Queueing Systems, 5 (1989) 113–130

QUEUES WITH NONSTATIONARY INPUTS

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Received 20 June 1988; revised 17 February 1989

Abstract

In this paper we study single server queues with independent and identically distributed service times and a general nonstationary input stream. We discuss several notions of "being in equilibrium". For queues with a doubly stochastic Poisson input we survey continuity and bounds of moments of some performance characteristics. We also discuss conjectures posed by Ross [34] to the effect that for a "more stationary" input we have a better performance characteristics. Some results are reviewed to typify a problem and then it is followed by a discussion, questions and related bibliography.

Keywords: $M_t/G/1$, DSP/G/1, G/G/1, stationary waiting time, stationary work-load, Ross' conjectures, equilibrium, a.m.s. sequence, tightness, continuity, uniform integrability.

1. Introduction

This study arose from considerations about a problem which can be formulated roughly as follows. Suppose we consider a class of queueing systems, say for example single server queues, with different inputs having the same asymptotic arrival rate, the service process being the same for all systems. For a given performance characteristic we ask which input minimizes this characteristic or, more generally, we want to find some monotonicity properties of performance characteristics regarding inputs. In this paper we consider doubly stochastic Poisson inputs with generic random measures satisfying with probability 1, $\Lambda([0, t])/t \to \bar{\lambda}$ for $t \to \infty$, where $\bar{\lambda}$ is a given arrival rate. To answer such questions we must first decide what we understand by the stationary performance characteristic. Note that in the classical theory of queues, where the input is a renewal process, it is quite obvious what we mean by stationary performance characteristics and the possible approaches are suggested by the theory of Markov chains, renewal theory and the theory of regenerative processes.

In this paper we review a few conjectures concerning the problem regarding which input is better, and some technical problems useful in such considerations. Thus we discuss several notions of stability and among others conditions for tightness of some performance processes. We also discuss some continuity results for moments and conditions for finiteness of moments of performance characteristics.

All random variables are defined on a common probability space $(\Omega, \mathscr{F}, P)$. We shall deal mostly with single server queues operating with the first come, first served discipline and unlimited capacity. The service time of the ith customer is S_i and unless otherwise stated, the sequence $\boldsymbol{S} = \{S_i\}$ consists of independent and identically distributed (i.i.d.) random variables (r.v.) independent of the input. In this case we assume $E[S] < \infty$ and let $\mu = (E[S])^{-1}$. The input process of the system is represented by the point process N on $\mathbb{R}_+$ defined by

$$N(s, t] = \sum_{i=1}^{\infty} 1(s < \tau_i \leqslant t) \; (s < t), \tag{1.1}$$

where $0 < \tau_1 < \tau_2 \ldots$ are the arrival epochs and $1(A)$ is the indicator function of the event A. The $N(s, t]$ is the number of arrivals in the interval $(s, t]$. The inter-arrival times are

$$T_1 = \tau_1 \text{ and } T_i = \tau_i - \tau_{i-1} \; (i = 2, 3, \ldots). \tag{1.2}$$

We also write $N(t) = N(0, t]$. In the special case of interest N is a homogeneous or non-homogeneous Poisson process Π_Λ with intensity function $\Lambda(\mathrm{d}t) = \lambda(t)\,\mathrm{d}t$. The intensity Λ can be also random, and then Π_Λ is said to be a doubly stochastic Poisson (DSP) process directed by the random measure Λ (for details see Kallenberg [20], Serfozo [36]). If $\lambda(t) \equiv \bar{\lambda}$ then Π_Λ is stationary and is denoted by $\Pi_{\bar{\lambda}}$. In the special case $\lambda(t) \equiv 1$, we write Π to denote the standard stationary Poisson process.

We are interested in the following performance processes: the waiting time sequence $\{W_i, i = 1, 2, \ldots\}$ given recursively by

$$W_{n+1} = (W_n + S_n - T_{n+1})_+ \; (n = 1, 2, \ldots) \tag{1.3}$$

with an arbitrary W_1, the work-load process $\{V(t), t \geqslant t\}$ defined by

$$V(t) = \begin{cases} (V(0) - t)_+, & 0 \leqslant t < \tau_1, \\ \left(W_{N(t)} + S_{N(t)} - \left(t - \tau_{N(t)}\right)\right)_+, & t \geqslant \tau_1, \end{cases} \tag{1.4}$$

and the number in system process $\{L(t), t \geqslant 0\}$ defined by

$$L(t) = \sum_i 1(\tau_i \leqslant t < \tau_i + W_i + S_i). \tag{1.5}$$

We sometimes refer to these processes defined on the entire real line $\mathbb{R}$, in which case the arrival times are subscripted such that $\ldots < \tau_0 \leqslant 0 < \tau_1 < \ldots$.

2. Long run behaviour and equilibrium

The notion of steady state or equilibrium state has several interpretations for non time-homogeneous Markov or non-stationary processes. We survey here some results in this regard that might lead to the understanding of more general cases.

(a) $M_t/G/1$ *queue*. Suppose that the arrival process N is non-homogeneous Poisson with intensity function (arrival rate function) $\lambda(t)$. A weaker requirement for a performance process is whether the process is "kept in some range" for all $t \geqslant 0$. A formal way of describing this is in terms of tightness. A real valued process $\{X(t),\ t \geqslant 0\}$ is tight if for each positive ϵ we can find reals a and b for which $P\{a \leqslant X(t) \leqslant b\} \geqslant 1 - \epsilon (t > 0)$. Tightness for discrete processes is similar.

PROPOSITION 2.1

If there are constants $\bar{\lambda} > 0$ and $d \geqslant 0$ such that $\bar{\lambda} E(S) < 1$ and

$$\int_s^t \lambda(v)\ \mathrm{d}v \leqslant \bar{\lambda}(t-s) + d, \quad s \leqslant t, \tag{2.1}$$

then $\{V(t),\ t \geqslant 0\}$ and $\{W_n,\ n = 1, 2, \dots\}$ are tight.

Proof

For simplicity we assume $W_1 = 0$ and $V(0) = 0$. We know that if

$$X_0 = 0, \quad X_n = S_1 + \dots + S_n - T_2 - \dots - T_{n+1} \quad (n = 1, 2, \dots)$$

then

$$W_{n+1} = \max\{X_n - X_i,\ i = 0, \dots, n\}$$

and if

$$X(t) = \sum_{i=1}^{N(t)} S_i - t$$

then

$$V(t) = \max\{X(t) - X(s),\ 0 \leqslant s \leqslant t\};$$

see e.g. Borovkov [6]. We can represent the input process by $N(t) = \Pi(\Lambda(t))$, where $\Pi(t) = \Pi[0, t]$, $\Lambda(t) = \int_0^t \lambda(s)\ \mathrm{d}s$. Let $\{M_i\}$ consists of independent random variables with the common standard exponential distribution. Define a sequence of standard Poisson processes indexed by $n = 1, 2, \dots$

$$\overleftarrow{\Pi}_n(t) = \sum_{-\infty < i \leqslant n} 1(M_i + \dots + M_n \leqslant t) \quad (t \geqslant 0)$$

where the consecutive inter-point distances are $M_n, M_{n-1}, \dots$. Set

$$\overleftarrow{\lambda}_n(t) = \lambda(T_1 + \dots + T_n - t) \text{ and } \overleftarrow{\Lambda}_n(t) = \int_0^t \overleftarrow{\lambda}_n(v)\ \mathrm{d}v.$$

Define a sequence of counting processes indexed by $n = 1, 2, \dots$

$$\overleftarrow{\Pi}_n \circ \overleftarrow{\Lambda}_n(t) = \sum_{i=1}^{n} 1(T_i + \dots + T_n \leqslant t), \ (0 \leqslant t < T_1 + \dots + T_n)$$

($n = 1, 2, \ldots$) which counts points of N beginning at $T_1 + \ldots + T_n$ leftwards. Thus

$$\Pi \circ \Lambda(T_1 + \ldots + T_n) - \Pi \circ \Lambda(T_1 + \ldots + T_n - t)) = \overleftarrow{\Pi}_n \circ \overleftarrow{\Lambda}_n(t),$$
$$(0 \leqslant t < T_1 + \ldots + T_n).$$

By (2.1) we have

$$\overleftarrow{\Pi}_n \circ \overleftarrow{\Lambda}_n(t) \leqslant \overleftarrow{\Pi}_n(\bar{\lambda} t + d), \ (t \geqslant 0). \tag{2.2}$$

Let $\{T_i^n, \ i = 1, 2, \ldots\}$ be a sequence defined by

$$T_i^n = 0, \quad \left(i = 1, \ldots, \overleftarrow{\Pi}_n(d)\right)$$

and

$$T_j^n + \overleftarrow{\Pi}_n(d) = M_j', \ (j = 1, 2, \ldots),$$

where $\{M_j', \ j = 1, \ldots\}$ is a sequence independent of $\overleftarrow{\Pi}_n(d)$ and $\{S_j\}$, consisting of independent and identically distributed random variables, each exponentially distributed with parameter $\bar{\lambda}$. We have for each $n = 1, 2, \ldots$

$$\overleftarrow{\Pi}_n(\bar{\lambda} t + d) = \sum_{i=1}^{\infty} 1(T_1^n + \ldots + T_i^n \leqslant t)$$

and hence by (2.2)

$$(T_{n+1}, \ldots, T_2 + \ldots + T_{n+1}) \geqslant_d (T_1^n, \ldots, T_1^n + \ldots + T_n^n).$$

This yields

$$W_n \leqslant_d \max(0, \ S_1 - T_1^n, \ldots, S_1 + \ldots + S_n - T_1^n - \ldots - T_n^n). \tag{2.3}$$

Now let $\{S_i'\} \overset{\mathrm{d}}{=} \{S_i\}$, $\overleftarrow{\Pi}_n(d) \overset{\mathrm{d}}{=} \Pi'(d)$ and $\{S_i\}$, $\{S_i'\}$, $\overleftarrow{\Pi}_n(d)$, $\{M_i'\}$ are independent and then the right hand side of (2.3) can be stochastically bounded above by

$$S_1 + \ldots + S_{\overleftarrow{\Pi}_n(d)} + \max(0, \ S_1' - M_1', \ldots, S_1' + \ldots + S_n' - M_1' - \ldots - M_n')$$

which does converge in distribution. This proves that $\{W_n\}$ is tight.

To prove that $\{V(t)\}$ is tight we write

$$V(t) = \max\left\{\sum_{i=1}^{N(s,t]} S_i - (t - s), \quad 0 \leqslant s \leqslant t\right\}.$$

By (2.1)

$$V(t) \leqslant_d \max\left\{\sum_{i=1}^{\Pi(\bar{\lambda}(t-s)+d)} S_i - (t - s), \quad 0 \leqslant s \leqslant t\right\}$$
$$\leqslant_d \sum_{i=1}^{\Pi'(d)} S_i' + \max\left\{\sum_{i=1}^{\Pi(\bar{\lambda} s)} S_i - s, \quad 0 \leqslant s \leqslant t\right\}.$$

Since the last expression is convergent in distribution we obtain that $\{V(t)\}$ is tight. □

REMARKS

(i) Condition (2.1) is satisfied when

$$\sup\left\{\int_0^t \lambda(v)\,\mathrm{d}v - \bar{\lambda}t,\ t>0\right\} = d/2. \tag{2.4}$$

Indeed

$$\sup\left\{\int_s^t \lambda(v)\,\mathrm{d}v - \bar{\lambda}(t-s),\ 0 \leqslant s < t\right\}$$

$$= \sup\left\{\left(\int_0^t \lambda(v)\,\mathrm{d}v - \bar{\lambda}t\right) - \left(\int_0^s \lambda(v)\,\mathrm{d}v - \bar{\lambda}s\right),\ s<t\right\} \leqslant d.$$

(ii) If λ is a periodic function or the sum of a finite number of periodic functions (with possibly different periods), then its mean

$$\bar{\lambda} = \lim_{t\to\infty} t^{-1} \int_0^t \lambda(v)\,\mathrm{d}v$$

exists and is finite. In this case (2.1) is satisfied.

(iii) Finite sums of continuous periodic functions are Bohr's almost periodic functions. Condition (2.1) can also be satisfied by other Bohr's almost periodic functions. More details and references can be found in Rolski [32].

There are few results in literature closely related to proposition 2.1. For example Heyman and Whitt [18] proved tightness of the queue length process $\{L(t),\ t \geqslant 0\}$ in $M_t/M/k$ queues under the following condition: For some t_0, ϵ and $T>0$

$$\int_{t_0+nT}^{t_0+(n+1)T} \lambda(s)\,\mathrm{d}s \leqslant (k\mu-\epsilon)T, \quad (n=1,2,\ldots).$$

Thorisson [41] considered more general $M_t/G/1$ queues with

$$\liminf_{t\to\infty} \sup_s \int_s^{s+t} \lambda(v)\,\mathrm{d}v < \mu \tag{2.5}$$

and, in particular mentioned that for the work-load $V_s(t)$ at time t when the system starts at $s<t$,

$$V^*(t) = \lim_{s\to-\infty} V_s(t), \text{ with probability } 1.$$

Condition similar to (2.1) or (2.5) appear also in Masey [23].

For the work-load process we have

$$V(t) = \max\left\{\sum_{i=1}^{N(s,\ t]} S_i - (t-s),\quad 0 \leqslant s \leqslant t\right\}$$

(see e.g. Borovkov [6]). Applying Jensen's inequality we get

$$E[V(T)] \geqslant \max\left\{E(S)\int_0^t \lambda(s)\,\mathrm{d}s-(t-s),\ 0\leqslant s\leqslant t\right\}$$
$$=v(t)\ \text{(say)}. \tag{2.6}$$

For a formal proof we use lemma 1 from Rolski [30] and Wald's identity. Here $v(t)$ is the solution of the following deterministic fluid flow problem in which $v(t)$ is the content of an initially empty container into which a fluid flows in a rate $\lambda(t)E(S)$ and flows out (if $v(t)>0$) at rate 1. Heyman and Whitt [18] showed that even for a bounded intensity function $\lambda(t)$ on R_+ satisfying

$$\lim_{t\to\infty} t^{-1}\int_0^t \lambda(s)\,\mathrm{d}s=\bar{\lambda} \quad\text{and}\quad \bar{\lambda}E(S)<1, \tag{2.7}$$

it is possible that

$$\lim_{t\to\infty} t^{-1}\int_0^t v(s)\,\mathrm{d}s=\infty.$$

In this case (2.6) yields

$$\lim_{t\to\infty} t^{-1}\int_0^t E[V(s)]\,\mathrm{d}s=\infty. \tag{2.8}$$

This raises the question whether it is also true that with probability 1

$$\lim_{t\to\infty} t^{-1}\int_0^t V(s)\,\mathrm{d}s=\infty. \tag{2.9}$$

On the other hand we can ask, what conditions we have to impose on $\lambda(t)$ and the generic service time S, besides (2.7), to ensure that with probability 1

$$\limsup_{t\to\infty} t^{-1}\int_0^t V(s)\,\mathrm{d}s<\infty.$$

Clearly from the proof of proposition 2.1 it suffices to assume (2.1) (recall that $\bar{\lambda}E(S)<1$) and $E(S^2)<\infty$. It is not clear whether these conditions can be weakened.

A second question that arises is whether in $M_t/G/k$ queues the tightness of one of the processes $\{V(t)\}$, $\{W_n(t)\}$, $\{L(t)\}$ implies this property for the other two. This question is also of interest for general $G/G/1$ queues without our usual assumptions on the service sequence $\{S_i\}$; here we need some additional assumptions to exclude trivial cases such as when $S_i=T_i=i$ $(i=1, 2,\dots)$.

Frequently an asymptotic property for the input data is preserved for a performance process; see Rolski [27], Rolski and Szekli [28], Szczotka [39]. Thus we can ask whether the tightness of $\{\theta_t(\Sigma_i S_i 1_{\tau_i}),\ t\geqslant 0\}$ implies the tightness of $\{V(t)\}$, $\{L(t)\}$ and $\{W_n\}$. We need tightness of $\{W_n\}$ to prove some stability results (see Szczotka [39] and the proof of our proposition 2.2).

(b) *Periodic and almost periodic Poisson queues*. Suppose that $\lambda(t)$ is a periodic or almost periodic function. An almost periodic function on $\mathbb{R}_+$ is the uniform closure of all continuous periodic functions and their linear combinations. It is known that the mean of such a function λ

$$\bar{\lambda} = \lim_{t\to\infty} t^{-1} \int_0^t \lambda(s)\,\mathrm{d}s$$

exists (see e.g. Corduneau [9]).

PROPOSITION 2.2

Suppose $\lambda(t)$ is an almost periodic function satisfying the hypothesis of proposition 2.1. Then there exists a stationary and ergodic sequence $\{T_i^0\}$ independent of $\{S_i\}$ such that for $j \to \infty$

$$(T_j, T_{j+1},\dots) \xrightarrow{d} (T_1^0, T_2^0,\dots) \tag{2.10}$$

$$(W_j, W_{j+1},\dots) \xrightarrow{d} (W_1^0, W_2^0,\dots), \tag{2.11}$$

where

$$W_i^0 = \max(S_{i-1} - T_i^0,\ S_{i-1} + S_{i-2} - T_i^0 - T_{i-1}^0,\dots)$$

and $\xrightarrow{d}$ denote the convergence in distribution. If $\lambda(t)$ is a periodic function then $(T_i, T_{i+1},\dots)$ and $(W_i, W_{i+1},\dots)$ converge in total variation to $(T_1^0, T_2^0,$ and $(W_1^0, W_2^0,\dots)$ respectively.

Proof

(2.10) was proved in Rolski [32]. By proposition 2.1 the sequence $\{W_i\}$ is tight and so the assumptions of theorem 1 of Szczotka [39] are satisfied and we can deduce (2.11). The total variation convergence of $(T_i, T_{i+1},\dots)$ was also proved in Rolski [32] from which we obtain immediately the same convergence for $(W_i, W_{i+1}\dots)$. □

In Rolski [32] for the proof of (2.10) we required a weaker condition than (2.1). The question arises whether in this case we also have (2.10).

The theory outlined in Rolski [32] provides a representation of $\{T_i^0\}$. In the periodic case, $\{T_i^0\}$ is the inter-arrival sequence in a DSP process with the random intensity function $\lambda^0(t) = \lambda(\theta^0 + t)$, where θ^0 has density function $\bar{\lambda}^{-1}\lambda(t)$ on $[0,\ p = \text{period of } \lambda]$. If θ^* is uniformly distributed over $[0, p]$ then $\lambda^*(t) = \lambda(\theta^* + t)$ is a random intensity function of a stationary DSP process, whose Palm version has inter-arrival times $\{T_i^0\}$. To see the structure of $\{T_i^0\}$ when λ is an almost periodic function, consider a simple case in which $\lambda = \lambda_1 + \lambda_2$, where λ_1 and λ_2 are periodic with periods p_1 and p_2 respectively and p_1, p_2 are incommensurable. We can decompose the Poisson process with intensity function $\lambda_1 + \lambda_2$ as the superposition of two independent Poisson processes with intensities λ_1, λ_2 respectively. If θ_1^* and θ_2^* are independent and uniformly distributed over $[0, p_1)$ and $[0, p_2)$ then $\lambda_1(\theta^* + t) + \lambda_2(\theta^* + t)$ is the intensity function of a stationary DSP process. Its Palm version is a DSP process with additional point at zero and having intensity function $\lambda_1(\theta_1^0+t)+\lambda_2(\theta_2^0+t)$, where (θ_1^0, θ_2^0) has the density function $(\lambda_1(t_1)+\lambda_2(t_2))/(\bar{\lambda}_1+\bar{\lambda}_2)$ over $[0, p_1) \times [0, p_2)$. This latter fact can be also proved using probabilistic arguments applying theorem 1.3.11 from Franken et al. [12] and theorem 2.4.2 from Grandell [13]. For an arbitrary almost periodic function $\lambda(t)$, the Bohr compactification of $\mathbb{R}$ with respect λ is used to define $\{T_i^0\}$; for details see Rolski [32]. The other papers related to the discussion in (b) are: Grillingeber [15], Böhme [5], Asmussen and Thorisson [1], Berbee [4], Karandikar and Kulkarni [19], Harrison and Lemoine [16], Thorisson [42], Lemoine [21].

(c) *General case.* Let $\boldsymbol{X}=\{X_i\}$ be a sequence of random variables that takes values in the Polish space $\mathbb{E}$. For $\boldsymbol{x}=(x_1, x_2,\dots)\in\mathbb{E}^\infty$ define the family of shifts $\{\sigma^n\}$ by $\sigma^n\boldsymbol{x}=(x_{n+1}, x_{n+2},\dots)$. We now introduce a class of asymptotically mean stationary (a.m.s.) sequences. It is said that a sequence $\boldsymbol{X}$ is a.m.s. if there exists a stationary sequence $\boldsymbol{X}^0=\{X_i^0\}$ such that for each B Borel subset of $\mathbb{E}^\infty$

$$\lim_{n\to\infty} n^{-1}\sum_{j=1}^{n} P\{\sigma^j\boldsymbol{X}\in B\}=P\{\boldsymbol{X}^0\in B\}.$$

This notion was independently introduced by Rolski [26], [27] and Gray and Kieffer [14]. It is clear that $\boldsymbol{X}^0$ must be stationary. If moreover $\boldsymbol{X}^0$ is ergodic then the individual ergodic theorem is satisfied: that is, for each measurable function f: $\mathbb{E}^\infty\to R_+$ with probability 1

$$\lim_{n\to\infty} n^{-1}\sum_{j=1}^{n} f(\sigma^j\boldsymbol{X})=E\big(f(\boldsymbol{X}^0)\big).$$

If $\boldsymbol{X}$ is an a.m.s. sequence then we write $\sigma^n\boldsymbol{X}\xrightarrow{\text{a.m.s.}}\boldsymbol{X}^0$ and we say that $\boldsymbol{X}^0$ is a stationary representation of $\boldsymbol{X}$. In this paper we tacitly assume that stationary representation $\boldsymbol{X}^0$ is always ergodic. The usefulness of a.m.s. notion can be illustrated by the following proposition. Note that the input data and the performance process are of the same type.

PROPOSITION 2.3 (ROLSKI [27])

If $\boldsymbol{T}$, $\boldsymbol{S}$ are general inter-arrival and service sequences respectively, such that for $n\to\infty$

$$\sigma^n(\boldsymbol{T}, \boldsymbol{S})\xrightarrow{\text{a.m.s.}}(\boldsymbol{T}^0, \boldsymbol{S}^0) \tag{2.12}$$

and $\rho=E(S_i^0)/E(T_i^0)<1$ then

$$\sigma^n(\boldsymbol{W}, \boldsymbol{T}, \boldsymbol{S})\xrightarrow{\text{a.m.s.}}(\boldsymbol{W}^0, \boldsymbol{T}^0, \boldsymbol{S}^0) \tag{2.13}$$

where

$$W_i^0=\max\big(0,\ S_{i-1}^0-T_i^0,\ S_{i-1}^0-T_i^0+S_{i-2}^0-T_{i-1}^0,\dots\big).$$

Note that there we do not assume anything on the independence of $\boldsymbol{S}$. Szczotka [39] proves similar results for other types of convergence including weak convergence and total variation convergence.

REMARKS

(i) Note that for an a.m.s. sequence $\boldsymbol{T}$ we have

$$\lim_{n\to\infty} n^{-1} \sum_{j=1}^{n} T_j = E\left(T_1^0\right) = \bar{\lambda}^{-1}. \tag{2.14}$$

The constant $\bar{\lambda}$ is called the asymptotic arrival rate.

(ii) Assumption (2.12) holds if

$$\sigma^n \boldsymbol{T} \xrightarrow{\text{a.m.s.}} \boldsymbol{T}^0,$$

where $\boldsymbol{T}^0$ is a stationary and ergodic sequence, $\boldsymbol{T}^0$ is independent of $\boldsymbol{S}$, and $\boldsymbol{S}$ consists of i.i.d. r.v.s. If $\boldsymbol{T}$ is the inter-arrival time sequence in a stationary and ergodic point process, then $\boldsymbol{T}$ is a.m.s. (see Rolski [27], [29]). An example of a stationary ergodic point process is a DSP process with stationary and ergodic intensity process $\{\lambda^*(t)\}$. It is not known whether in this case we have for $n \to \infty$, $W_n \xrightarrow{d} W^0$, where $\xrightarrow{d}$ denote the convergence in distribution.

(iii) The following inter-arrival times are a.m.s. and have the same stationary representations: $\{T_i\}$ is the periodic Poisson process with arrival rate $\lambda(t)$ and $\{T_i^*\}$ is the DSP process with arrival rate function $\lambda^*(t) = \lambda(\theta + t)$, where θ is a uniformly distributed over $[0, p]$, where p is the period of λ. For both these sequences the stationary representation $\{T_i^0\}$ is a DSP process with arrival rate function $\lambda^0(t) = \lambda(\theta^0 + t)$, where θ^0 has the density function $\bar{\lambda}^{-1}\lambda(t)$ over $[0, p]$.

(iv) Rolski and Szekli [28] proved that under the hypotheses of proposition 2.3, the inter-departure times form an a.m.s. sequence.

Proposition 2.3 justifies the use of known identities valid for queues with stationary and ergodic input data, also for queues satisfying the hypotheses of proposition 2.2. For example the mean stationary waiting time equals with probability 1 to

$$E(W^0) = \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} W_i,$$

and the mean sojourn time is given by

$$E(W^0 + S^0) = \lim_{n\to\infty} n^{-1} \sum_{j=1}^{n} (W_i + S_i).$$

If $E(W^0) < \infty$ then we have with probability 1

$$\lim_{n\to\infty} W_n/n = \lim_{n\to\infty} W_n^0/n = 0.$$

Hence from Stidham [37] we obtain that there exists with probability 1

$$\bar{L} = \lim_{t \to \infty} t^{-1} \int_0^t L(s)\, \mathrm{d}s$$

and the Little formula $\bar{L} = \bar{\lambda} E(W^0 + S^0)$ holds. Similarly we can prove the following version of Brumelle identity

$$\lim_{t \to \infty} t^{-1} \int_0^t V(s)\, \mathrm{d}s = \bar{\lambda}\left(E(S)E(W^0) + \tfrac{1}{2}E(S^2)\right), \tag{2.15}$$

which is satisfied for $\boldsymbol{T}$ and $\boldsymbol{S}$ independent, $\boldsymbol{T}$ being a.m.s. and $\boldsymbol{S}$ consisting of independent and identically distributed random variables. For further discussion of the points raised here, see Rolski [27], [29], Szczotka [39], Miyazawa [24], Rolski and Szekli [28], Grey and Kieffer [14], Borovkov [6], [7], Loynes [22], Asmussen and Thorisson [1], Stidham [37], Brumelle [8].

3. The DSP/G/1 queue

As we saw in section 2, even for a queue with a non-stationary, deterministic arrival rate function, its stationary representation is built on the interarrival times of a DSP process. There are also other examples such as queues with Markov modulated input in which DSP arrivals are generated by the arrival rate $\lambda(t) = a(X(t))$, which is a function of $X(t)$, where $X(t)$ is a finite, irreducible continuous time Markov process. Rolski [27] showed that if $\boldsymbol{T} = \{T_i\}$ are the interarrival times of such a Markov modulated input, then $\boldsymbol{T}$ is an a.m.s. sequence with stationary representation $\{T_i^0\}$ being interarrival times of the Palm version of the DSP process with the arrival rate function $\lambda^*(t) = a(X^*(t))$. Here $X^*(t)$ is the stationary Markov process with the same generator as $X(t)$. Note that finiteness of the state space of the process $X(t)$ is not essential.

In this section we suppose that the arrival intensity function $\lambda^*(t)$ is stationary and ergodic. In such a case $\boldsymbol{T}$ is an a.m.s. sequence. We also assume that $\rho = \bar{\lambda} E(S) < 1$, where

$$\bar{\lambda} = \lim_{t \to \infty} t^{-1} \int_0^t \lambda^*(s)\, \mathrm{d}s \quad \text{with probability } 1.$$

Continuity. We study here a rather technical, but important property of continuity of performance characteristics. We are also interested in the continuity of the mean performance characteristics which seems to be less understood for general G/G/1 queues. Consider a sequence of DSP/G/1 queues. Suppose that the arrivals at the k-th queue have the arrival rate $\lambda_k^*(t)$, which is stationary and ergodic and such that with probability 1

$$\lim_{t \to \infty} t^{-1} \int_0^t \lambda_k^*(s)\, \mathrm{d}s = \bar{\lambda}_k < \infty.$$

Consider the random measure $\Lambda_k^*(B)=\int_B \lambda_k^*(t)\,\mathrm{d}t$ and suppose that

$$\Lambda_k^* \xrightarrow{d} \Lambda^* \quad \text{and} \quad \bar{\lambda}_k \to \bar{\lambda} = E(\Lambda^*[0,1]),$$

where $\Lambda^*(B)=\int_B \lambda^*(s)\,\mathrm{d}s$. Here $\xrightarrow{d}$ denote the convergence in distribution of random measures. Then by Kallenberg [20, exercise 4.5] or Serfozo [35], it follows that for arrival processes

$$\Pi_{\Lambda_k^*} \xrightarrow{d} \Pi_{\Lambda^*}.$$

Thus the conditions of theorem 3.2.1 from Franken et al. [12] (see also Borovkov [7]) are satisfied and if V_k is the stationary work-load in the k-th queue and V^* in the queue with arrivals generated by Π_{Λ^*} then

$$V_k \xrightarrow{d} V^*. \tag{3.1}$$

Recall that in all queues the service process is the same. For the convergence of means in (3.1) we need to prove the uniform integrability of $\{V_k\}$. For this we define

$$D_k = \sup\{\Lambda_k^*[0,t] - \gamma t,\ t \geq 0\},$$

where $\lim_{k\to\infty}\bar{\lambda}_k < \gamma < (E(S))^{-1}$, which by the ergodic theorem is a finite r.v..

PROPOSITION 3.1

If $E(S^3)<\infty$ and

$$\sup_k E\left(D_k^2\right) < \infty, \tag{3.2}$$

then for $k\to\infty$, $E(V_k)\to E(V^*)$.

Proof

Since (3.1) is fulfilled, we have to prove the uniform integrability of $\{V_k\}$, for which it suffices to show

$$\sup_k E\left(V_k^2\right) < \infty. \tag{3.3}$$

Following the proof of proposition 1 of Rolski [31] we can prove that

$$\tfrac{1}{2}E\left(V_k^2\right) \leq E(D_k)\left(E(S^2)-E^2(S)\right) + E\left(D_k^2\right)E^2(S) + \frac{\gamma E(S^3)}{3(1-\gamma E(S))} + \frac{\gamma^2 E^2(S^2)}{2(1-\gamma E(S))}$$

which yields (3.3) in view of (3.2). □

The essence of D_k is transparent when we write down the following inequality given in Rolski [31, lemma 2]

$$V_k =_d \sup\left\{ \sum_{i=1}^{\Pi_{\Lambda_k}(t)} S_i - t,\ t \geqslant 0 \right\}$$

$$\leqslant_{st} \sum_{i=1}^{\Pi'_{\Lambda_k}(D_k)} S_i' + \sup\left\{ \sum_{i=1}^{\Pi(\gamma t)} S_i - t,\ t \geqslant 0 \right\}, \tag{3.4}$$

where $\Pi'_{\Lambda_k}(D_k)$, $\{\Pi(\gamma t)\}$, $\{S_i\}$, $\{S_i'\}$ are independent, $\{S_i'\} =_d \{S_i\}$ and $\Pi'_{\Lambda_k}(D_k) =_d \Pi_{\Lambda_k}(D_k)$. This inequality is also useful for proving the finiteness of moments of V_k. Thus for example, we can prove as in Rolski [33, proposition 3] that $E(V_k) < \infty$ provided $E(S^2) < \infty$ and $E(D_k) < \infty$. Unfortunately we are able to find bounds for moments of D_k in a few cases only, namely for queues with a periodic intensity function and almost periodic intensity fulfilling (2.4). Rolski [31] found bounds for $E(D_k)$ and $E(D_k^2)$ in some particular cases of the Markov modulated input. These bounds were needed to approximate a periodic Poisson queue by a sequence of Markov modulated queues. However for a general stationary and ergodic intensity process $\{\lambda(t)\}$ or even for the Markov modulated case we do not know bounds or finiteness conditions for moments of D_k.

Ross [34] posed several conjectures about the following phenomena in queues with DSP arrivals: roughly speaking, it can be said that for λ being "closer" to stationarity we have smaller performance characteristics. The following is an example.

PROPOSITION 3.2 (ROLSKI [27], [29])

In a DSP/G/1 queue, if $\lambda(t)$ is either periodic or Markov modulated or stationary and ergodic, then for the mean stationary waiting time W^0 and each nondecreasing and convex function f: $\mathbb{R}_+ \to R_+$, we have $E(f(W^0)) \geqslant E(f(\tilde{W}))$, where $\tilde{W}$ is the stationary waiting time in the standard M/G/1 queue having the arrival rate $\bar{\lambda}$ and the generic service time $\boldsymbol{S}$. In particular

$$E(W^0) \geqslant \frac{\bar{\lambda} E(S^2)}{2(1 - \bar{\lambda} E(S))} = \omega(\bar{\lambda}). \tag{3.5}$$

Unfortunately, in general we are unable to say what is meant by "less stationary arrival rate". What we can say is that the constant arrival rate is the most stationary among all arrivals with the same asymptotic rate $\bar{\lambda}$. We now give a few plausible conjectures which, if true, would throw some light on this problem.

Conjectures

(i) Ross [34]). Let $\lambda_c^*(t)$ be a two state (λ_1, λ_2) Markov process with intensity matrix

$$\begin{pmatrix} -c\alpha_1 & c\alpha_1 \\ c\alpha_2 & -c\alpha_2 \end{pmatrix} \tag{3.6}$$

and $w(c)$ denote the mean waiting time in the queue with arrival rate $\lambda_c^*(t)$. Clearly, for each $c > 0$

$$\bar{\lambda} = \frac{\lambda_1\alpha_2 + \lambda_2\alpha_1}{\alpha_1 + \alpha_2} = \lim_{t\to\infty} t^{-1}\int_0^t \lambda_c^*(s)\,\mathrm{d}s \quad \text{with probability 1.}$$

The conjecture is that $w(c)$ is a decreasing function as $c \to \infty$. Notice that $\lambda_1^*(ct)$ is a Markov process with intensity matrix (3.6).

(ii) Let $\lambda^*(t)$ be a stationary and ergodic arrival rate function and let $\lambda_c^*(t) = \lambda^*(ct)$. We denote by $w(c)$ the mean stationary waiting time with arrival rate $\lambda_c^*(t)$. When does $w(c)$ decrease? Observe that $w(c)$ is not always strictly decreasing. Consider a periodic Poisson/D/1 queue with a periodic arrival rate function $\lambda^*(t)$ having period 1 and the service times equal to 1. Following Heyman [17] we can prove that $w(1) = w(2) = \dots$. However we conjecture that in a periodic Poisson/M/1, function $w(c)$ is strictly decreasing.

(iii) Under the hypotheses of proposition 3.2 we conjecture that the similar property holds for the variance as for the mean that is

$$D^2(W^0) \geqslant D^2(\tilde{W}) = \frac{\lambda E(S^3)}{3(1 - \bar{\lambda}E(S))} + 2\omega^2(\bar{\lambda}) \tag{3.7}$$

provided $E(S^3) < \infty$. We can prove (3.7) if $\lambda^*(t)$ assumes two values $\lambda_1 = 0$ and $\lambda_2 > 0$ with intensity matrix (3.6), using arguments similar to Ross [34] and Stoyan [38, theorem 5.22].

PROPOSITION 3.3

Suppose that $\lambda(t)$ is a stationary and ergodic random process and for some $\bar{\lambda} < \gamma < (E(S))^{-1}$

$$E(D^2) = E\left[\sup\left\{\int_0^t \lambda(s)\,\mathrm{d}s - \gamma t, \quad t \geqslant 0\right\}\right]^2 < \infty$$

and $E(S^2) < \infty$. Let $w(c)$ be the mean stationary waiting time in the system with the arrival rate function $\lambda_c(t) = \lambda(ct)$. Then $w(c)$ is a continuous function on $(0, \infty]$ and

$$w(\infty) \leqslant w(c) \leqslant w(0) = E[\omega(\lambda(0))]. \tag{3.8}$$

If moreover

$$P\{\lambda(t) \leqslant \gamma\} = 1 \tag{3.9}$$

then $w(c)$ is continuous on $[0, \infty]$.

Proof

By Brumelle's formula (2.15) it suffices to consider the mean stationary work-load. We have $\bar{\lambda}_c = E(\lambda_c(0)) = \bar{\lambda}$. Note that if

$$D_c = \sup\left\{\int_0^t \lambda_c(s)\,\mathrm{d}s - \gamma t, \quad t \geqslant 0\right\}$$

then $D_c \leqslant D/c$. Thus by (3.8), for each $c_0 > 0$

$$\sup_{c > c_0} E\left(D_c^2\right) < \infty,$$

which coupled with proposition 3.1 and proposition A of the Appendix shows the continuity of $w(c)$ on $(0, \infty]$. The stationary waiting time $W(c)$ has the representation

$$W(c) =_d \sup\left\{\sum_{i=1}^{\Pi_{\Lambda_c}(t)} S_i - t, \quad t \geqslant 0\right\},$$

where Π_{Λ_c} and $\{S_i\}$ are independent. If (3.9) is satisfied then $W(c)$ is stochastically bounded above by

$$W = \sup\left\{\sum_{i=1}^{\Pi_{\gamma}(t)} S_i - t, \quad t \geqslant 0\right\}.$$

Since $E(W) < \infty$ and

$$W(c) \xrightarrow{d} \sup\left\{\sum_{i=1}^{\Pi_{\lambda(0)}(t)} S_i - t, \quad t \geqslant 0\right\}, \quad c \to 0$$

we obtain $w(c) \to E(\omega(\lambda(0)))$. Inequalities in (3.9) follows from (3.5) and the result of Rolski [30]. □

For further discussion of points raised here see Fond and Ross [11], Niu [25], Rolski [30], Chang and Pinedo [10], Svoronos and Green [40], Baccelli and Makowski [2]

Appendix

POINT PROCESSES AND RANDOM MEASURES

We denote by $\mathscr{M}(\mathbb{E})$ ($\mathscr{N}(\mathbb{E})$) the space of locally finite nonnegative (integer valued) measures on $\mathbb{E} = \mathbb{R}$ or $\mathbb{R}_+$ endowed with the vague topology (for details see Kallenberg [20]). A random measure (point process) is a random element

$$M\colon (\Omega, \mathscr{F}, P) \to (\mathscr{M}, b(\mathscr{M})),$$

$$N\colon (\Omega, \mathscr{F}, P) \to (\mathscr{N}, b(\mathscr{N})),$$

where $b(\cdot)$ denote the σ-field of Borel subsets. For $\mathbb{E} = \mathbb{R}$, the family of shifts $\{\theta_t\}$ is defined by

$$\theta_t \nu(\cdot) = \nu(\cdot + t).$$

A point process Π_Λ on $\mathbb{R}$ or $\mathbb{R}_+$ is said to be a non-homogeneous Poisson process with intensity function $\Lambda(\mathrm{d}t) = \lambda(t)\,\mathrm{d}t$ if for each disjoint bounded Borel sets $B_1, \ldots, B_k$ random variables $\Pi_\Lambda(B_1), \ldots, \Pi_\Lambda(B_k)$ are independent and

$$P(\Pi_\Lambda(B) = k\} = \frac{\Lambda^k(B)}{k!}\,\mathrm{e}^{-\Lambda(B)}.$$

The intensity measure Λ can be generalized to be random, and then Π_Λ is said to be a doubly stochastic Poisson process. If $\lambda(t) \equiv \lambda(0)$, where $\lambda(0)$ is a random variable then Π_Λ is called a mixed Poisson process and is denoted by $\Pi_{\lambda(0)}$.

PROPOSITION A

Let $\lambda(t)$ be a stationary and ergodic random intensity function and define $\lambda_c(t) = \lambda(ct)$. If Λ_c is the random measure with intensity function $\lambda_c(t)$ and $\bar{\lambda} = E[\lambda(0)] < \infty$ then the family $\{\Pi_{\Lambda_c}, 0 < c \leqslant \infty\}$ is continuous in distribution and for $c \to \infty$

$$\Pi_{\Lambda_c} \xrightarrow{d} \Pi_{\bar{\lambda}}.$$

If the trajectories of $\lambda(t)$ are right continuous then for $c \to 0$

$$\Pi_{\Lambda_c} \xrightarrow{d} \Pi_{\lambda(0)}.$$

Proof

In view of Kallenberg [20, exercise 4.5] or Serfozo [35], it suffices to prove the continuity in distribution of the family $\{\Lambda_c, 0 < c \leqslant \infty\}$. By Kallenberg [20, theorem 4.2] it suffices to prove that for each $c_0 \in (0, \infty]$

$$\left(\Lambda_c[a_1, b_1], \ldots, \Lambda_c[a_k, b_k]\right) \xrightarrow{d} \left(\Lambda_{c_0}[a_1, b_1], \ldots, \Lambda_{c_0}[a_k, b_k]\right), \quad c \to c_0. \tag{A.1}$$

However

$$\Lambda_c([a, b]) = \frac{1}{c}\int_{ac}^{bc} \lambda(s)\,\mathrm{d}s = \frac{b}{cb}\int_0^{bc} \lambda(s)\,\mathrm{d}s - \frac{a}{ca}\int_0^{ac} \lambda(s)\,\mathrm{d}s$$

and (A.1) follows for $0 < c_0 < \infty$. For $c_0 = \infty$ we have to apply the individual ergodic theorem to obtain (A.1). For $c_0 = 0$ define $\Lambda_0(\mathrm{d}t) = \lambda(0)\mathrm{d}t$ and (A.1) follows because for right continuous $\lambda(t)$ with probability 1

$$\lim_{t \downarrow 0} t^{-1} \int_0^t \lambda(s)\,\mathrm{d}s = \lambda(0).$$

For further references on point processes see Rolski [26], Franken et al. [12], Baccelli and Bremaud [3], Serfozo [36].

Acknowledgement

This research was done while the author was visiting the Department of Mathematics and Statistics at Case Western Reserve University.

Note added in proof:

Conjecture (i) from section 3 has been recently proved by C.S. Chang, X.L. Chao and M. Pinedo: Monotonicity results for queues with doubly stochastic Poisson arrivals: Ross's conjecture.

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