$$K = [1-ut^2 4a/(x-x')^3]^{1/2}(2\pi i t)^{-1/2} exp\{i[(x-x')^2/2t-2aut/(x-x')]\} \quad (4.2)$$

and the reflection coefficient

$$R = erf(-\sigma p_0) + 4ua/(p_0^3 t). \quad (4.3)$$

We see that the first term in (3.11) and (4.3) corresponds to free motion, and the expansion parameters are the ratios of the potential energy to the kinetic and of the barrier width to the distance traversed. The last result can be generalized to any potential $U(x)$ that falls off rapidly enough at infiity

$$R = erf(-\sigma p_0) + 2\int_{-\infty}^{+\infty} U(x)dx/(p_0^3 t). \quad (4.4)$$

At large values of $\sigma\rho_0$, the last term plays the decisive role.

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# QUANTUM UNCERTAINTY AND ITS ROLE IN NONLINEAR PROPAGATION OF A NONLINEAR SOLITON IN A LIGHTGUIDE


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*An exact solution of the Schrödinger quantum equation is used to investigate the evolution of a fundamental optical soliton in its proper waveguide having a Kerr nonlinearity. It is established that the quantum fluctuations grow unceasingly over the entire length of the nonlinear propagation, so that the soliton is ultimately annihilated. A four-photon interaction model is used to clarify the physical nature of this phenomenon. It is shown that the effects considered restrict the possibility of producing quantum squeezed states of a light pulse.*


## 1. Introduction

One of the most attractive features of optical Schrödinger solitons produced and propagating in nonresonant cubically nonlinear media is their stability. The forms and regularity of the phase of a fundamental soliton follows from the classical Schrödinger equation. Moreover, the soliton is stable against initial noise modulation and "clears itself" of the fluctuation components in the course of nonlinear propagation [1, 2].

The quantum picture, unfortunately, is not so optimistic. Thus, one of the results of the development of a consistent quantum theory of the pulse evolution in nonlinear lightguides [3-10] is the conclusion that the phase and amplitude uncertainties increase and the soliton spread dispersively [11-13]. However, the approximations used in the cited references restrict the validity of these statements only to the initial nonlinear-propagation state.


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What happens next? Does the classical "self-clearing" property compensate for the growth of the quantum fluctuations on going to the far zone? After all, elimination of soliton noise is observed precisely at considerable propagation paths. Or does the destabilizing influence of the quantum uncertainty grow continuously and leads ultimately to destruction rather than formation of an ideal soliton? The answers to these and other questions is the subject of the present paper.

## 2. Basic Relations

The evolution of the electric field of one-dimensional radiation entering a transparent medium having a cubic nonlinearity can be described, in second-order dispersion theory, by the equation [3-13]

$$(\partial/\partial z + u^{-1}\partial/\partial t)E^{(+)}(z,t) = [(i/2)g\partial^2/\partial t^2 + (ik\varepsilon_{HЛ}/2\varepsilon_0) \times E^{(-)}(z,t)E^{(+)}(z,t)]E^{(+)}(z,t). \tag{2.1}$$

Here $E^{(+)}(z, t)$ and $E^{(-)}(z, t)$ are the operators of the positive- and negative-frequency parts of the field in the Heisenberg representation and vary slowly with time, the z-axis is directed along the propagation path, $t$ is the time, $u = (\partial k/\partial\omega)^{-1}$is the group velocity at the carrier frequency $\omega$, $k$ is the carrier wave number, the parameter $g = \partial^2k/\partial\omega^2$is indicative of the group-velocity dispersion, and $\varepsilon_{nl}$and $\varepsilon_0$ are the nonlinear and linear parts of the dielectric constant of the medium. It is assumed that the interaction is collinear, the propagation mode is planar, and the nonlinearity is instantaneous. A rather detailed derivation of (2.1) can be found, for example, in [4, 9] and we shall therefore not dwell on it in detail. We note only that for a transition to the classical equation [1] it suffices to replace $E^{(+)}$ and $E^{(-)}$ by a pair of complex amplitudes $A$ and $A^+$.

Relation (2.1) reduces to a nonlinear quantum Schrödinger equation by introduction of the dimensionless variables

$$x = ut - z,\quad S = gu^2z/2,$$
$$\phi(S,x) = E^{(+)}(z,t)/|A_0|,\quad c = -k\varepsilon_{n\ell}|A_0|^2/2u^2g\varepsilon_0, \tag{2.2}$$

where $x$ is the deviation from the crest of a pulse propagating with velocity $u$, $S$ is the normalized distance traveled by the pulse, $\phi(S, x)$is the normalized photon-annihilation operator at the points $S$ and $x$, and $A_0$ is the pulse amplitude at its crest.

We adhere to the notation widely used in the literature, we replace the variable $S$ by $t$, which has in fact the meaning of the normalized propagation time. We obtain then

$$i\partial\phi(t,x)/\partial t = -\partial^2\phi(t,x)/\partial x^2 + 2c\phi^+(t,x)\phi(t,x)\phi(t,x). \tag{2.3}$$

This equation, as well as the operators it contains, is written in the Heisenberg representation. The following commutation relations should then be satisfied:

$$[\phi(t,x),\phi^+(t,x')] = \delta(x-x'),$$
$$[\phi(t,x),\phi(t,x')] = [\phi^+(t,x),\phi^+(t,x')] = 0. \tag{2.4}$$

It is more convenient, however, to obtain an exact solution by using the Schrödinger representation. The transformation of the system state vector $|\psi\rangle$ is then described by

$$i\hbar d|\phi\rangle/dt = H|\phi\rangle, \tag{2.5}$$

with a Hamiltonian

$$H = \hbar[\int\phi_x^+(x)\phi_x(x)dx + c\int\phi^+(x)\phi^+(x)\phi(x)\phi(x)dx]. \tag{2.6}$$

From now on, unless otherwise stipulated, the integration is between the infinite limits.

A soliton-like solution of (2.5) exists only for negative $c$. It has the form a superposition of Fock states $|n, p\rangle$ with a definite number $n$ of photons and a momentum $p$ [11]

$$|\phi\rangle = \sum_n a_n\int g_n(p)e^{-iE(n,p)t}|n,p\rangle dp. \tag{2.7}$$

Here $|n, p\rangle$ are eigenstates for the Hamiltonian (2.6):

$$|n,p\rangle = (n!)^{-1/2}\int f_{np}(x_1,\ldots,x_n)\phi^+(x_1)\ldots\phi^+(x_n)dx_1\ldots dx_n|0\rangle, \tag{2.8}$$

$$f_{np}(x_1,\ldots,x_n) = N_n\exp[ip\sum_{j=1}^n x_j + (c/2)\sum\sum_{1\leqslant i<j\leqslant n}|x_j - x_i|, \tag{2.9}$$

and the normalization factor $N_n$ is determined from the condition $\langle n', p'|n, p\rangle = d_{nn'}\delta(p-p')$:

$$N_n^2 = |c|^{n-1}(n-1)!/2\pi, \quad (2.10)$$

with

$$\int |f_{np}(x_1,\ldots,x_n,t)|^2 dx_1 \ldots dx_n = 1. \quad (2.11)$$

The energies are eigenvalues for the Hamiltonian (2.6) and the state (2.8):

$$E(n,p) = np^2 - c^2 n(n^2-1)/12. \quad (2.12)$$

If the pulse entering the lightguide is an aggregate of coherent modes, the weighting factors $a_n$ and functions $g_n$ have respectively Poisson and Gaussian distributions:

$$a_n = \alpha_0^n \exp(-n_0/2)/(n!)^{1/2},$$
$$g_n(p) = \pi^{-1/4}\Delta p^{-1/2}\exp[-(p-p_0)^2/2\Delta p^2 - inpx_0], \quad (2.13)$$

where $n_0 = |\alpha_0|^2$ is the average number of photons in the pulse, and

$$\sum_n |a_n|^2 = 1, \quad \int |g_n(p)|^2 dp = 1. \quad (2.14)$$

We determine first the average amplitude of the pulse as in the course of its nonlinear propagation (a derivation is given in Appendix 1)

$$\langle\psi|\phi(x)|\psi\rangle \simeq \sum_n [n(n+1)/|c|q_1]^{1/2}\ a_n^* a_{n+1}\exp[itc^2 n(n+1)/4 +$$

$$+ [ip_0(x-x_0-p_0t) - (x-x_0)^2\Delta p^2/4]/q_1\}\int\exp\{(-[\Delta p^{-2}+i2t+$$

$$+4t^2 n(n+1)\Delta p^2]p^2 + i2(n+1/2)(x-x_0-2p_0t)p\}/q_1\}\operatorname{sech}(2\pi p/|c|)dp$$

$$q_1 = 1 + it\Delta p^2. \quad (2.15)$$

In contrast to [11] this equation was derived without any restrictions on the path length (the parameter t) and on other parameter. The approximate-equality sign applies only the average number $n_0$ of the photons, which exceed unity substantially, as is the case in practice.

During the initial stage ($t\Delta p^2 << 1$) and under the condition

$$|c| \ll \Delta p \ll n_0|c| \quad (2.16)$$

relation (2.15) reduces to a superposition of pulses with envelopes in the form of a hyperbolic secants, i.e., classical solitons. If, however, the spread in the number $n$ of photons and in the momentum $p$ is neglected, we obtain, putting $n = n_0$ and $p = p_0$, the fundamental classical soliton in pure form:

$$\langle\psi|\phi(x)|\psi\rangle \simeq 2^{-1}(n_0-1)|c|^{1/2}\exp[i(n_0-1)^2c^2t/4 + ip_0(x-x_0-p_0t)] \times$$
$$\times \operatorname{sech}[2^{-1}(n_0-1)|c|(x-x_0-2p_0t)]. \quad (2.17)$$

The distinctive feature of quantum treatment, however, is that exact knowledge of the number of photons $n$ and of the momentum $p$ makes absolutely unknown the phases and coordinates of the pulse. This means in turn that its mean amplitude is zero and the envelope is independent of $x$.

The necessary condition for the existence of a soliton is thus the presence of energy and momentum spreads. These, as we shall verify below, lead to rather sad consequences in the course of nonlinear propagation.

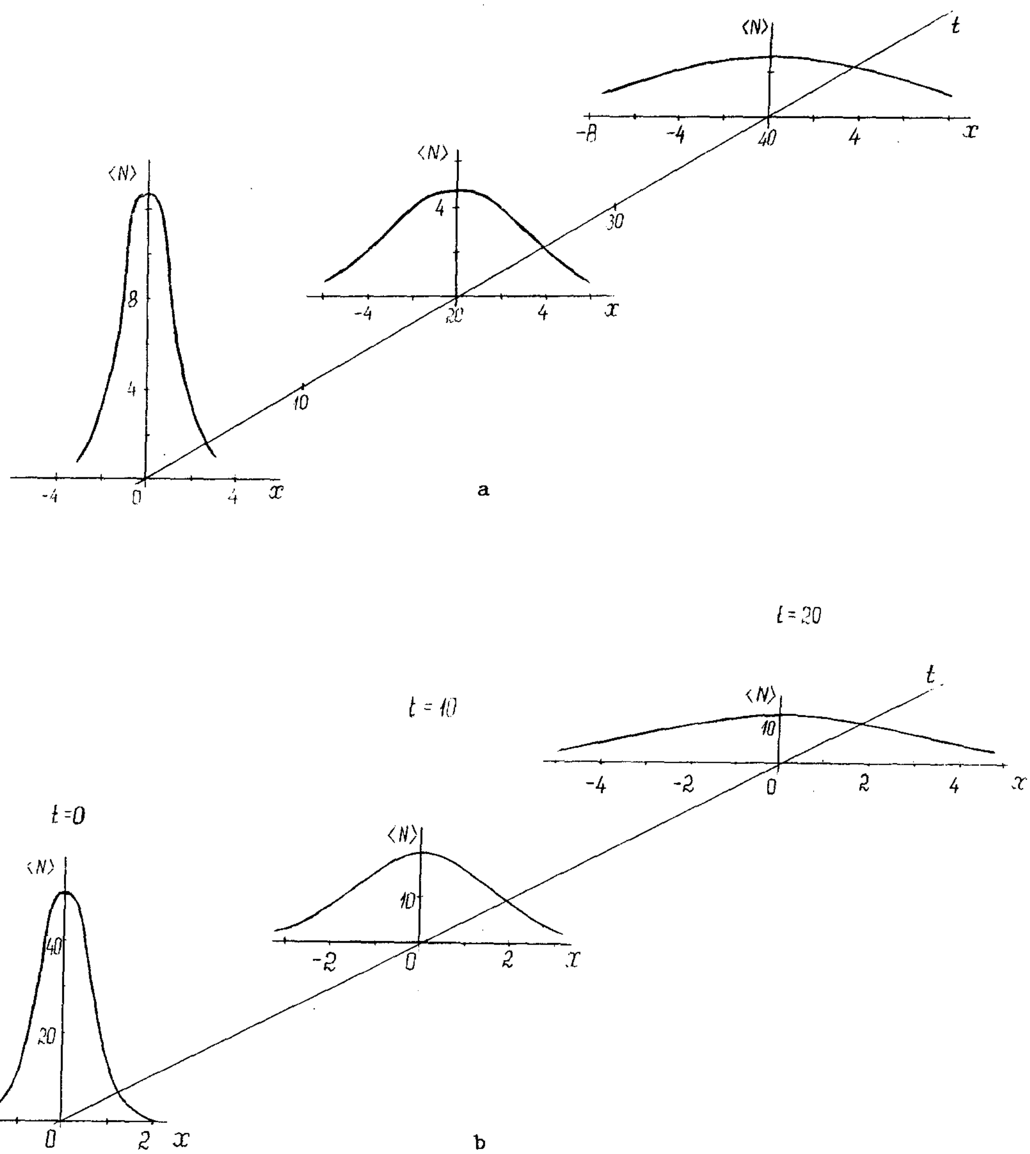


Fig. 1. Evolution of envelope of a nonlinearly propagating soliton: (a) $n_0 = 40$, (b) $n_0 = 80$. The remaining parameters are the same for both cases: $|c| = \pi/100$, $\Delta p = 0.1$, $p_0 = x_0 = 0$.

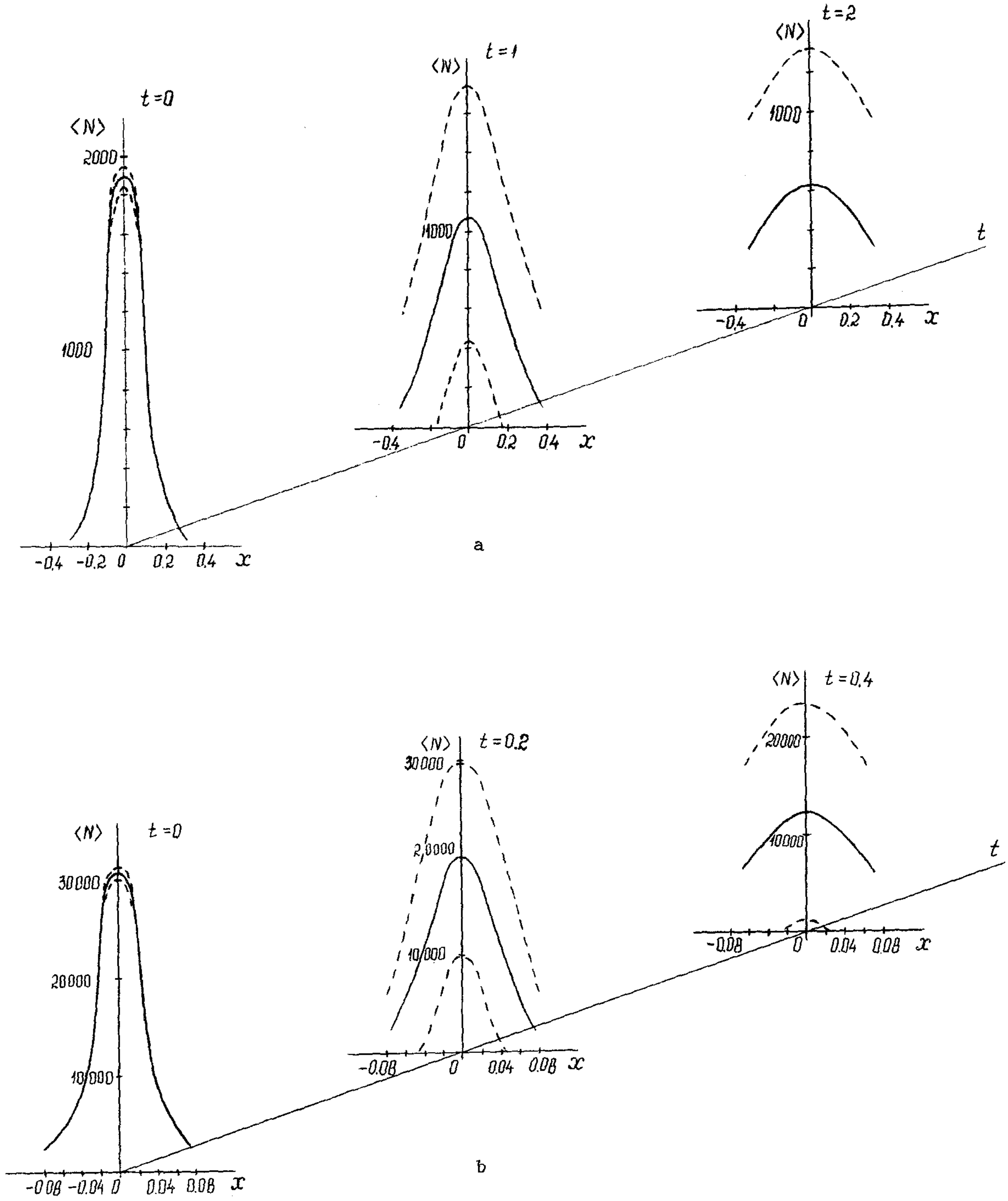


Fig. 2. Plots illustrating the disintegration of a nonlinearly propagating soliton. The dashed lines are the boundaries of the region of the photon-number quantum uncertainty, as calculated from (3.2) and (3.6). (a): $n_0 = 500$, (b): $n_0 = 2000$. The remaining parameters are the same as in Fig. 1.

## 3. Evolution of Soliton Shape and of the Photon Fluctuations

To make clear the pulse dynamics, we calculate first its average intensity:

$$\langle N(x)\rangle \equiv \langle\psi|\phi^{+}(x)\phi(x)|\psi\rangle = 2|c|^{-1}\exp(-n_0)\sum_n (n^2 n_0^n/n!)\int G_n(x,p)dp, \tag{3.1}$$

$$G_n(x,p) \simeq p\,\mathrm{sh}^{-1}(2\pi p/|c|)\,\exp[-(\Delta p^{-2}+4t^2n^2\Delta p^2)p^2+ + i2n(x-x_0-2p_0t)p]. \tag{3.2}$$

The approximate equality symbol in (3.2) applies only to the average number of photons in the pulse: $n_0 >> 1$. Just as in (2.15), no constraints are imposed on the path length or on any other quantities. Relations for them are given in Appendix 2.

To estimate the soliton evolution analytically, we assume initially that the spread of its components with respect to n plays a secondary role in this process, and assume $n = n_0$. The integrand in (3.1), apart from the exponential phase factor, is then the spectrum of the pulse. It can be seen that this spectrum becomes narrower with time, in view of the terms $\exp[-(2tnp\Delta p)^2]$, which leads in turn to a continuous spreading of the soliton in the course of its nonlinear propagation.

Further analysis of (3.1) and (3.2) shows that the characteristic time of twofold broadening depends on the ratio of the parameter $c$ to the range $\Delta p$ of the distribution function of the momenta. We can distinguish then between three regimes, for which

$$t_{char} \simeq 2/n_0|c|\Delta p \quad \text{for} \quad \Delta p \gg |c|, \tag{3.3}$$

$$t_{char} \simeq 2^{1/2}/n_0|c|\Delta p \quad \text{for} \quad \Delta p \simeq |c|, \tag{3.4}$$

$$t_{char} \simeq 3^{1/2}/2n_0\Delta p^2 \quad \text{for} \quad \Delta p \ll |c|. \tag{3.5}$$

Expression (3.3) was obtained earlier in [11], but only in the small-$t$ approximation. Here, however, it is generalized to include arbitrary times.

The validity of the assumption made and of these estimates is confirmed by numerical calculations in accordance with (3.1) and (3.2). The results are illustrated in Figs. 1 and 2. It can be seen that the soliton disintegrates completely in time. We become even more certain of the validity of this conclusion by determining the variance of the photon-number fluctuations:

$$\langle\Delta N^2(x)\rangle \equiv \langle\psi|\phi^{+}(x)\phi^{+}(x)\phi(x)\phi(x)|\psi\rangle + \langle N(x)\rangle - \langle N(x)\rangle^2 \simeq$$

$$\simeq [\exp(-n_0)/3]\sum_n n^2(n^2-1)(n_0^n/n!)\int(1+4p^2/c^2)G_n(x,p)dp + + \langle N(x)\rangle - \langle N(x)\rangle^2. \tag{3.6}$$

This relation is derived in Appendix 3.

The results of the numerical calculation are shown in Fig. 2. Unfortunately, expression (3.6) turns out to be more sensitive than (3.1) and (3.2) to satisfaction of the condition $n_0 >> 1$. The ranges of the photon-number quantum uncertainty are therefore not indicated in Fig. 1. The approximate relation (3.6), however, is perfectly applicable for $n_0 = 500$ and more.

It follows from the foregoing that nonlinear propagation of a soliton is accompanied not only by spreading of the envelope, but also by a continuous growth of the amplitude fluctuations, so that the pattern of its gradual annihilation becomes even more aggravated. Thus, the initial Poisson statistics of the photons, with $\langle\Delta N^2(x)\rangle = \langle N(x)\rangle$, becomes super-Poisson with

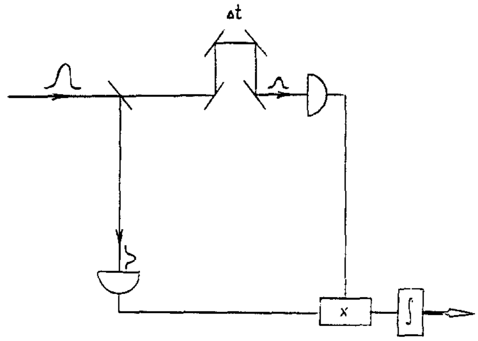


Fig. 3. Simplified correlator scheme. The pulse in one of the channels is delayed prior to detection by a time interval $\Delta t$ relative to the passage through the second channel.

$\langle \Delta N^2(x)\rangle > \langle N(x)\rangle$. This important conclusion could not be drawn in the earlier studies [3-13] because inadequate models were used for large ranges. For example, both the Hartree approximation [11] and the quasistatic given-channel approximation [8-10] lead to the conclusion that the photon statistics remains unchanged in nonlinear propagation and retains its Poisson character.

The results in Fig. 2 lead also to the conclusion that the pulse becomes "all noise" just when its width is doubled.

What is the cause of this behavior? Why does not the soliton get rid of the fluctuations by classical "self-clearing" [1, 2]? We explain the resultant situation by using the following model.

The initial soliton entering the light guide and constituting an aggregate of modes in coherent states having various amplitudes can be represented as a superposition of a classical envelope in the form of a hyperbolic secant (regular component of the signal) and quantum vacuum fluctuations (noise).

The initial noise modulation, present only during the soliton lifetime, is "dumped" to the wings in the course of propagation, and the soliton is gradually "self-cleared." It is unable, however, to be free of the stationary vacuum noise, since such a "dumping" is accompanied by "inflow" of the fluctuations initially present in the solitons.

Why, then, are these oppositely acting phenomena not balanced ultimately and stabilize the picture at some level of the growing fluctuations? To answer this question we analyze the nonlinear evolution of vacuum noise in the presence of an intense regular soliton component. We use the Heisenberg representation of the Schrödinger equation (2.1) or (2.3). We linearize it with respect to the fluctuation components and consider for simplicity and clarity a single-mode interaction regime. As a result we find that the average number of noise photons is [13]

$$\langle N_{III}\rangle = \Psi^2, \tag{3.7}$$

where $\Psi = t(cn_0)^2/2$ is the nonlinear phase shift in the propagation, and $n_0$ the average number of photons in the mode.

Linearization in terms of the fluctuation components means in fact the use of the model of four-photon parametric interaction in a specified classical pump field (regular signal). In accordance with (3.7), the noise intensity increases continuously through transfer of photons from the regular component to the fluctuation one by four-photon parametric amplification.

This is not the only cause of destabilization. On going to the far zone, where the linear approximation no longer holds, the increase of the fluctuations is accompanied by depletion of the soliton itself, a process that pumps parametrically amplified vacuum noise, since the total number of photons must remain unchanged.

These irreversible processes cause the soliton to spread out gradually and become ultimately degraded.

By now the reader is apparently asking himself a valid and fundamental question not yet raised by us: After all, calculation of $\langle N(x)\rangle$ and $\langle \Delta N^2(x)\rangle$ provides no valid grounds for distinguishing between real soliton spreading, on the one hand, and simple uncertainty with respect to $x$. Does the soliton actually spread out, or is the $\langle N(x)\rangle$ broadening due to averaging, over the ensemble of solitons that begin to propagate at the same instant of time and are then subject to a quantum scatter in the coordinate $x$, i.e., to different time delays? Quantum-mechanical averaging does not distinguish in this case between two such unlike evolutions.

The situation, however, is not hopeless. To clarify the true state of matters we can calculate the intensity correlation function

$$K(\Delta x) = \int \langle N(x) N(x+\Delta x)\rangle dx. \tag{3.8}$$

This correlation function is physically realized, for example, in measurements of ultrashort-pulse durations. A rough scheme is shown in Fig. 3. The result of measurements in this variant are independent of the absolute instant of pulse arrival, but is determined only by the delay time. Consequently, if the soliton does not spread during the nonlinear propagation, the correlation function after the soliton passes through the fiber will be exactly the same as the input (at $t = 0$), since the $x$ uncertainty should not come into play in this case. If the pulse spreads, however, $K(\Delta x)$should broaden in accordance with the degree of this spreading.

Thus,

$$\langle N(x) N(x+\Delta x)\rangle \equiv \langle\psi|\phi^+(x)\phi^+(x+\Delta x)\phi(x+\Delta x)\phi(x)|\psi\rangle + N(x)\delta(\Delta x) =$$

$$= \pi^{-1/2}\Delta p^{-1}\exp(-n_0) \sum_n (n_0^n/n!) \iint \exp(-[(p-p_0)^2+(p'-p_0)^2]/2\Delta p^2+$$

$$+in[x_0(p'-p)+t(p'^2-p^2)]]F_n(x,\Delta x,p-p')dp'dp + N(x)\delta(\Delta x), \tag{3.9}$$

where the matrix element is

$$F_n(x,\Delta x,p-p') = \langle n,p'|\phi^+(x)\phi^+(x+\Delta x)\phi(x+\Delta x)\phi(x)|n,p\rangle.$$

Direct calculations of $F_n$ are unfortunately very difficult and do not yield in general analytic results. We use therefore the following indirect estimate.

We change to new variables $p_1 = (p - p')/2$, $p_2 = (p + p')/2$ and integrate the resultant relation with respect to $p_2$. The result, with the subscript of $p_1$ dropped, is

$$K(\Delta x) = n_0\delta(\Delta x) + 2\exp(-n_0) \sum_n (n_0^n/n!) \int \exp(-[(\Delta p^{-2}+4t^2n^2\Delta p^2)p^2+$$
$$+ i2n(2p_0t+x_0)p]][\int F_n(x,\Delta x,2p)dx]dp. \tag{3.10}$$

We can put $x_0 = p_0 = 0$ without loss of generality. Integration with respect to the momentum $p$ means then that the expression under the inner integral sign is in fact approximately the soliton spectrum. The validity of this approximation was confirmed by the calculations of $\langle N(x)\rangle$ and of the characteristic spreading times (3.3)-(3.5). Consequently, the spectrum becomes narrower durring the time $t$, and $K(\Delta x)$ should correspondingly broaden. This means that the soliton spreads nevertheless! Otherwise $K(\Delta x)$ would not change when $t$ is increased.

## 4. Phase and Frequency Fluctuations. Squeezed States

The processes considered so far concerned only the evolution of the amplitude characteristics. The spreading and the noise-intensity growth are far from all the concomitant phenomena. Thus the increase of the phase fluctuations during nonlinear soliton propagation turns out to be much swifter and manifests itself strongly even during the initial stage, i.e., in the near zone.

It is shown in [12, 13] that in the quasistatic approximation, which is valid for a given channel at small t, before the dispersive spreading considered above comes into play, the dispersion of the soliton phase fluctuations increases like

$$\langle\Delta\theta^2\rangle = [1 + 4\Psi^2(t)]/4n_0 \tag{4.1}$$

Here $\Psi(t) = t(cn_0)^2/2$ is double the nonlinear phase, i.e., the nonlinear phase advance in the absence of dispersion. A similar relation can be obtained also from estimates made in [11] using the Hartree approximation, which likewise ignores the dispersion spreading.

If the pulses are recorded with a square-law detector, the quantum uncertainty of the phase does not by itself influence directly the measurement results. It leads, however, to at least two undesirable consequences, even disregarding interference experiments, in which the phase stability is of primary significance.

First of all, the phase scatter leads inevitably to a corresponding frequency destabilization. Such a phenomenon was considered within the scope of the classical approach in [15], where the propagation of the fundamental soliton periodically enhanced to compensate for the loss in the fiber, was analyzed. Amplified with the soliton is also the concomitant spontaneous noise. The noise accumulates, increases, and the first symptom of its presence are random departures of the carrier frequency and the ensuing changes of the propagation velocity. This is extremely undesirable in information-carrying communication lines, for which the optical solitons are in fact mainly intended. Therefore certain limits are imposed on the operating range of the information channel.

The dispersion of the frequency fluctuations turned out therefore in the classical approximation to be [15]

$$\langle \Delta\omega^2 \rangle_{c\ell} = (e^{\gamma t} - 1)A/3n_0 \simeq A\gamma t/3n_0, \tag{4.2}$$

where $\gamma$ is the growth rate of the gain compensating for the losses, and $A = (n_0 - 1)|c|^{1/2}/2$ is the normalized soliton amplitude [see (2.17)].

The simple relation (4.2), which is convenient for practical computations, can be generalized also to take quantum fluctuations into account. This can be done by virtue of the correspondence principle [16], which is valid in the considered situation of linear amplification. Account must be taken, however, of the additional "noise making" due to the losses (for the gain to cancel the noise completely the growth rate must be doubled), and also the vacuum fluctuations (the factor 1/2). The result is

$$\langle \Delta\omega \rangle_{qu} \simeq A(2\gamma t + 1/2)/3n_0. \tag{4.3}$$

It is estimated [15] that the random frequency deviations should set in at distances exceeding 1000 km.

Another unpleasant consequence of the growth of the phase uncertainty is its destructive effect on the preparation of squeezed states.

Quantum squeezed states are known to permit suppression of photodetection shot noise and extend correspondingly the capabilities of various systems in which photons are the information carriers (see, e.g., [10, 17-20]. One of the most promising methods of preparing such states is precisely the use of optical solitons [3-10].

We introduce the quadrature component

$$x = \phi e^{-i\varphi} + \phi^+ e^{i\varphi}, \tag{4.4}$$

where $\varphi$ is a free phase parameter, the choice of which can optimize the depth of suppression of the dispersion of the quadrature fluctuations:

$$\langle \Delta x^2 \rangle \equiv \langle x^2 \rangle - \langle x \rangle^2 \equiv$$
$$\equiv 1 + (\langle \phi^2 \rangle e^{-i2\varphi} + \langle \phi^+\phi \rangle - \langle \phi \rangle^2 e^{-i2\varphi} - |\langle \phi \rangle|^2 + \text{K.C.}). \tag{4.5}$$

The criterion for the onset of a squeezed state is satisfaction of the condition $\langle \Delta x^2 \rangle < 1$, i.e., lowering the variance of the fluctuations below the vacuum level.

For the fundamental soliton we have

$$\langle \psi | \phi^2(x) | \psi \rangle \simeq [2\alpha_0^2 \exp(-n_0)/|c|q_2^{1/2}] \sum_n (n_0^n (n+1)^2/[n(n!)(n+$$
$$+1)!]^{-1/2}\} \exp(itc^2(n+1)^2/2 + [i2p_0(x-x_0-p_0t) - (x-x_0)^2\Delta p^2]/q_2\} \times$$
$$\times \int p \,\text{sh}^{-1}(2\pi p/|c|) \exp((-[\Delta p^{-2} + i4t + 4t^2 n(n+2)\Delta p^2]p^2 +$$

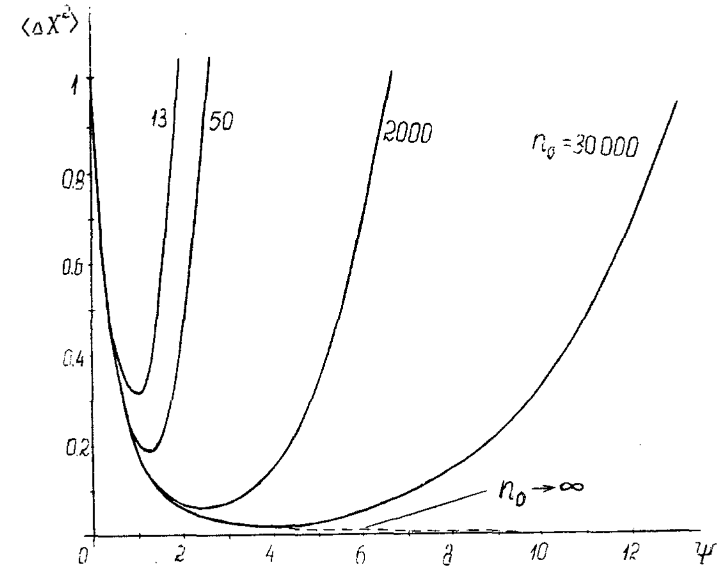


Fig. 4. Plots indicative of the evolution of the squeezing limit in linear propagation, i.e., with increase of $\Psi$, for different $n_0 = 13, 50, 2000$, and 30000. The dashed curve corresponds to the limit $n_0 \Rightarrow \infty$, i.e., to the ideal case of zero phase fluctuations.

$$+i2(n+1)(x-x_0-2p_0t)p]/q_2\}dp, \quad q_2 = 1 + i2t\Delta p^2. \tag{4.6}$$

This relation is derived in Appendix 4. Note only that the approximation sign is related to the condition $n_0 >> 1$.

Analyzing the structure of (4.6), we can conclude that as $t$ increases the modulus of $\langle\phi^2\rangle$ decreases much more intensively than $\langle\phi^+\phi\rangle$ defined in accordance with (3.1) and (3.2). This is due primarily to the presence in (4.6) of the phase term $\exp[itc^2(n+1)^2/2]$. In fact,

$$\sum_n [n_0^n \exp(-n_0)/n!]\exp[itc^2(n+1)^2/2] \simeq q^{-1/2}\exp[itc^2(n_0+1)^2/2q], \quad q = 1 - itc^2 n_0, \tag{4.7}$$

and an increase of $t$ is accompanied by a decrease of the absolute value of (4.7). We have assumed in these calculations that $n_0 >> 1$ and used a Poisson distribution of the form in (A.9).

However, the lowering of $|\langle\phi^2\rangle|$ in advance of $\langle\phi^+\phi\rangle$ decreases, according to (4.5), the maximally attainable squeezing. Thus, the effective generation of squeezed states during the initial stage of nonlinear propagation [3-11] should give way to their degradation.

One can point to at least two causes of this behavior.

In the framework of the foregoing four-photon model of nonlinear interaction with a soliton passing through a fiber, the regular component is so to speak a pumping of parametrically amplified vacuum fluctuations. In this case the pump first becomes depleted by transfer of the phonons to the noise component, and it is this which degrades the squeezing. A similar effect takes place, for example, also in three-photon parametric amplification [21].

The second cause is more substantial and is manifested already in an early stage of the propagation. It constitutes a growth of the soliton phase (i.e., pump) fluctuations whose destructive influence is also similar to the generation of squeezed states in parametric amplifiers, considered in [22-24] where, however, synchronous amplification was analyzed. In our case the interaction is fundamentally asynchronous in account of the presence of a nonlinear phase advance of the pump. More details on this difference can be found, e.g., in [13].

To obtain analytic estimates of the influence of phase fluctuations, we use the following simple model.

In the framework of the quasistatic approximation of the specified channel, the variance of the quadrature component for the soliton vertex is equal to [8-10]

$$\langle \Delta x^2(t,0)\rangle = 1 - 2\Psi \sin 2(\varphi - \Psi/2) + 4\Psi^2 \sin^2(\varphi - \Psi/2). \tag{4.8}$$

Optimal squeezing, i.e., a minimum of $\langle \Delta x^2 \rangle$ is obtained by choosing a phase parameter $\varphi = \varphi_0$ such that

$$\mathrm{tg}(2\varphi_0 - \Psi) = \Psi^{-1}. \tag{4.9}$$

The quantum spread of the soliton phase, however, makes satisfaction of the last condition impossible. It can thus be concluded that even in the optimum case

$$\langle \Delta x^2(t,0)\rangle_{min} \simeq 1 - 2\Psi \sin(2\varphi_0 + \langle \Delta\theta^2 \rangle^{1/2} - \Psi) + 4\Psi^2 \sin^2[(2\varphi_0 + \langle \Delta\theta^2 \rangle^{1/2} - \Psi)/2], \tag{4.10}$$

where $\langle \Delta\theta^2 \rangle$ is defined in accordance with (4.1).

The results of a numerical calculation of the maximum attainable suppression of quantum fluctuations of the quadrature are shown in Fig. 4. Evidently, an effective production of squeezed states of an optical polaron is possible only on a certain interval of its range, above which destructive effects set in. However, high-intensity solitons with $n_0 > 10^5$ can make the fluctuation suppression depth quite substantial and hardly limited by the effects above. Nonetheless, the presence of a quantum limit seems important from the fundamental point of view.

## 5. Conclusion

We have thus ascertained that the quantum effects accompanying the propagation of a Schrödinger soliton in a proper nonlinear waveguide cause gradual but steady disintegration of the soliton. We have also shed light on the four-photon nature of this phenomenon. Examining the practical aspects of the results, let us estimate the maximum possible range of a soliton in a fiber.

In accordance with (2.15), (3.1), and (3.2), a fundamental soliton adequately described classically, i.e., with an envelope in the form of a hyperbolic secant (2.17), can exist in a fiber only if $|c| < \Delta p$. Assuming $\Delta p \cong |c|$in the limiting case corresponding to minimum spreading, we get from (3.4)

$$t_{\mathrm{lim}} = 2^{1/2}/n_0 c^2 \equiv 2^{1/2} n_0 T/8\pi \simeq n_0 T/20. \tag{5.1}$$

The soliton period $T = 3\pi/(n_0 c)^2$is here the time during which a nonlinear phase advance equal to $2\pi$ is accumulated.

Recognizing that, as established by us, the soliton is practically annihilated by the growth of the noise when the profile of its average intensity is doubled, we can conclude that $t_{\mathrm{lim}}$ does indeed determine the maximum possible propagation route.

Next, $t_{lim}$ exceeds $T$ substantially, since $n_0 >> 1$ in real situations. This means that the amplitude quantum effects come into play only for extra long paths, or else in media with high nonlinearity, i.e., under conditions when a stronger destabilizing influence can be exerted by other factors, such as losses and the gain needed to offset them [13, 15], inhomogeneity of the fiber [4], dispersive effects of third and higher orders, and a finite nonlinear-response time [25]. The disclosed quantum disintegration of the solitons imposes nonetheless a fundamental restriction on the limiting length of its propagation path, a fact of undisputed importance.

The arguments advanced concerned only the average intensity and the amplitude noise. The growth of the phase fluctuation, on the other hand, is swifter. Considering in addition random deviations of the carrier frequency, a clear picture of the soliton destabilization emerges. And all this seems to be caused by the insignificant quantum uncertainty!

Note also that for detection of ultrashort pulses account must be taken of the concomitant interesting peculiarities of the photocount statistics [26].

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## APPENDIX 1

### Determination of Average Pulse Amplitude

This appendix illustrates the derivation of (2.15). In accordance with (2.7) we have

$$\langle\psi|\phi(x)|\psi\rangle = \sum_{n'}\sum_{n} a^*_{n'} a_n \iint g^*_{n'}(p')g_n(p)\, e^{it[E(n',p')-E(n,p)]} \times$$

$$\times \langle n',p'|\phi(x)|n,p\rangle dp dp' \tag{A.1}$$

Calculation of the matrix element $\langle n', p'|\phi(x)|n+1, p\rangle$ yields [11]

$$\langle n',p'|\phi(x)|n+1,p\rangle = \delta_{nn'}(2\pi)^{-1}[n(n+1)]^{1/2}(n-1)!n!|c|^{2n-1/2} \times$$

$$\times \{\prod_{r=1}^{n}[(p-p')^2+c^2(2n-2r+1)^2/4]^{-1}\}\, e^{i[(n+1)p-np']x} \simeq \tag{A.2}$$

$$\simeq \delta_{nn'}2^{-1}|c|^{-1/2}[n(n+1)]^{1/2} e^{i[(n+1)p-np']}\operatorname{sech}[\pi(p-p')/|c|]. \tag{A.3}$$

The approximate part (A.3) is produced following the transition to the limit $n \Rightarrow \infty$, but actually at $n > 10$-$500$ it depends on the ratio of the parameters $|c|$ and $p - p'$.

We substitute (A.3) in (A.1) and introduce the new variables

$$p_1 = (p-p')/2,\quad p_2 = (p+p')/2. \tag{A.4}$$

Taking (2.13) into account, we obtain then

$$\langle\psi|\phi(x)|\psi\rangle = \sum_n [|c|^{-1}n(n+1)]^{1/2}\, a^*_n a_{n+1}\pi^{-1/2}\Delta p^{-1}\, e^{itc^2 n(n+1)/4} \times$$

$$\times \iint \exp(-[p_1^2+(p_2-p_0)^2/\Delta p^2]+i\{[2p_1(n+1/2)+p_2](x-x_0)-$$

$$-t[4p_1p_2(n+1/2)-p_1^2-p_2^2]\})\operatorname{sech}(2\pi p/|c|)dp_2dp_1. \tag{A.5}$$

Integrating with respect to $p_2$ and omitting the subscript of $p_1$, we arrive at the final expression (2.15).

## APPENDIX 2

### Determination of the Shape of the Pulse Envelope

In this section we derive relations (3.1) and (3.2), which we recast in a form useful for practical calculations. According to (2.7),

$$\langle\psi|\phi^+(x)\phi(x)|\psi\rangle = \sum_{n'}\sum_{n} a^*_{n'} a_n \iint g^*_{n'}(p')g_n(p) e^{it[E(n',p')-E(n,p)]} \times$$

$$\times \langle n',p'|\phi^+(x)\phi(x)|n,p\rangle dp dp', \tag{A.6}$$

with [11]

$$\langle n',p'|\phi^+(x)\phi(x)|n,p\rangle = \delta_{nn'}(2\pi)^{-1}c^{2(n-1)}(n!)^2\, e^{in(p-p')x} \times$$

$$\times \prod_{j=1}^{n-1}[(jc)^2+(p-p')^2]^{-1} \simeq \tag{A.7}$$

$$\simeq \delta_{nn'}[n^2 e^{in(p-p')x}/2|c|](p-p')\ \mathrm{sh}^{-1}[\pi(p-p')/|c|]. \tag{A.8}$$

The approximate part (A.8) is valid for $n >> 1$.

The succeeding transformation is similar to that described in Appendix 1: variables of type (A.4) and integration over $p_2$ are introduced. The result is (3.1) and (3.2), but these equations are not too convenient for practical calculations. We therefore transform them somewhat, using the condition $n_0 >> 1$. The Poisson distribution can then be replaced by a Gaussian one (see, e.g., [14]):

$$\exp(-n_0)\ n_0^n/n! \simeq (2\pi n_0)^{-1/2}\exp[-(n-n_0)^2/2n_0], \tag{A.9}$$

and the summation by integration over $n$ with infinite limits, getting

$$\langle N(x)\rangle \simeq 4n_0|c|^{-1}\int_0^\infty p[(U+n_0-V^2/n_0)\cos(U/V)-2V\sin(U/V)]U^{-5/2}\ \times$$

$$\times\ \mathrm{sh}^{-1}(2\pi p/|c|)\ \exp([n_0(1-U)-V^2/n_0]/2U-p^2/\Delta p^2)dp,$$

$$U = 1+8n_0t^2p^2\Delta p^2, \quad V = 2pn_0(x-x_0-2p_0t). \tag{A.10}$$

The seeming complexity notwithstanding, this expression is more suitable for numerical estimates since there is no need to sum over $n$.

## APPENDIX 3

We derive here Eq. (3.6). Thus,

$$\langle\phi|\phi^+(x)\phi^+(x)\phi(x)\phi(x)|\phi\rangle = \sum_{n'}\sum_{n} a^*_{n'}a_n\int\int g^*_{n'}(p')g_n(p)\ \times$$

$$\times\ e^{it[E(n',p')-E(n,p)]}\langle n',p'|\phi^+(x)\phi^+(x)\phi(x)\phi(x)|n,p\rangle dp dp'. \tag{A.11}$$

The matrix element is

$$M \equiv \langle n',p'|\phi^+(x)\phi^+(x)\phi(x)\phi(x)|n,p\rangle = \delta_{nn'}(n!)^{-1}\int\int f^*_{np'}(x'_1,x'_2,\ldots$$

$$\ldots,x'_n)f_{np}(x_1,x_2,\ldots,x_n)\langle 0|\phi(x'_1)\ldots\phi(x'_n)\phi^+(x_1)\ldots\phi^+(x_n)|0\rangle\ \times$$

$$\times\ dx'_1\ldots dx'_n dx_1\ldots dx_n = \tag{A.12}$$

$$=\delta_{nn'}n(n-1)\int f^*_{np'}(x,x,x_1,\ldots,x_{n-2})f_{np}(x,x,x_1,\ldots,x_{n-2})dx_1\ldots dx_{n-2}. \tag{A.13}$$

Substitution of (2.9) in (A.13) yields (we omit hereafter the symbol $\partial_{nn'}$, assuming that $n = n'$)

$$M = N_n^2 n(n-1)\int\exp[i(p-p')\sum_{j=1}^{n-2}x_j - 2|c|\sum_{j=1}^{n-2}|x-x_j|-|c|\sum\sum_{1\leqslant i<j\leqslant n-2}|x_i-$$

$$-x_j|]dx_1\ldots dx_{n-2}. \tag{A.14}$$

Since the function $f_{np}$ is symmetric, the integration is possible only in the region $-\infty \le x_1 \le x_2 \le \ldots \le x_m \le x \le x_{m+1} \le x_{m+2} \le \ldots \le x_{n-2} \le \infty$ since the integrals over the other regions are obtained only by all possible permutations of $x$ and $x_j$, which do not influence the values of $f_{np}$. We can therefore represent (A.14) in the form

$$M=N_n^2 n!\, e^{i2x(p-p')}\sum_{m=0}^{n-2}\int_{-\infty}^{x}dx_m\int_{-\infty}^{x_m}dx_{m-1}\cdots\int_{-\infty}^{x_2}dx_1\int_{x}^{\infty}dx_{m+1}\int_{x_{m+1}}^{\infty}dx_{m+2}\cdots\int_{x_{n-3}}^{\infty}dx_{n-2}\,\mathrm{X}$$

$$\mathrm{X}\ \exp[i(p-p')\sum_{j=1}^{n-2}x_j+2|c|\sum_{j=1}^{m}x_j-2|c|\sum_{j=m+1}^{n-2}x_j+2|c|(n-2m-2)x+$$

$$+\ |c|\sum_{j=1}^{n-2}(n-2j-1)x_j]. \tag{A.15}$$

The integration yields

$$M=N_n^2 n!\, e^{in(p-p')x}|c|^{-(n-2)}\sum_{m=0}^{n-2}\{m!(n-2-m)!\prod_{r=1}^{m}[n-r+i(p-p')/|c|]\ \mathrm{X}$$

$$\prod_{r=1}^{n-2-m}[n-r-i(p-p')/|c|]\}^{-1}. \tag{A.16}$$

Note that the sum in (A.16) can be written in the form

$$\sum_{m=0}^{n-2}\ldots=\prod_{r=2}^{n-1}[r^2+(p-p')^2/c^2]^{-1}\sum_{m=0}^{n-2}[m!(n-2-m)!]^{-1}\prod_{r=2}^{m+1}[(r-$$

$$-i(p-p')/|c|]\prod_{r=2}^{n-1-m}[r+i(p-p')/|c|], \tag{A.17}$$

where

$$\sum_{m=0}^{n-2}[m!(n-m)!]^{-1}\prod_{r=2}^{m+1}[r-i(p-p')/|c|]\prod_{r=2}^{n-1-m}[r+i(p-p')/|c|]=\sum_{m=1}^{n-1}m(n-m). \tag{A.18}$$

But

$$\sum_{m=1}^{n-1}m(n-m)=n(n^2-1)/6. \tag{A.19}$$

We have thus, taking (2.10) into account

$$M=(12\pi)^{-1}(n^2-1)(n!)^2|c|e^{in(p-p')x}\prod_{r=2}^{n-1}[r^2+(p-p')^2/c^2]^{-1}. \tag{A.20}$$

As n $\Rightarrow\infty$ the finite product in (A.20) becomes infinite and can in turn be represented by a hyperbolic sine:

$$\pi x\prod_{j=1}^{\infty}(1+x^2/j^2)=\mathrm{sh}\,\pi x. \tag{A.21}$$

As a result we have

$$\langle n,p'|\phi^+(x)\phi^+(x)\phi(x)\phi(x)|n,p\rangle\simeq[n^2(n^2-1)(p-p')/12][1+$$

$$+(p-p')^2/c^2]\ \mathrm{sh}^{-1}[\pi(p-p')/|c|]\ e^{in(p-p')x}. \tag{A.22}$$

Substitution of this relation in (A.11), a change to new variables of the form (A.4), and integration over $p_2$ yield (3.6).

## APPENDIX 4

We elucidate here the derivation of (4.6). Thus,

$$\langle\psi|\phi^2(x)|\psi\rangle = \sum_n a_n^* a_{n+2} \iint g_n^*(p') g_{n+2}(p) \langle n,p'|\phi^2(x)|n+2,p\rangle \times$$

$$\times e^{it[E(n,p')-E(n+2,p)]}\, dp\, dp'. \qquad (A.23)$$

The matrix element is

$$\langle n,p'|\phi^2(x)|n+2,p\rangle = [(n+2)(n+1)]^{1/2} \int f^*_{np'}(x_1,\ldots,x_n) \times$$

$$\times f_{n+2,p}(x_1,\ldots x_n,x,x)\,dx_1\ldots dx_n =$$

$$= [(n+2)(n+1)]^{1/2} N_n N_{n+2} n! \sum_{m=0}^{n} \int_{-\infty}^{x} dx_m \int_{-\infty}^{x_m} dx_{m-1} \ldots \int_{-\infty}^{x_2} dx_1 \int_{x}^{\infty} dx_{m+1} \times$$

$$\times \int_{x_{m+1}}^{\infty} dx_{m+2} \ldots \int_{x_{n-1}}^{\infty} dx_n \exp[i2px + i(p-p')\sum_{j=1}^{n} x_j + (|c|/2)\sum_{j=1}^{n}(n-2j+1)x_j +$$

$$+(|c|/2)\sum_{j=1}^{m}(n-2j+3)x_j + (|c|/2)\sum_{j=m+1}^{n}(n-2j-1)x_j + |c|(n-2m)x] =$$

$$= [(n+2)(n+1)]^{1/2} N_n N_{n+2} n! e^{ix[n(p-p')+i2p]} \sum_{m=0}^{n} \prod_{r=1}^{m} \{[|c|(n-r+1)+$$

$$+i(p-p')]r\}^{-1} \prod_{r=1}^{n-m} \{[|c|(n-r+1)-i(p-p')]r\}^{-1} \qquad (A.24)$$

Summing in (A.24) it is possible to arrive at the form

$$\sum_{m=0}^{n} \ldots = \prod_{r=1}^{n} [r^2+(p-p')^2/c^2]^{-1} \sum_{m=0}^{n} [m!(n-m)!]^{-1} \prod_{r=1}^{m} [r-i(p-p')/|c|] \times$$

$$\prod_{r=1}^{n-m} [r+i(p-p')/|c|] = (n+1)/\prod_{r=1}^{n} [r^2+(p-p')^2/c^2]. \qquad (A.25)$$

Then the form

$$\langle n,p'|\phi^2(x)|n+2,p\rangle = (2\pi)^{-1}(n+1)^2[(n+2)/n]^{1/2} e^{ix[(n+2)p-np']} \times$$

$$\times \prod_{r=1}^{n} [1+(p-p')/r^2c^2]^{-1}. \qquad (A.26)$$

Taking the limit as $n \Rightarrow \infty$ (and actually at $n_0 >> 1$)

$$\langle n,p'|\phi^2(x)|n+2,p\rangle \simeq 2^{-1}(n+1)^2[(n+2)/n]^{1/2} e^{ix[(n+2)p-np']} \times$$

$$\times (p-p')/|c|\, sh[\pi(p-p')/|c|]. \qquad (A.27)$$

Substitution of (A.27) in (A.23) and integration with respect to one of the variables yield (4.6)

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