Math. Z. 113, 106–112 (1970)

Rings with the Minimum Condition for Principal Right Ideals Have the Maximum Condition for Principal Left Ideals

DAVID JONAH

Introduction

A ring R is said to have the ascending chain condition on cyclic left modules if each ascending chain of cyclic submodules of a module terminates. If a ring R has this property then clearly it has the ascending chain condition on principal left ideals. The ring of integers shows that the converse is false.

Main Theorem. *A ring R is perfect if and only if it has the ascending chain condition for cyclic left modules.*

Bass [1] called a ring perfect if each left R-module has a projective cover. He showed that a ring was perfect if and only if it satisfied the descending chain condition for principal right ideals; furthermore, he showed that this was equivalent to the Jacobson radical N being T-nilpotent and R/N having the descending chain condition. An ideal is called T-nilpotent if for each sequence $\{r_n\}$ of elements of the ideal there is an integer k such that the product $r_1 r_2 \ldots r_k$ is zero. Thus the rings with descending chain condition are perfect rings; in this case the radical is nilpotent. There are examples of perfect rings for which the radical is not nilpotent.

The author is indebted to his colleague Richard Courter for many encouraging conversations; in particular, for his matrix proof that full matrix rings over perfect local rings have the maximum condition for left principal ideals. The corresponding property for perfect rings was a natural conjecture. An examination of the "matrix" rings

$$\begin{bmatrix} R & M \\ 0 & S \end{bmatrix} = \left\{ \begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \middle| r \in R,\ s \in S,\ m \in M \right\}$$

where M is an $R-S$ bimodule and where R and S are perfect rings, quickly showed that if the perfect rings $\begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ were going to have the ascending chain condition on principal left ideals then all left R-modules would have the ascending chain condition on cyclic submodules.

Section 1. Perfect ⇒ acc-*n*

For the purposes of the proof we find it convenient to introduce the following provisional definition:

A ring R is said to have acc-n iff each ascending chain of left R-modules each generated by less than or equal to n elements terminates.

In order to work efficiently with the definition we will use the direct limit of a sequence of modules as explained below. Using it we will see that:

Lemma. (i) *The factor ring of a ring with* acc-n *has* acc-n*; and*
(ii) *the direct product of finitely many rings with* acc-n *again has* acc-n.

Morita techniques will then show:

Proposition 1.1. *R has* acc-n *iff the ring $M_n(R)$ of $n\times n$ matrices over R has* acc-1.

Corollary. *A skew field has* acc-n *for all positive integers n.*

From the Wedderburn theorem and the observation that the finite direct product of rings with acc-1 has acc-1 we have:

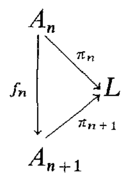
Corollary. *A semi-simple ring with the descending chain condition has* acc-n *for all n.*

The precision implied by the definition of acc-n does not exist because a simple corollary of the main theorem is that acc-1 always implies acc-n, as it is well known that the ring of $n\times n$ matrices over a perfect ring is again perfect.

We will also use the notion and a few simple properties of the direct limit of a sequence

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \to \cdots \to A_n \xrightarrow{f_n} A_{n+1} \to \cdots \tag{1}$$

of left R-module homorphisms. Recall that a *limit* is a family $\{\pi_n\colon A_n \to L\}_n$ of homomorphisms which satisfy

(L$_1$) 

is commutative for all $n\geqq 1$;

(L$_2$) For any family $\{\rho_n\colon A_n \to M\}_n$ also satisfying $\rho_{n+1} f_n = \rho_n$, $n=1,2,\ldots$, there is a unique homomorphism $\theta\colon L\to M$ such that

$$\theta\pi_n = \rho_n, \qquad n=1,2,\ldots.$$

Because of the usual uniqueness proof we will talk about *the* direct limit. The second condition can be replaced by

(L$_2'$) (i) L is the union of the images of the π_n;
(ii) if $\pi_n(x)=0$, then for some $k\geqq n$ the composition

$$f_k \cdots f_{n+1} f_n$$

is zero on x.

We will use the fact that the direct limit is an exact functor from the category of sequences to the category of R-modules and that it commutes with the tensor product; the latter means that if B is an $S-R$ bimodule, then the direct limit of the sequence

$$B\otimes_R A_1 \to \cdots B\otimes_R A_n \to \cdots,$$

where the maps are $1\otimes f_n$, is

$$\{1\otimes \pi_n\colon B\otimes_R A_n \to B\otimes_R L\}_n,$$

where

$$\{\pi_n\colon A_n \to L\}_n$$

is the direct limit of the sequence (1).

A module M is said to have property T if for each sequence (1) where

$$A_n = M, \qquad \text{all } n,$$

there is an integer k such that

$$\pi_k\colon M \to L$$

is onto, where $\{\pi_n\colon M \to L\}_n$ is the limit.

Clearly, if the limit L is always finitely generated, then by (L_2') (i) M has property T.

Proposition 1.2. *The ring R has* acc-*n if and only if the left R-module $R^n = R\times\cdots\times R$ (n copies) has property T.*

Simple Morita theory then allows up to prove Proposition 1.1.

Lemma. *The ideal I is T-nilpotent* $\Leftrightarrow$

$$\textit{for all } L, \quad R/I\otimes_R L = 0 \Rightarrow L = 0.$$

Proof. $R/I\otimes_R L = L/IL$. Bass shows that $IL = L$ for $L \neq 0$ implies L is not T-nilpotent [1; Proof of Lemma 2.6]. Conversely, if I is not T-nilpotent there is a sequence $\{r_n\}$ of elements of I such that the products

$$r_1 r_2 \ldots r_k, \qquad k = 1, 2, \ldots, \tag{2}$$

are never zero.

Consider a sequence (1) where each $A_n = R$ and f_n is right multiplication by r_n. As the products (2) are never zero, Condition (L_2') (ii) shows that the direct limit L is not zero. But as each $r_n \in I$ each

$$1\otimes f_n\colon R/I\otimes_R R \to R/I\otimes_R R$$

is zero, giving $R/I\otimes_R L = 0$ with $L \neq 0$.

Corollary. *If I is a T-nilpotent 2-sided ideal of R, then a left R-module homomorphism*

$$f\colon M \to N$$

is an epimorphism if and only if the morphism

$$R/I\otimes f\colon R/I\otimes_R M \to R/I\otimes_R N$$

is an epimorphism.

Proposition 1.3. *If $S=R/I$ has* acc-1 *where I is a 2-sided T-nilpotent ideal of R, then R has* acc-1.

Proof. By Proposition 1.2 we need only show that R has property T. Let $\{f_n\}$ be a sequence of left R-module morphisms $R\to R$. As I is a two-sided ideal they induce S-module morphisms $\bar{f}_n\colon S\to S$; treating S as an R-module via the natural map $R\to S$ these morphisms are also R-module morphisms. The direct limit of the sequence $\{\bar{f}_n\}$ as R-modules is the direct limit as S-modules; in fact, we have a commutative diagram:

$$\begin{array}{ccc} S\otimes_R R & \xrightarrow{1\otimes f_n} & S\otimes_R R \\ \downarrow & & \downarrow \\ S & \xrightarrow{\bar{f}_n} & S \end{array}$$

of R-modules, where the vertical arrows represent the natural isomorphism $S\otimes_R R\to S$. Then as R-modules the direct limit of $\{\bar{f}_n\}$ is isomorphic to the direct limit of the sequence $\{1\otimes f_n\}$, i.e., is

$$1\otimes\pi_n\colon S\otimes_R R\to S\otimes_R L.$$

But as S has acc-1 it has property T as an R-module. Hence there is an integer k such that

$$1\otimes\pi_k\colon R/I\otimes_R R\to R/I\otimes_R L$$

is an epimorphism. As I is T-nilpotent π_k is an epimorphism as desired.

This completes the proof that a perfect ring has acc-1.

Section 2. acc-1 ⇒ Perfect

The proof that rings with acc-1 are perfect – for our needs, the Jacobson radical N is T-nilpotent and R/N has the descending chain condition – will start with the proposition:

Proposition 2.1. acc-1 *implies that the radical is T-nilpotent.*

As the factor ring of a ring with acc-1 again has acc-1 we may restrict ourselves to rings with the Jacobson radical zero; such rings are subdirect products of primitive rings. The proof then proceeds as follows.

Proposition 2.2. *Primitive and* acc-1 *implies simple and the descending chain condition.*

Proposition 2.3. *R having* acc-1 *and being the subdirect product of simple rings makes R the direct product of finitely many simple rings. In particular, if each of the simple rings has the descending chain condition then R itself has the descending chain condition.*

These three propositions show that acc-1 implies perfect, completing our main theorem.

Proof of Proposition 2.1. Let $\{r_n\}_n$ be a sequence of elements of the radical, let $A_n = R$ and let f_n be multiplication by r_n for $n = 1, 2, \ldots$. The direct limit L is certainly finitely generated as R has acc-1. Just as clearly, $R/N \otimes_R L = 0$. By Nakayama's lemma, $L = 0$.

For the next two more ring theoretic proofs we find it advisable to convert to an "elemental" statement concerning acc-1.

Lemma. *For a ring R the following statements are equivalent.*

(1) *R has* acc-1.

(2) *For each sequence $\{r_n\}_{n=1}^{\infty}$ of elements of R there are integers j, k, with $k > j$ and an element s of R such that*

$$s\, r_j r_{j+1} \ldots r_k = r_{j+1} \ldots r_k. \tag{3}$$

(3) *For each sequence $\{r_n\}_{n=1}^{\infty}$ of elements of R there is an integer N such that for each $j \geqq N$ there are an element s of R and an integer $k > j$ such that Eq.* (3) *holds.*

Proof of Proposition 2.2. If the primitive ring R does not have the descending chain condition then it can be considered as a dense ring of linear transformations over an infinite dimensional vector space V. We will construct a sequence of elements $\{r_n\}$ of R such that Eq. (3) never holds.

As V is assumed to be infinite dimensional let $\{x_i\}_{i=0}^{\infty}$ be a countable sequence of linearly independent elements of R. Because R is dense in the space of all linear transformations $V \to V$ there are linear transformations r_i representing elements of R which satisfy

$$r_i(x_j) = \begin{cases} x_j & \text{for } 0 \leqq j \leqq i-1 \\ 0 & \text{for } i = j \end{cases} \qquad i = 1, 2, \ldots. \tag{4}$$

Let $k > j$ be positive integers and let s be an element of R; then Eqs. (4) show that

$$s\, r_j r_{j+1} \ldots r_k(x_j) = 0,$$

while

$$r_{j+1} \ldots r_k(x_j) = x_j,$$

which is certainly not zero as the sequence $\{x_j\}$ is linearly independent.

Proof of Proposition 2.3. By assumption there is a nonempty family P of ideals $\mathscr{A}$ of R such that

$$R/\mathscr{A} \quad \text{is simple}$$

and

$$\bigcap P = 0.$$

We want to know that if such a ring R also has the acc-1, then it has a finite subfamily F such that

$$\bigcap F = 0,$$

for as is well known this will say that R is actually a finite product of simple rings.

Actually we set out to prove the contrapositive: If for every finite subset F of P the intersection is non-zero then R does not satisfy acc-1.

As the intersection of each finite subset is nonzero and as the total intersection is zero we may choose by induction a sequence $\{\mathscr{A}_n\}_{n=0}^{\infty}$ of (non-zero) elements of the family of ideals P for which

$$\mathscr{A}_0 \cap \mathscr{A}_1 \cap \cdots \cap \mathscr{A}_{i-1} \not\subset \mathscr{A}_i \qquad \text{for } i=1,2,\ldots.$$

Thus for each positive integer i there is an element c_i of R such that the image $c_i(j)$ of c_i in $R/\mathscr{A}_j$ satisfies

$$c_i(j)=\begin{cases}0 & \text{for } 0\leqq j\leqq i-1\\ \text{nonzero} & \text{for } i=j.\end{cases}$$

Since $c_i(i)$ is a nonzero element of a simple ring $R/\mathscr{A}_i$ there are elements $\{d_i\}_{i=1}^{\infty}$ of R such that

$$d_i(j)=\begin{cases}0 & \text{for } 0\leqq j\leqq i-1\\ 1 & \text{for } i=j.\end{cases}$$

Using these we may define a sequence of elements $\{r_i\}_{i=1}^{\infty}$ of R such that

$$r_i(j)=\begin{cases}1 & \text{for } 0\leqq j\leqq i-1\\ 0 & \text{for } i=j.\end{cases}$$

Then acc-1 does not hold in R because for any s in R and any pair of positive integers j, k with $j\leqq k$ we have

$$s\, r_j r_{j+1} \ldots r_k(j)=0$$

while

$$r_{j+1}\ldots r_k(j)=1\neq 0.$$

Thus R does not have acc-1. Thus if a ring R with acc-1 is a subdirect product of simple rings then it is a finite product of simple rings.

Section 3

McCoy [5] called a ring π-regular if for each element x of the ring some positive power x^n of it is (von Neumann) regular; Fuchs and Rangaswamy [3] call a ring $\bar{\pi}$-regular if for each element x there is a positive integer m such that x^n is regular for all $n\geqq m$. As the following proposition shows, perfect rings satisfy a stronger property.

Proposition 3.1. *If $\{r_n\}_{n=1}^{\infty}$ is a sequence of elements of a perfect ring R then there is a pair of integers j, m with $j\leqq m$ such that for each $k\geqq m$ the product*

$$r_j \ldots r_k$$

is regular.

Proof. Let $A_n = R$ and let $f_n: A_n \to A_{n+1}$ be multiplication by r_n for $n \geqq 1$. By Bass [1] the direct limit L is projective, and by our main theorem there is an integer j such that

$$\pi_j: A_j = R \to L$$

is an epimorphism where $\{\pi_n: A_n \to L\}_n$ is the direct limit. As L is projective there is a homomorphism $s: L \to R$ such that $\pi_j s = 1_L$. Let $e = s\pi_j(1)$.

Then

$$\pi_j(e) = \pi_j(1).$$

This will mean that there is an integer $m \geqq j$ such that

$$(1-e)\, r_j \dots r_k = 0 \quad \text{for } k \geqq m. \tag{$*$}$$

From $e = s\pi_j(1)$ and the compatibility relation

$$\pi_{n+1} f_n = \pi_n \qquad \text{which gives } r_n \pi_{n+1}(1) = \pi_n(1)$$

we have

$$e = r_j \dots r_k\, y_k \qquad \text{for } k \geqq m, \text{ where } y_k = s(\pi_k(1)). \tag{$**$}$$

Hence

$$\begin{aligned} &r_j \dots r_k\, y_k\, r_j \dots r_k \\ &\quad = e\, r_j \dots r_k \qquad \text{by } (**) \\ &\quad = r_j \dots r_k \qquad \text{by } (*). \end{aligned}$$

References

1. Bass, H.: Finitistic dimension and a homological generalization of semi-primary rings. Trans. Amer. Math. Soc. **95**, 466–488 (1960).
2. Faith, C.: Rings with minimum condition on principal ideals. Arch. der Math. **10**, 327–330 (1959) [cf. remarks added in proof].
3. Fuchs, L., Rangaswamy, K.M.: On generalized regular rings. Math. Z. **107**, 71–81 (1968).
4. Kaplansky, I.: Topological representation of algebras. II. Trans. Amer. Math. Soc. **68**, 62–75 (1950) [cf. Section 8].
5. McCoy, N.H.: Generalized regular rings. Bull. Amer. Math. Soc. **45**, 175–178 (1939).

Dr. David Jonah
Department of Mathematics
Wayne State University
Detroit, Michigan 48202, USA

(Received April 21, 1969)