 2\left(1 - \dfrac{\lambda_0(0)}{\mu_0(0)}\right)\left(1 - \dfrac{\lambda_1(0)}{\mu_1(0)}\right)}$$

$$m_i^{(2)}(0) = \int_0^\infty b_i(z, 0) z^2 \mathrm{d}z, \qquad i = 0, 1.$$

Note that $\psi''(0) > 0$ since $r'(0) < 0$. Thus, the WKB approximation to the stationary density is given by

$$\begin{pmatrix} p(w,0) \\ p(w,1) \end{pmatrix} \sim \left(\frac{2\psi''(0)}{\pi\epsilon}\right)^{1/2} \frac{K(w)}{K(0)}\, \mathrm{e}^{-\frac{1}{\epsilon}\psi(w)} \begin{pmatrix} \dfrac{\beta(w)}{\beta(w) + \theta_0(w)} \\[2ex] \dfrac{\alpha(w)}{\alpha(w) + \theta_1(w)} \end{pmatrix}, \qquad w \gg \epsilon. \tag{2.32}$$

We now determine the probability $A = A_0 + A_1$ of finding the system empty by matching the local solutions $Q_i(\eta)$ to the WKB expansion. First, we note that $\psi'(0) = 0$ implies $\theta_0(0) = \alpha(0)$, $\theta_1(0) = \beta(0)$ so that for $w \ll 1$ the WKB expansion reduces to

$$\begin{pmatrix} p(w,0) \\ p(w,1) \end{pmatrix} \sim \left(\frac{2\psi''(0)}{\pi\epsilon}\right)^{1/2} \frac{1}{\alpha(0) + \beta(0)} \begin{pmatrix} \beta(0) \\ \alpha(0) \end{pmatrix}. \tag{2.33}$$

Since $r(0) = 1$, $s = 0$ is a double root of $D(s, 0) = 0$ (i.e. $D(0, 0) = D_s(0, 0) = 0$) and hence the local solutions $Q_i(\eta)$ tend to a constant as $\eta \to \infty$. Using $D_{ss}(0, 0) = 2r'(0)[\alpha(0) + \beta(0)]/\psi''(0)$, we obtain

$$\lim_{\eta \to \infty} \begin{pmatrix} Q_0(\eta) \\ Q_1(\eta) \end{pmatrix} = \lim_{s \to 0} \begin{pmatrix} s\hat{Q}_0(s) \\ s\hat{Q}_1(s) \end{pmatrix} = \frac{-\psi''(0)\ (A_0 + A_1)}{\epsilon(\alpha(0) + \beta(0))\ r'(0)} \begin{pmatrix} \beta(0) \\ \alpha(0) \end{pmatrix}. \tag{2.34}$$

Matching (2.33) to (2.34) determines A as

$$A = A_0 + A_1 = \left(\frac{2\epsilon}{\pi\psi''(0)}\right)^{1/2} |r'(0)|. \tag{2.35}$$

Finally, using (2.35), (2.3), and (2.22)–(2.23) we obtain the constants A_i as

$$A_0 = \frac{\Delta_0(0)}{\alpha(0) + \Delta_0(0)} |r'(0)| \left(\frac{2\epsilon}{\pi\psi''(0)}\right)^{1/2},$$

$$A_1 = \frac{\Delta_1(0)}{\beta(0) + \Delta_1(0)} |r'(0)| \left(\frac{2\epsilon}{\pi\psi''(0)}\right)^{1/2}. \tag{2.36}$$

This completes the analysis of case 2.

Case 3. Here $r(w_0) = 1$ for $w_0 > 0$. The stationary density is now peaked in an $O(\sqrt{\epsilon})$ neighborhood of w_0. For this case the local solutions $Q_j(\eta)$ represent only a distortion in the exponentially small tail of the WKB solution. As in case 2, we may normalize the WKB solution to fix the constant C in (2.13), thus obtaining

$$\begin{pmatrix} p(w, 0) \\ p(w, 1) \end{pmatrix} \sim \left(\frac{\psi''(w_0)}{2\pi\epsilon}\right)^{1/2} \frac{K(w)}{K(w_0)} e^{-\frac{1}{\epsilon}\psi(w)} \begin{pmatrix} \dfrac{\beta(w)}{\beta(w) + \theta_0(w)} \\ \dfrac{\alpha(w)}{\alpha(w) + \theta_1(w)} \end{pmatrix} \tag{2.37}$$

for $w \gg \epsilon$. For small w, (2.37) looks like

$$\begin{pmatrix} p(w, 0) \\ p(w, 1) \end{pmatrix} \sim \begin{pmatrix} \dfrac{\beta(0)}{\beta(0) + \theta_0(0)} \\ \dfrac{\alpha(0)}{\alpha(0) + \theta_1(0)} \end{pmatrix} \left(\frac{\psi''(w_0)}{2\pi\epsilon}\right)^{1/2} \frac{K(0)}{K(w_0)} e^{-\frac{1}{\epsilon}\psi(0)} e^{-\psi'(0)w/\epsilon}. \tag{2.38}$$

The large η expansions of the local solutions Q_j are given by

$$Q_0(\eta) \sim \frac{e^{s^*(0)\eta}}{D_s(s^*(0),0)} \left[\Gamma_1(s^*(0),0)\left(Q_0(0) - \frac{\lambda_0(0)}{\epsilon} A_0 \hat{b}_0(s^*,0)\right)\right.$$

$$\left. - \beta(0)\left(Q_1(0) - \frac{\lambda_1(0)}{\epsilon} A_1 \hat{b}_1(s^*,0)\right)\right]$$

(2.39)

$$Q_1(\eta) \sim \frac{e^{s^*(0)\eta}}{D_s(s^*(0),0)} \left[\Gamma_0(s^*(0),0)\left(Q_1(0) - \frac{\lambda_1(0)}{\epsilon} A_1 \hat{b}_1(s^*,0)\right)\right.$$

$$\left. - \alpha(0)\left(Q_0(0) - \frac{\lambda_0(0)}{\epsilon} A_0 \hat{b}_0(s^*,0)\right)\right].$$

Matching (2.38) to (2.39), noting that $s^*(0) = -\psi'(0) > 0$ since $r(0) > 1$, and using (2.3) and (2.22) we obtain

$$A_0 = \frac{\Delta_0(0)\; e^{-\frac{1}{\epsilon}\psi(0)}}{[\alpha(0) + \theta_1(0)]\;[\Delta_0(0) + \theta_0(0)]} \left(\frac{\epsilon\psi''(w_0)}{2\pi}\right)^{1/2} \frac{K(0)}{K(w_0)} \frac{|D_s(s^*(0),0)|}{s^*(0)}$$

(2.40)

$$A_1 = \frac{\Delta_1(0)\; e^{-\frac{1}{\epsilon}\psi(0)}}{[\beta(0) + \theta_0(0)]\;[\Delta_1(0) + \theta_1(0)]} \left(\frac{\epsilon\psi''(w_0)}{2\pi}\right)^{1/2} \frac{K(0)}{K(w_0)} \frac{|D_s(s^*(0),0)|}{s^*(0)}.$$

(2.41)

Thus, the probability A of finding an empty system is exponentially small

$$\left(O\left(\sqrt{\epsilon}\; e^{-\frac{1}{\epsilon}\psi(0)}\right)\right)$$

for $\epsilon \ll 1$. Note that for case 3, $\psi(w_0) = 0$ and $\psi(0) > 0$.

We have thus completed the computation of the stationary density. The results can be summarized as follows.

RESULT

Asymptotic expansions for the stationary density

$$\begin{pmatrix} p(w,0) \\ p(w,1) \end{pmatrix} \sim \begin{cases} C\,\dfrac{K(w)}{K(w^*)}\,\mathrm{e}^{-\psi(w)/\epsilon}\begin{pmatrix} \dfrac{\beta(w)}{\beta(w)+\theta_0(w)} \\[2ex] \dfrac{\alpha(w)}{\alpha(w)+\theta_1(w)} \end{pmatrix} + \ldots, & w \gg \epsilon \\[6ex] \begin{pmatrix} Q_0(w/\epsilon) \\ Q_1(w/\epsilon) \end{pmatrix} + \ldots, & w = O(\epsilon) \end{cases}$$

where $\psi'(w)$ is the unique solution of $\theta_0(w)\,\theta_1(w) = \alpha(w)\,\beta(w)$ that satisfies $\psi'(w) <$ or > 0 when $r(w) <$ or > 1 and $\psi(w^*) = 0$. The Laplace transforms of $Q_i(\eta)$ are given by (2.21) in terms of A_i and $Q_i(0)$. $Q_i(0)$ are given in terms of A_i in (2.22)–(2.23) and the A_i are given below. Here

$$\theta_0(w) = \psi'(w) + \alpha(w) + \lambda_0(w) - \lambda_0(w)\int_0^\infty b_0(z,w)\,\mathrm{e}^{z\psi'(w)}\mathrm{d}z$$

$$\theta_1(w) = \psi'(w) + \beta(w) + \lambda_1(w) - \lambda_1(w)\int_0^\infty b_1(z,w)\,\mathrm{e}^{z\psi'(w)}\mathrm{d}z$$

$$K(w) = K(w^*)\,\exp\left\{\int_{w^*}^{w} [(\xi_1(z) + \xi_2(z))/\xi_3(z)]\,\mathrm{d}z\right\}$$

with ξ_i defined in (2.12). The constants A_i, w^*, and C are given by

Case 1: $r(w) < 1$, $w \geqq 1$. C is given by (2.30), $w^* = 0$, and

$$A_0 = \frac{\beta(0)\,(1 - r(0))}{\beta(0) + \Delta_1(0)}, \qquad A_1 = \frac{\alpha(0)\,(1 - r(0))}{\alpha(0) + \Delta_0(0)}.$$

Case 2: $r(w) < 1,\ w > 0;\ r(0) = 1,\ r'(0) < 0$. Here $w^* = 0$,

$$C = [2\psi''(0)/\pi\epsilon]^{1/2}$$

$$A_0 = \frac{\Delta_0(0)}{\alpha(0) + \Delta_0(0)} |r'(0)| \left[\frac{2\epsilon}{\pi\psi''(0)}\right]^{1/2}$$

$$A_1 = \frac{\Delta_1(0)}{\beta(0) + \Delta_1(0)} |r'(0)| \left[\frac{2\epsilon}{\pi\psi''(0)}\right]^{1/2} .$$

Case 3: $r(w_0) = 1$. Here $w^* = w_0$, A_j is given by (2.40),

$$C = [\psi''(w_0)/2\pi\epsilon]^{1/2}.$$

The $\Delta_i(0)$ are given (for all three cases) by

$$\Delta_0(0) = c(0) - \alpha(0) - \lambda_0(0) + \lambda_0(0)\ \hat{b}_0(c(0), 0)$$

$$\Delta_1(0) = c(0) - \beta(0) - \lambda_1(0) + \lambda_1(0)\ \hat{b}_1(c(0), 0)$$

where $c(0)$ is the largest positive root of the transcendental equation $D(s, 0) = 0$ (see (2.21)). Note that $D(s, 0) = 0$ has a unique positive solution for cases 1 and 2 and has two positive solutions for case 3.

3. Remarks

Our analysis indicates the wide range of behaviors that are possible for the state-dependent model depending on the nature of the traffic intensity function $r(w)$. For case 1 we found that $A = 1 - r(0) + O(\epsilon)$ which is similar to the result for state-independent systems. This is reasonable since if the traffic intensity $r(w) < 1$ for all $w \geqq 0$, and the mean jumps in unfinished work are small, the total work backlog in the system remains an $O(\epsilon)$ quantity for most times. Higher order corrections to A, i.e. $O(\epsilon^M)$ terms with $M = 1, 2, \ldots$, could be obtained by systematically computing the higher order terms in the WKB expansion and the expansions of the local solutions (2.16), and requiring the matching condition (2.19) to hold to all orders in ϵ. For example, the $O(\epsilon)$ terms in the expansion of A would involve derivatives of the functions r, α, β, and λ_i, evaluated at $w = 0$.

In case 2, $A = O(\sqrt{\epsilon})$. Again the density is peaked near $w = 0$, but since $r(0) = 1$, the stationary density is concentrated in an $O(\sqrt{\epsilon})$ region about $w = 0$, rather than the thinner $O(\epsilon)$ region for case 1. If we had not assumed that $r'(0) < 0$, but rather that $r'(0) = r''(0) = \ldots = r^{(M-1)}(0) = 0$ and $r^{(M)}(0) < 0$ for some M, a similar analysis would show that $A = O(\epsilon^{M/(M+1)})$. In case 3, A is exponentially small for $\epsilon \ll 1$ which is also intuitively reasonable since in this case the stationary density is sharply peaked about w_0 and the system only very rarely becomes empty.

Finally, we observe that when α, β, λ_i and b_i are all independent of w, the WKB expansion (2.4), as well as the local expansion (2.16) truncate after one term since $\psi'' = 0$ and the correction terms involve derivatives with respect to w of α, β, λ_i and b_i, which are now zero. When all the functions are independent of w, we must assume that the (constant) traffic intensity $r < 1$. Thus, the state-independent model is contained in case 1. Furthermore, the leading term in the local expansion (2.16) is in fact an exact solution to the problem, while the WKB expansion gives the exponential tail of the stationary distribution. The exact state-independent solution is discussed in more detail in [24].

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