$M/G/1$ retrial queueing systems with two types of calls and finite capacity

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We consider an $M/G/1$ priority retrial queueing system with two types of calls which models a telephone switching system and a cellular mobile communication system. In the case that arriving calls are blocked due to the server being busy, type I calls are queued in a priority queue of finite capacity K whereas type II calls enter the retrial group in order to try service again after a random amount of time. In this paper we find the joint generating function of the numbers of calls in the priority queue and the retrial group in closed form. When $\lambda_1 = 0$, it is shown that our results are consistent with the known results for a classical retrial queueing system.

Keywords: Retrial queue; finite capacity; non-preemptive priority.

1. Introduction

Retrial queueing systems are characterized by the feature that arriving calls who find the server busy join the retrial group to try again for their requests in random order and at random intervals. Retrial queues have been widely used to model many problems in telephone switching systems, computer and communication systems. For comprehensive surveys of retrial queues, see Yang and Templeton [9] and Falin [5].

Most retrial queues deal with one type of calls. But there are some practical models which deal with several types of calls as follows. One example is a telephone switching system (Falin et al. [6]). In modern telephone exchanges, subscriber lines are usually connected to the so-called subscriber line modules. These modules serve both incoming and outgoing calls. An important difference between these two types of calls lies in the fact that in the case of blocking due to all channels busy in the module, outgoing calls can be queued, whereas incoming calls get busy signal and must be retried in order to establish the connection. As soon as the channel is free, an outgoing call, if present, occupies the channel immediately. Therefore incoming calls may not establish the connection as long as there are outgoing calls waiting. This fact implies that outgoing calls have non-preemptive priority over incoming calls.

Another example is a mobile cellular radio communication system (Yoon and

Un [2]). For an efficient use of frequency channels, the service area is divided into a certain number of cells so that the base station in each cell can reuse the channels used in the other cells at the same time. The base station in a cell handles two types of calls. One type is the call initiated in its cell (originating cell). A subscriber with a blocked cell usually reinitiates his attempt after random time. The other type arises when a subscriber holding the line enters the cell from adjacent cells (handoff call). If the base station fails to assign an idle channel until the subscriber gets out of the overlap region of the cells, he suffers from a breakdown during the conversation. The degradation of the quality of the telephone service caused by such a breakdown is more serious than that caused by a blocking of an originating call. Thus the base station may give priority to a handoff call by assigning a queue. In the mobile cellular radio communication, the loss of handoff call and the time needed for an originating call to get a channel are the important factors for the quality of service.

Choi and Park [1] modeled the above systems as $M/G/1$ priority retrial queue where service times for both type of calls are independent and identically distributed and priority queue has infinity capacity. Khalil et al. [4] investigated above model at Makovian level in detail. Later Falin et al. [6] extended Choi and Park's model to the case where two types of calls may have different service time distributions.

In this paper, we consider Choi and Park's model in which priority queue has finite capacity. Because of finite capacity of priority queue, we need different method from one used in Choi and Park [1]. We obtain in closed form the joint distribution of the numbers of calls in priority queue and in retrial group. From these, we explicitly present some performance measures including the loss probability for type I calls. Moreover, in order to show the feasibility of the computation of these measures, we offer some numerical examples in the last section.

This paper is organized as follows. In section 2, we describe the mathematical queueing model and consider the stability condition of the system. In section 3, we obtain the joint distribution of the numbers of calls in priority queue and in retrial group. In section 4, we compute some performance measures. Finally we give some numerical examples in section 5.

2. Mathematical model

As a mathematical model of a telephone switching system and a base station in a mobile cellular radio communication, we consider an $M/G/1$ priority retrial queueing system in which type I and type II calls arrive according to Poisson process with rate λ_1 and λ_2, respectively. Type I calls (type II calls, respectively) can be identified as outgoing calls (incoming calls, respectively) in the telephone switching system and as handoff calls (originating calls, respectively) in the mobile cellular radio communication system.

If a type II call upon arrival finds the server free, he immediately occupies the server and leaves the system after service. If a type II call finds the server busy on his arrival, the call enters the retrial group in order to seek service again after a random amount of time (see fig. 1). The call persists this way until he succeeds the connection. The retrial time (the time interval between two consecutive attempts made by a call in the retrial group) is exponentially distributed with mean $1/\alpha$ and is independent of all previous retrial times and all other stochastic processes in the system.

Type I calls are queued in a priority queue of capacity K after blocking if there is any empty waiting position in priority queue, otherwise the call is lost. As soon as the server is free, one of the cells, if any, in the priority queue is served, so the calls in the retrial group will be served only when there are no calls in the priority queue. According to the above rule, type I calls have non-preemptive priority over type II calls. The calls in priority queue consist of type I calls and the cells in the retrial group consist of type II calls. We sometimes refer to the type I calls in the priority queue as the calls in the priority queue and refer to the type II calls in the retrial group as the calls in the retrial group.

Service times are independent and identically distributed and have the same distribution for both calls. We denote the service time by S. In order to use the supplementary variable method, we assume that the service time distribution has a probability density function (p.d.f.) $b(x)$. Let

$$b^*(\theta) \equiv \int_0^\infty e^{-\theta x} b(x)\, dx.$$

By following the usual argument using mean drift (Falin [7]), we can show that the system is stable if

$$m_K + \rho_2 < 1, \tag{2.1}$$

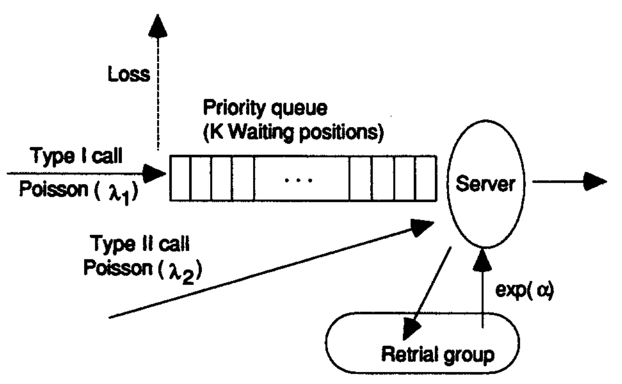

Fig. 1. Mathematical model of the queueing system.

where

$$m_K = \begin{cases} K - \sum_{k=0}^{K} (K-k) E\left(\dfrac{e^{-\lambda_1 S}(\lambda_1 S)^k}{k!}\right), & K < \infty, \\ \rho_1, & K = \infty, \end{cases}$$

$$\rho_i = \lambda_i E(S), \quad i, 2.$$

Note that (2.1) is also a necessary condition on the stable system when $K = 0$ (Falin [5]) or $K = \infty$ (Choi and Park [1]).

In the remainder of this paper, we always assume that the system is stable.

3. The joint distributions of queue sizes

We derive the joint distribution of the numbers of calls in the priority queue and the retrial group at an arbitrary time by using the supplementary variable method (Choi and Park [1], Hokstad [8]). To do so, let us define random variables on the system in steady state.

$N_p \equiv$ the number of calls in the priority queue,
$N_r \equiv$ the number of calls in the retrial group,
$\tilde{S} \equiv$ the residual service time of the call in service,
$\xi \equiv \begin{cases} 0 & \text{when the server is idle,} \\ 1 & \text{when the server is busy.} \end{cases}$

We define the related probabilities for $x \geq 0$,

$$q_j \equiv \mathrm{P}(N_r = j, \xi = 0), \quad j \geq 0,$$

$$p_{i,j}(x)\Delta t \equiv \mathrm{P}(N_p = i, N_r = j, \tilde{S} \in (x, x + \Delta t], \xi = 1),$$
$$i = 0, 1, 2, \ldots, K, \quad j \geq 0,$$

and define their Laplace transforms

$$p^*_{i,j}(\theta) \equiv \int_0^\infty e^{-\theta x} p_{i,j}(x)\, dx, \quad i = 0, 1, 2, \ldots, K, \quad j \geq 0.$$

We treat the case $K \geq 1$, for the system with $K = 0$ was investigated by Falin [5] as a model with impatient subscribers; however, our results are consistent with $K = 0$ (see Remark 3.2b)).

The usual arguments lead to the following differential difference equations

(Choi and Park [1]):

$$p_{0,j}(0) = (\lambda + j\alpha)q_j, \quad j \geq 0, \tag{3.1a}$$

$$-p'_{0,j}(x) = -\lambda p_{0,j}(x) + \lambda_2 p_{0,j-1}(x) + b(x)p_{1,j}(0) + (j+1)\alpha b(x)q_{j+1} + \lambda b(x)q_j, \quad j \geq 0, \tag{3.1b}$$

$$-p'_{i,j}(x) = -\lambda p_{i,j}(x) + \lambda_2 p_{i,j-1}(x) + \lambda_1 p_{i-1,j}(x) + b(x)p_{i+1,j}(0), \quad i = 1,2,\ldots,K-1, \quad j \geq 0, \tag{3.1c}$$

$$-p'_{K,j}(x) = -\lambda_2 p_{K,j}(x) + \lambda_2 p_{K,j-1}(x) + \lambda_1 p_{K-1,j}(x), \quad j \geq 0 \tag{3.1d}$$

and the normalization condition

$$\sum_{j=0}^{\infty}\sum_{i=0}^{K}\int_0^{\infty} p_{i,j}(x)\,dx + \sum_{j=0}^{\infty} q_j = 1, \tag{3.1e}$$

where

$$\lambda = \lambda_1 + \lambda_2 \quad \text{and} \quad p_{i,-1}(x) \equiv 0, \quad i = 0,1,\ldots,K.$$

By taking the Laplace transform of eqs. (3.1b)–(3.1d) and then multiplying by z^j respectively and summing over j (Choi and Park [1]), we have the following basic system of equations:

$$P_0(0,z) = \lambda Q(z) + \alpha z Q'(z), \tag{3.2a}$$

$$(\theta - \lambda + \lambda_2 z)P_0^*(\theta,z) = P_0(0,z) - b^*(\theta)\{P_1(0,z) + \alpha Q'(z) + \lambda Q(z)\}, \tag{3.2b}$$

$$(\theta - \lambda + \lambda_2 z)P_i^*(\theta,z) = P_i(0,z) - b^*(\theta)P_{i+1}(0,z) - \lambda_1 P_{i-1}^*(\theta,z), \quad i = 1,2,\ldots,K-1, \tag{3.2c}$$

$$(\theta - \lambda_2 + \lambda_2 z)P_K^*(\theta,z) = P_K(0,z) - \lambda_1 P_{K-1}^*(\theta,z), \tag{3.2d}$$

where for $|z| \leq 1$

$$Q(z) \equiv \sum_{j=0}^{\infty} q_j z^j,$$

$$P_i^*(\theta,z) \equiv \sum_{j=0}^{\infty} p_{i,j}^*(\theta) z^j, \quad i = 0,1,2,\ldots,K,$$

$$P_i(0,z) \equiv \sum_{j=0}^{\infty} p_{i,j}(0) z^j, \quad i = 0,1,2,\ldots,K.$$

Furthermore, we get the normalization condition from (3.1e),

$$\sum_{i=0}^{K} P_i^*(0,1) + Q(1) = 1. \tag{3.2e}$$

We note that $Q(z)$ and $P_i^*(0,z)$ are the probability generating functions of the number of calls in the retrial group in steady state when the server is idle and when the server is busy with i calls in the priority queue, respectively.

The rest of this section is devoted to obtain the solution $Q(z)$ and $P_i^*(0,z)$ from the above system equations in closed form. The first thing to do is to formulate the differential equation of order one for $Q(z)$ (see (3.11)) and then solve it. By substituting $\theta = \lambda_2 - \lambda_2 z$ in (3.2a)–(3.2d) and summing them, we obtain

$$\begin{aligned} 0 = (1 - b^*(\lambda_2 - \lambda_2 z)) &\sum_{i=1}^{K} P_i(0,z) \\ &+ \lambda(1 - b^*(\lambda_2 - \lambda_2 z))Q(z) - \alpha(b^*(\lambda_2 - \lambda_2 z) - z)Q'(z). \end{aligned} \tag{3.3}$$

Moreover, the left hand sides of (3.2b) and (3.2c) vanish at $\theta = \lambda - \lambda_2 z$, thus we have

$$P_0(0,z) = b^*(\lambda - \lambda_2 z)(P_1(0,z) + \alpha Q'(z) + \lambda Q(z)), \tag{3.4a}$$

$$P_i(0,z) = b^*(\lambda - \lambda_2 z)P_{i+1}(0,z) + \lambda_1 P_{i-1}^*(\lambda - \lambda_2 z, z), \quad i = 1,2,\ldots,K-1. \tag{3.4b}$$

Next we have another expression of $P_i^*(\lambda - \lambda_2 z, z)$ in terms of $P_i(0,z)$, $Q'(z)$ and $Q(z)$ (see (3.6)). For convenience, we employ the following notations,

$$b^{*(m)}(s) \equiv \left.\frac{d^m b^*(\theta)}{d\theta^m}\right|_{\theta=s}, \qquad \mathrm{Re}\,(s) > 0,$$

$$P_i^{*(m)}(s,z) \equiv \left.\frac{\partial^m P_i^*(\theta,z)}{\partial\theta^m}\right|_{\theta=s}, \qquad \mathrm{Re}\,(s) > 0, \quad i = 0,1,\ldots,K, m \geq 0.$$

Differentiating by $(m+1)$ times both sides of eqs. (3.2b)–(3.2c) with respect to θ and evaluating at $\theta = \lambda - \lambda_2 z$ yields for $m = 0,1,2,\ldots$,

$$(m+1)P_0^{*(m)}(\lambda - \lambda_2 z, z) = -b^{*(m+1)}(\lambda - \lambda_2 z)(P_1(0,z) + \alpha Q'(z) + \lambda Q(z)), \tag{3.5a}$$

$$\begin{aligned} (m+1)P_i^{*(m)}(\lambda - \lambda_2 z, z) &= -b^{*(m+1)}(\lambda - \lambda_2 z)P_{i+1}(0,z) - \lambda_1 P_{i-1}^{*(m+1)}(\lambda - \lambda_2 z, z), \\ &i = 1,2,\ldots,K-1. \end{aligned} \tag{3.5b}$$

By starting with $m = 0$ in (3.5b) and successive substitution of (3.5b) with

$m = 1, \ldots, i$, we get

$$\begin{aligned} P_i^*(\lambda - \lambda_2 z, z) \equiv P_i^{*(0)}(\lambda - \lambda_2 z, z) \\ &= -\sum_{j=1}^{i+1} \frac{(-\lambda_1)^{i+1-j} b^{*(i+2-j)}(\lambda - \lambda_2 z)}{(i+2-j)!} P_j(0,z) \\ &\quad - \frac{(-\lambda_1)^i b^{*(i+1)}(\lambda - \lambda_2 z)}{(i+1)!} (\alpha Q'(z) + \lambda Q(z)). \end{aligned} \tag{3.6}$$

Note from (3.5a) that (3.6) also holds for $i = 0$.

By substituting (3.6) into (3.4b), we have the following recursive form about the boundary function $P_i(0,z)$:

$$P_i(0,z) = \begin{cases} \dfrac{1}{b^*(\lambda - \lambda_2 z)} P_0(0,z) - (\alpha Q'(z) + \lambda Q(z)), & i = 1, \\ c_1(z) P_1(0,z) + \dfrac{\lambda_1 b^{*(1)}(\lambda - \lambda_2 z)}{b^*(\lambda - \lambda_2 z)} (\alpha Q'(z) + \lambda Q(z)), & i = 2, \\ \displaystyle\sum_{j=1}^{i-1} c_{i-j}(z) P_j(0,z) + c_{i-1}(z)(\alpha Q'(z) + \lambda Q(z)), & i = 3, 4, \ldots, K, \end{cases} \tag{3.7}$$

where

$$c_i(z) \equiv \begin{cases} \dfrac{1 + \lambda_1 b^{*(1)}(\lambda - \lambda_2 z)}{b^*(\lambda - \lambda_2 z)}, & i = 1, \\ -\dfrac{(-\lambda_1)^i b^{*(i)}(\lambda - \lambda_2 z)}{i! b^*(\lambda - \lambda_2 z)}, & i \geq 2. \end{cases} \tag{3.8}$$

LEMMA

Let $x_i(z)$ be the coefficient of η^i in Taylor series expansion of

$$\frac{b^*(\lambda - \lambda_2 z - \lambda_1 \eta)}{b^*(\lambda - \lambda_2 z - \lambda_1 \eta) - \eta}, \quad |z| \leq 1, \quad i = 0, 1, \ldots.$$

Then we have that for $i = 1, 2, \ldots, K$,

$$P_i(0,z) = \alpha(z x_i(z) - x_{i-1}(z)) Q'(z) + \lambda(x_i(z) - x_{i-1}(z)) Q(z). \tag{3.9}$$

Proof

It is known (Choi and Park [1]) that for each z, $|z| \le 1$, $b^*(\lambda - \lambda_2 z - \lambda_1 \eta) = \eta$ has no solution in a neighborhood of $\eta = 0$. So $x_i(z)$, $i = 0, 1, \dots$, are well defined for $|z| \le 1$.

$x_i(z)$'s satisfy the following recurrence relation:

$$\begin{aligned} &x_0(z) = 1, \quad x_1(z) = \frac{1}{b^*(\lambda - \lambda_2 z)} \\ &x_i(z) = \sum_{j=1}^{i-1} c_{i-j}(z) x_j(z), \quad i = 2, 3, \dots . \end{aligned} \tag{3.10}$$

Using this relation, we readily verify by induction that for $i = 0, 1, \dots, K$,

$$P_i(0, z) = x_i(z) P_0(0, z) - x_{i-1}(z)(\alpha Q'(z) + \lambda Q(z)).$$

Then (3.9) easily follows from (3.2a). □

Substituting (3.9) into (3.3) yields the following differential equation for $Q(z)$:

$$Q'(z) = \frac{\lambda}{\alpha} \times \frac{x_K(z)(1 - b^*(\lambda_2 - \lambda_2 z))}{D_K(z)} Q(z), \tag{3.11}$$

where

$$D_K(z) = (1 - z) - (1 - b^*(\lambda_2 - \lambda_2 z)) \left\{ x_K(z) - (1 - z) \sum_{l=1}^{K} x_l(z) \right\}.$$

The general solution of the differential equation (3.11) is given by

$$Q(z) \equiv C \exp \left\{ -\frac{\lambda}{\alpha} \int_z^1 \frac{x_K(s)(1 - b^*(\lambda_2 - \lambda_2 s))}{D_K(s)} ds \right\},$$

where C is a constant.

To find C which is the probability that the server is idle, we evaluate (3.2b)–(3.2d) at $\theta = 0$ and sum over i, then we have from (3.2a) that

$$\lambda_2 \sum_{i=0}^{K} P_i^*(0, z) = \alpha Q'(z). \tag{3.12}$$

Using (3.12) and the normalization condition (3.2e), we find

$$C = \frac{1 - \rho_2 x_K(1)}{1 + \rho_1 x_K(1)}. \tag{3.13}$$

Thus we have obtained the following results.

THEOREM 3.1

In steady state, when the server is idle, the probability generating function $Q(z)$ of the number of calls in the retrial group is given by

$$\begin{aligned} Q(z) &= E(z^{N_r}; \xi = 0) \\ &= \frac{1 - \rho_2 x_K(1)}{1 + \rho_1 x_K(1)} \exp\left\{ -\frac{\lambda}{\alpha} \int_z^1 \frac{x_K(s)(1 - b^*(\lambda_2 - \lambda_2 s))}{D_K(s)} ds \right\}. \end{aligned} \tag{3.14}$$

□

Remark 3.1

As we expected, the probability C that the server is idle converges to $1 - \rho$, $\rho = \rho_1 + \rho_2$ as $K \to \infty$. Note that the probability C that the server is idle is equal to $1 - \rho$ when $K = \infty$ (Choi and Park [1]). The function $\sum_{i=0}^{\infty} x_i(1)\eta^i = b^*(\lambda_1 - \lambda_1\eta)/(b^*(\lambda_1 - \lambda_1\eta) - \eta)$ is analytic in $|\eta| < 1$, because the equation $b^*(\lambda_1 - \lambda_1\eta) = \eta$ has no solution in $|\eta| < 1$ provided that $\rho_1 < 1$ (Choi and Park [1]). Thus by the Abelian theorem (Cohen [3]), we have

$$\lim_{K \to \infty} x_K(1) = \lim_{\eta \uparrow 1} (1 - \eta) \frac{b^*(\lambda_1 - \lambda_1\eta)}{b^*(\lambda_1 - \lambda_1\eta) - \eta} = \frac{1}{1 - \rho_1}.$$

Therefore we conclude that

$$C = \frac{1 - \rho_2 x_K(1)}{1 + \rho_1 x_K(1)} \to 1 - \rho, \quad \text{as } K \to \infty.$$

□

By substituting (3.14) into (3.2a) for $i = 0$ and (3.9) for $i = 1, 2, \ldots, K$, we obtain explicit expression for the boundary function $P_i(0, z)$.

$$P_0(0, z) = \frac{\lambda(1 - z)B_K(z)Q(z)}{D_K(z)}, \tag{3.15a}$$

and for $i = 1, 2, \ldots, K$,

$$P_i(0, z) = \frac{\lambda(1 - z)Q(z)}{D_K(z)} \{(x_i(z) - x_{i-1}(z))B_K(z) - x_{i-1}(z)x_K(z)(1 - b^*(\lambda_2 - \lambda_2 z))\}, \tag{3.15b}$$

where

$$B_K(z) = 1 + (1 - b^*(\lambda_2 - \lambda_2 z))\left(\sum_{l=1}^{K} x_l(z) - x_K(z)\right).$$

Finally, by substituting (3.14) and (3.15) into (3.2b)–(3.2d), we obtain $P_i^*(0, z)$.

THEOREM 3.2

The probability generating function $P_i^*(0, z)$ of the number of calls in the retrial group when the server is busy and there are i calls in the priority queue is given by (i)

$$P_0^*(0, z) \equiv E(z^{N_r}; \xi = 1) = \frac{\lambda(1 - z)(1 - b^*(\lambda - \lambda_2 z))B_K(z)Q(z)}{(\lambda - \lambda_2 z)b^*(\lambda - \lambda_2 z)D_K(z)}, \tag{3.16a}$$

(ii) for $i = 1, 2, \ldots, K - 1$,

$$\begin{aligned}
P_i^*(0, z) \equiv E(z^{N_r}; N_p = i, \xi = 1) &= \frac{\lambda(1 - z)Q(z)}{b^*(\lambda - \lambda_2 z)D_K(z)} \\
&\times \left\{ \frac{\lambda_1^i}{(\lambda - \lambda_2 z)^{i+1}} (1 - b^*(\lambda - \lambda_2 z))B_K(z) + b^*(\lambda - \lambda_2 z) \right. \\
&\times \sum_{j=1}^{i} \frac{\lambda_1^{i-j}}{(\lambda - \lambda_2 z)^{i+1-j}} (B_K(z)(x_{j+1}(z) - 2x_j(z) + x_{j-1}(z)) \\
&\left. - (1 - b^*(\lambda_2 - \lambda_2 z))x_K(z)(x_j(z) - x_{j-1}(z))) \right\},
\end{aligned} \tag{3.16b}$$

(iii)

$$\begin{aligned}
P_K^*(0, z) \equiv E(z^{N_r}; N_p = K, \xi = 1) &= \frac{\lambda Q(z)}{\lambda_2 b^*(\lambda - \lambda_2 z)D_K(z)} \\
&\times \left\{ \left(\frac{\lambda_1}{\lambda - \lambda_2 z}\right)^K (1 - b^*(\lambda - \lambda_2 z))B_K(z) + b^*(\lambda - \lambda_2 z) \sum_{j=1}^{K} \left(\frac{\lambda_1}{\lambda - \lambda_2 z}\right)^{K-j} \right. \\
&\times (B_K(z)(x_{j+1}(z) - 2x_j(z) + x_{j-1}(z)) \\
&- (1 - b^*(\lambda_2 - \lambda_2 z))x_K(z)(x_j(z) - x_{j-1}(z))) \\
&\left. - b^*(\lambda - \lambda_2 z)((x_{K+1}(z) - x_K(z))B_K(z) - (1 - b^*(\lambda_2 - \lambda_2 z))x_K^2(z)) \right\}.
\end{aligned} \tag{3.16c}$$

□

Remark 3.2

Special cases of our model:

(a) When $\lambda_1 = 0$, our model is reduced to the classical retrial queueing system (Falin [5]). In this case, we have that $x_i(z) = (b^*(\lambda_2 - \lambda_2 z))^{-i}$, $i = 0, 1, \ldots$. Thus (3.14) and (3.16a) lead to

$$Q(z) \equiv E(z^{N_r}; \xi = 0) = (1 - \rho_2) \exp\left\{ -\frac{\lambda}{\alpha} \int_z^1 \frac{1 - b^*(\lambda_2 - \lambda_2 s)}{b^*(\lambda_2 - \lambda_2 s) - s} ds \right\},$$

$$P_0^*(0, z) \equiv E(z^{N_r}; \xi = 1) = \frac{(1 - b^*(\lambda - \lambda_2 z))Q(z)}{b^*(\lambda - \lambda_2 z) - z}.$$

These agree with eqs. (10) and (11) in Falin [5].

(b) When $K = 0$, our model is identical to the $M/G/1$ retrial queueing system with impatient subscribers (Falin [5]). In this case, with convention that $\sum_{j=1}^{0} x_j(z) \equiv 0$, we have $D_0(z) = b^*(\lambda_2 - \lambda_2 z) - z$ and $B_0(z = b^*(\lambda_2 - \lambda_2 z)$. Equations (3.14) and (3.16c) become

$$E(z^{N_r}; \xi = 0) = \frac{1 - \rho_2}{1 + \rho_1} \exp\left\{ -\frac{\lambda}{\alpha} \int_z^1 \frac{1 - b^*(\lambda_2 - \lambda_2 s)}{b^*(\lambda_2 - \lambda_2 s) - s} ds \right\},$$

$$E(z^{N_r}; \xi = 1) = \frac{\lambda(1 - b^*(\lambda_2 - \lambda_2 z))Q(z)}{\lambda_2(b^*(\lambda_2 - \lambda_2 z) - z)},$$

which agree with eqs (66) and (67) in Falin [5]. □

4. Performance measures

We first evaluate $Q'(1)$ frequently appearing in the computation of the performance measures.

$$Q'(1) = \frac{\lambda}{\alpha} x_K(1) \frac{1 - \rho_2 x_K(1)}{1 + \rho_1 x_K(1)} \lim_{z \uparrow 1} \frac{1 - b^*(\lambda_2 - \lambda_2 z)}{D_K(z)} = \frac{\lambda}{\alpha} \frac{x_K(1)\rho_2}{1 + x_K(1)\rho_1}, \tag{4.1}$$

where we use L'Hospital's rule in the last equality.

(1) *The loss probability P_l of type I calls:*

PASTA property implies that

$$P_l = P_K^*(0, 1) = \frac{\lambda}{\lambda_1}\left(1 - \frac{x_K(1)}{1 + \rho_1 x_K(1)}\right). \tag{4.2}$$

Remark 4.1

Another way to obtain P_l is to use the relation $\lambda_2 + (1 - P_l)\lambda_1 = \sum_{i=0}^{K} P_i(0, 1)$, which means that the effective arrival rate must be equal to the effective departure rate in steady state. From (3.2a) and (3.3) we can calculate $\sum_{i=0}^{K} P_i(0, 1)$ and obtain the same result as (4.2). □

(2) *Blocking probability* $1 - C$ *of type II calls:*

When the server is busy, type II calls are blocked and return to retrial group in order to get service again. The probability C that the server is idle was given by (3.13).

(3) *Mean number* EN_p *of calls in priority queue:*

From Theorem 2(ii), we have

$$\sum_{i=1}^{K-1} iP_i^*(0,1) = \frac{\lambda}{\lambda_1} \frac{1 - \rho_2 x_K(1)}{1 + \rho_1 x_K(1)} \left(\lim_{z \uparrow 1} \frac{1 - z}{D_K(z)} \right) \sum_{i=1}^{K-1} i(x_{i+1}(1) - x_i(1))$$
$$= \frac{\lambda}{\lambda_1} \frac{K x_K(1) - \sum_{i=1}^{K} x_i(1)}{1 + \rho_1 x_K(1)}.$$

Therefore, we get

$$EN_p = \sum_{i=0}^{K} iP_i^*(0,1) = \sum_{i=1}^{K-1} iP_i^*(0,1) + KP_l$$
$$= \frac{\lambda}{\lambda_1} \left\{ K - \frac{\sum_{i=1}^{K} x_i(1)}{1 + \rho_1 x_K(1)} \right\}. \tag{4.3}$$

(4) *Mean number* EN_r *of calls in retrial group:*

Since $EN_r = \sum_{i=0}^{K} (d/dz) P_i^*(0, z)|_{z=1} + Q'(1)$, we may use Theorem 2 for $P_i^*(0, z)$. However there is an easy way to get the first term in the above equation by using (3.12) and (3.2a):

$$EN_r = \frac{\alpha}{\lambda_2} Q''(1) + Q'(1) = \frac{1}{\lambda_2} \left(\frac{d}{dz} P_0(0, z) \bigg|_{z=1} - (\lambda_1 + \alpha) Q'(1) \right).$$

Applying L'Hospital's rule twice in (3.15a) yields

$$EN_r = \frac{\lambda\{b(x_K'(1) + \rho_2 x_K(1) \sum_{i=1}^{K} x_i(1)) + \lambda_2 x_K(1) E(S^2)/2\}}{(1 + \rho_1 x_K(1))(1 - \rho_2 x_K(1))} + \frac{\lambda \rho_2 x_K(1)}{\alpha(1 - \rho_2 x_K(1))}. \tag{4.4}$$

Remark 4.2

The calculation of $x_i'(1)$ and $x_i(1)$ can be easily carried out by a computer from the following recursive properties derived from (3.10) for $i = 2, 3, \ldots,$

$$x_i(1) = \sum_{j=1}^{i-1} c_{i-j}(1) x_j(1), \tag{4.5a}$$

$$x_i'(1) = \sum_{j=1}^{i-1} \left(\frac{\lambda_2}{\lambda_1} (i-j) x_{j+1}(1) + x_j'(1) \right) c_{i-j}(1) + \frac{\lambda_2}{\lambda_1 b^*(\lambda_1)} (i c_i(1) - x_i(1)). \tag{4.5b}$$

□

(5) *Mean waiting time $W_p(W_r)$ of type I (type II) calls:*
By applying Little's formula, we get

$$W_p = \frac{EN_p}{\lambda_1(1 - P_l)} \quad \text{and} \quad W_r = \frac{EN_r}{\lambda_2}. \tag{4.6}$$

5. Numerical examples

In this section, we present some numerical examples on the performance measures derived in section 4. Throughout this section, we let the mean service time be a unit time and retrial rate α be 0.3. We consider the following service time distributions:

Hyper-exponential distribution (H_2): $b^*(\theta) = \frac{1}{3}\frac{1/2}{1/2+\theta} + \frac{2}{3}\frac{2}{2+\theta}$,

Exponential distribution (E): $b^*(\theta) = \frac{1}{1+\theta}$,

Deterministic distribution (D): $b^*(\theta) = e^{-\theta}$.

In fig. 2 the loss probability P_l of type I calls is plotted on logarithmic scale as the arrival rate λ_2 of type II calls varies with $\lambda_1 = 0.3\lambda_2$ as in Yoon and Un [2] which reflects the fact that the handoff call rate is proportional to the originating call rate in the mobile cellular radio communication system. As we expect, the larger the variance of a service time is the larger the loss probability P_l of type I calls is.

In the following numerical examples we let $\lambda_2 = 0.1$ and take the service time distribution as H_2.

Figure 3 plots the loss probability P_l of type I calls with K and λ_1 varying. In fig. 3, capacity of priority queue greater than 5 does not improve the loss probability

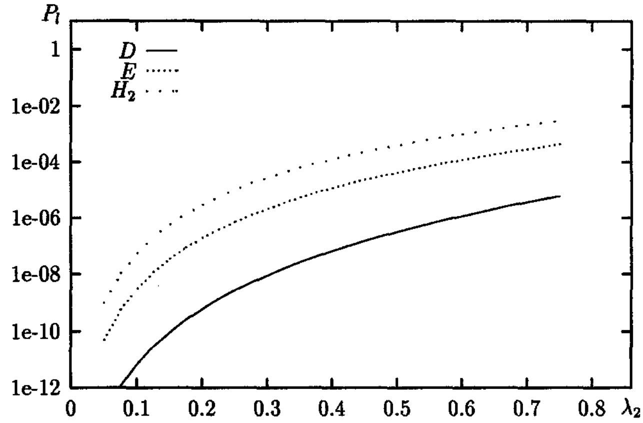

Fig. 2. Loss probability P_l of type I calls: $K = 5$, $\lambda_1 = 0.3\lambda_2$, $\alpha = 0.3$.

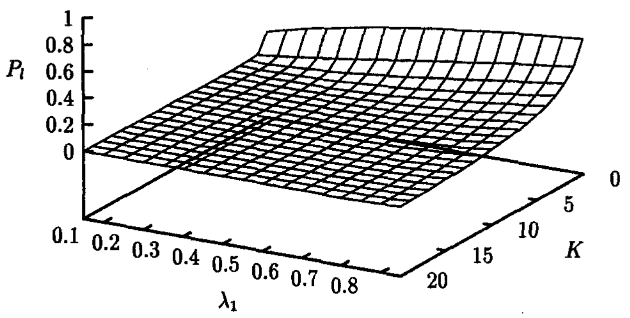

Fig. 3. Loss probability P_l of type I calls: H_2 service time, $\alpha = 0.3$, $\lambda_2 = 0.1$.

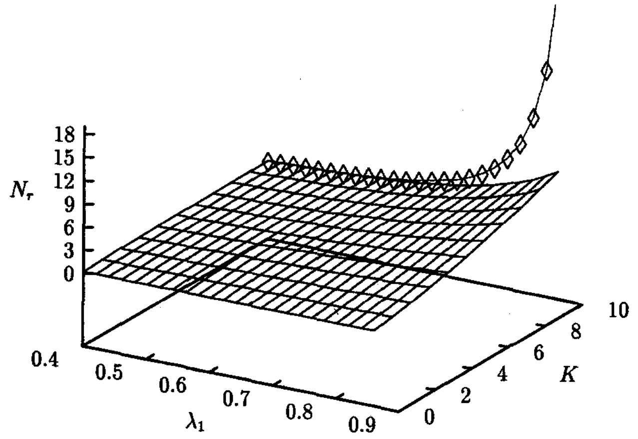

Fig. 4. Mean number N_r of calls in retrial group: H_2 service time, $\alpha = 0.3$, $\lambda_2 = 0.1$. $\diamond$ indicates mean number of calls in retrial group when $K = \infty$.

P_l of type I calls significantly. Thus $K = 5$ is a sufficient capacity of the priority queue with a reasonably low loss probability P_l of type I calls.

As a final numerical example, fig. 4 displays the mean number N_r of calls in retrial group with K and λ_1 varying. To check how good approximations of a system with infinite buffer to a system with finite buffer are, we also plot the mean number N_r of calls in the retrial group of the system with infinite buffer (Choi and Park [1]) by the symbol $\diamond$. We see that for $\lambda_1 < 0.7$, there is a little difference in the mean number of calls in the retrial group between the system with infinite buffer and the system with $K = 10$, but for $\lambda_1 \geq 0.7$, there is rather a big difference as we expected.

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