DENSITY OF AN (r, R)-SYSTEM

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In this paper we give a complete geometrical theory for the study of the exact lower bound of the density of n-dimensional lattices. For arbitrary (r, R)-systems we prove an analog of well known theorems due to Rogers from the theory of packings, and also from this same theory, an analog of a theorem due to Coxeter, Few, and Rogers. Several special examples are treated.

1°. There are two basic extremal problems of the geometry of positive quadratic forms that are rather widely known. The first of these is the classical problem of the densest lattice packing of n-dimensional balls while the second of these is the problem, posed in the last ten years, concerning the least dense lattice coverings of space by equal balls.

Considerable progress, achieved recently in the solution of these problems,* has made it possible to arrive at a solution of still other extremal problems, for example, the problem concerning a many-dimensional ζ-function (see [8, 7]). The present paper is devoted to one of such problems, namely, the problem concerning the density of an (r, R)-system, and is a detailed exposition of a portion of the results announced by the author in [6]. We make substantial use here of the methods and constructions employed in [5-7]; we assume, therefore, that the reader has knowledge of these papers.

2°. A set $\mathscr{E} \subset E^n$ is said to be a uniformly discrete system or an (r, R)-system [9] if numbers r and R exist satisfying the following two conditions:

1. In the open ball of radius r, circumscribing an arbitrary point of the set $\mathscr{E}$, there are no other points of this set.

2. In an arbitrary (closed) ball of radius R there is necessarily a point of the set $\mathscr{E}$.

An arbitrary n-dimensional lattice serves as an example of a uniformly discrete system. In particular, the lattice constructed on a cube of edge equal to two is an (r, R)-system for arbitrary positive $r \leq 1$ and $R \geq \sqrt{n}$.

We consider an arbitrary uniformly discrete system $\mathscr{E} \subset E^n$ and we find numbers r^* and R^*, which are, respectively, upper and lower bounds of numbers r and R for which the system $\mathscr{E}$ is an (r, R)-system. It is obvious that the system $\mathscr{E}$ is also an (r^*, R^*)-system. We remark that for a lattice the number R^* is the radius of the covering and the number r^* is the length of the minimal vector (twice the radius of the corresponding packing).

We denote the ratio R^*/r^* by $\varkappa(\mathscr{E})$ and call it the density of the system.

We denote by $\varkappa(n)$ the greatest lower bound of the ratio R^*/r^* over all uniformly discrete systems of the space E^n, and we denote by $\varkappa_\Gamma(n)$ the greatest lower bound of the ratio R^*/r^* over all lattice (r, R)-systems, i.e., over all lattices of the space E^n.

*See the book [1] and the survey [2], and also the papers [3-7] not included in the survey, in which, in particular, a more detailed bibliography is given of the most recent papers.

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The fundamental problem in the theory of the density of n-dimensional (r, R)-systems consists in finding the number $\varkappa(n)$ and determining the (r, R)-system (if it exists) which gives rise to this number. The fundamental problem in the theory of the densest n-dimensional lattice (r, R)-systems consists in finding the number $\varkappa_\Gamma(n)$ and determining a lattice realizing this number.

It is obvious that $\varkappa_\Gamma(n) \geq \varkappa(n)$; however, examples of nonlattice (r, R)-systems denser than lattice systems are not known to the author. Closely related questions have been studied by A. N. Kolmogorov, A. G. Vitushkin, and other authors (see A. G. Vitushkin's book [10]) in connection with a study of the concept "to within ε."

To some extent the simple results of Sec. 3°, concerned with estimates of the number $\varkappa(n)$, are left "hanging in air." The trivial result concerning the fact that the densest two-dimensional lattice (r, R)-system is the lattice Γ_1^2 constructed on a right triangle (since it gives the maximum number r* and the minimum number R* for a fixed discriminant) is well known. However there are no results even for three-dimensional lattice (r, R)-systems. As will be clarified (Sec. 3°), this may be explained by the inherent difficulty of the problem, which, to some extent, turns out to be solved only after the polyhedra $\mu(m)$ (see [6]) and the surface $\rho(r)$ (see [5]) have been constructed.

3°. Bounds for the Number $\varkappa(n)$.

THEOREM 1. The inequality $\varkappa(n) < 1$ is valid.

This inequality is proved by the following widely known (see, for example, [10]) reasoning showing the existence of (1, 1)-systems. We choose an arbitrary point of the space E^n and we construct a ball U_m with center at this point and with radius $m > 0$. In this ball we locate a number of points in order that the resulting system in the ball U_m will satisfy the condition 1) in the definition of a (1, 1)-system. If the system does not satisfy condition 2) in the definition of a (1, 1)-system, we can then find a point in the ball which is at a distance of at least one from all the points of the system. Adjoining this point to the initial system, we again obtain a system satisfying the condition 1). Repeating this construction a finite number of times, we arrive at a system satisfying both of the (1, 1)-conditions in the ball considered. Going over now to a ball with the same center but with a radius of $m + 1$, we repeat the foregoing reasoning. Since such balls exhaust the space E^n, we construct in this way a (1, 1)-system in the whole space.

LEMMA. The length of the smallest edge of an arbitrary n-dimensional simplex imbedded in the n-dimensional ball U_R of radius R cannot be larger than $\sqrt{2(n+1)/n}R$. This latter value is attained only on a regular simplex inscribed in this ball.

Proof. We note, first of all, that for any simplex S, imbedded in the ball U_R, we can find a simplex inscribed in this ball and having an edge not less than the corresponding edge of the simplex S. In fact, let A be one of those vertices of the simplex S which does not lie on the surface of the ball U_R, then, moving it along the extension of the altitude dropped onto the opposite face S_A, away from the face S_A, we increase all the edges emanating from the vertex A. We can perform this operation with all simplexes obtained in succession this way, continuing the process so long as we have not arrived at an inscribed simplex possessing the properties we require.

We assume now that our lemma is true for n−1 and we prove it for n. Thus, let S be a simplex inscribed in the ball U_R and having edges not less than $\sqrt{2(n+1)/n}R$. We take an arbitrary vertex A of the simplex S and we circumscribe about it a ball of radius $\sqrt{2(n+1)/n}R$; all other vertices of the simplex S lie in the smaller of the two spherical caps of the ball U_R into which U_R is divided by the new ball. But this cap can be imbedded in a ball of radius $\sqrt{(n^2-1)/n^2}R$, and, by the same token, the face S_A, opposite the vertex A, is imbedded in an (n−1)-dimensional ball of radius $\sqrt{(n^2-1)/n^2}R$. Since, however, all the edges of the simplex S_A are not less than $\sqrt{2(n+1)/n}R$, the simplex must be regular and inscribed in an (n−1)-dimensional ball of radius $\sqrt{(n^2-1)/n^2}R$. Since the vertex A was arbitrary, all the faces of our simplex are regular; i.e., the simplex is itself regular. Thus our lemma is completely proved.

THEOREM 2. The number $\varkappa(n)$ cannot be less than the ratio of the radius of the ball, circumscribed about a regular n-dimensional simplex, to the edge of the simplex, i.e.,

$$\varkappa(n) \geqslant \sqrt{n/2(n+1)}.$$

Proof. We take an arbitrary (r, R)-system $\mathscr{E} \subset E^n$ and we show that, of necessity, we can find a pair of points in it separated by a distance not greater than $\sqrt{2(n+1)/n}R$, i.e., such that $r < \sqrt{2(n+1)/n}R$. Actually, we select an arbitrary body L from the decomposition $\{L\}$, corresponding to the system $\mathscr{E}$ (see [9], 1-4). This body L is, by definition, inscribed in an empty (devoid of points of the system $\mathscr{E}$) ball of some radius R'; however, according to the definition of an (r, R)-system, the empty ball cannot have a radius larger than R, i.e., $R' \leq R$. Thus our body L can be imbedded in a ball of radius R. Since the body L is a convex finite n-dimensional polyhedron, a simplex, imbedded in the body L, can be found whose vertices are among the vertices of the body L. Thus we have found a simplex S, imbedded in a ball of radius R, whose vertices are points of the system $\mathscr{E}$. By virtue of the previous lemma the simplex S has at least one edge of length not greater than $\sqrt{2(n+1)/n}R$. Thus we have found the required pair of points and the proof of the theorem is complete.

We remark that Theorem 2 is the analog of a theorem of Rogers (see [1], Chap. 7) in the theory of packing and of a theorem of Coxeter, Few, and Rogers (see [1], Chap. 8) in the theory of coverings. Although our theorem is easily derivable from these theorems, we prefer to give an independent proof of it since it is the simplest of all three of these theorems.

COROLLARY. The lattice constructed on a right triangle is a unique two-dimensional (r, R)-system for which the value $\varkappa(2) = \sqrt{1/3}$ is attained.

This follows from Theorem 2 and the uniqueness of the decomposition of the plane into right triangles (to within similarity), and also from the biunique correspondence [9] between the decompositions $\{L\}$ and (r, R)-systems.

4°. A Generally Exact Theory of the Density of Lattice (r, R)-Systems. The discussion to follow will be conducted not in the space of an (r, R)-system but in the space E^N, where $N = n(n+1)/2$, the space of the coefficients of quadratic forms in n variables [6, 8, 9]. To positive quadratic forms in this space there correspond points of some convex conical set, the cone of positivity being denoted by **K**. We denote by $\mathfrak{G}$ the group of (affine) transformations of the cone **K** into itself, generated by all the integral unimodular transformations of the variables in the quadratic forms considered. We also employ here the theory of lattice types (parallelohedra) [11], the information needed from which is presented, for example, in [3] and [5].

THEOREM 3. Let Δ denote some n-dimensional lattice-type domain, closed relative to the cone **K**. Points, to which there correspond lattices with the smallest possible value, for a given type, of the number R^*/r^*, fill up a convex cone Q of the space E^N (with vertex at the coordinate origin).

Proof. We consider the bounded body $W(\Delta)$ constructed in [9] and the polyhedron $M(m)$ constructed in [6] (see also [7]). Both of these bodies are convex. For some value of the parameter m these bodies do not intersect, but since on each ray belonging to the domain Δ and issuing from the coordinate origin, points can be found belonging to both the polyhedron $M(m)$ for arbitrary m and the body $W(\Delta)$, it follows that there exists a value of the parameter m for which the intersection $W(\Delta) \cap M(m)$ is not empty. Consequently, as can be easily seen, we can, in fact, by virtue of the closure of these bodies, find a smallest such m_0. According to the geometric meaning of the surfaces of our two bodies the intersection $Q' = W(\Delta) \cap M(m_0)$ consists of points corresponding to lattices of our type, having a covering radius equal to one and a minimal vector (of length $\sqrt{m_0}$), the smallest possible for such lattices. We remark that the set Q' has no points on the surface of the cone K, since no such points exist in the polyhedron $M(m_0)$. Noting that, as the intersection of convex bodies, the set Q' is convex and that the required set Q is a cone with vertex at the coordinate origin, constructed on Q', we have proved our theorem completely since the absence of other (weakly) extremal points is obvious by virtue of the convexity of the bodies $W(\Delta)$ and $M(m_0)$.

We remark that in our theory Theorem 3 is the analog of a theorem of Barnes and Dickson [3].

THEOREM 4. We denote by $\mathfrak{G}_\Delta$ the subgroup of those transformations of the group $\mathfrak{G}$, which transform the domain Δ into itself. Then among the points of the set Q we can find points invariant relative to the group $\mathfrak{G}_\Delta$.

Proof. We consider an arbitrary point $f \in Q$ and its images $\mathfrak{g}f$ under all the transformations $\mathfrak{g}$ from the group $\mathfrak{G}_\Delta$. We also consider the centroid f' of this system of points. Since the set Q is convex, the point f' belongs to the set Q. Since for each transformation $\mathfrak{g} \in \mathfrak{G}_\Delta$ the points $\mathfrak{g}f$ merely change places, we have $\mathfrak{g}f' = f'$ for an arbitrary transformation $\mathfrak{g} \in \mathfrak{G}_\Delta$. Thus the theorem is completely proved.

We remark that this theorem also is an analog of a theorem of Barnes and Dickson [3] in the theory of coverings.

5°. Examples. Partial Results. Suppose now that the domain Δ is not an arbitrary type domain, but a principal domain of the first type [11, 9]. The group $\mathfrak{G}_\Delta$ for this domain is such [11, 3] that there is only one absolutely fixed line λf relative to the transformations of this group, where by f we mean the principal form of the first type

$$nx_1^2 + nx_2^2 + \ldots + nx_n^2 - 2x_1x_2 - 2x_1x_3 - \ldots - 2x_{n-1}x_n.$$

It follows from this that the smallest value of the ratio R^*/r^* for lattices of the first type is attained on the lattice Γ_1^n with the metric form f; this ratio can be easily calculated and is equal to

$$\sqrt{(n+2)/12}.$$

We can make various conclusions from this equation.

a) For an n-dimensional lattice we have

$$\varkappa_\Gamma(n) \leqslant \sqrt{(n+2)/12},$$

which for $n > 10$ is poorer than the available estimate for an arbitrary system.

b) For an arbitrary (r, R)-system, since $\varkappa_\Gamma(n) \geq \varkappa(n)$, we have

$$\varkappa(n) \leqslant \sqrt{(n+2)/12} \quad \text{for} \quad n \leqslant 10.$$

c) For $n = 2, 3$ we have $\varkappa_\Gamma(n) = \sqrt{(n+2)/12}$, and, since there is only one lattice type for these dimensions, there are no locally extremal values of the number R^*/r^* in these dimensions.

6°. As may readily be seen, when $n = 2$ the cone Q degenerates to a point. It would be of interest to clarify the form of the cones Q for $n > 2$. It would also be interesting to find all numbers $\varkappa(n)$ and $\varkappa_\Gamma(n)$, which it is entirely possible to do for the number $\varkappa_\Gamma(4)$. We note that when $n = 4$ a lattice of centered cubes yields the same bound $\sqrt{1/2}$ for the number $\varkappa_\Gamma(4)$ as does the lattice Γ_1^4.

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