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JACOBI-TYPE CONDITIONS FOR THE PROBLEM OF BOLZA WITH INEQUALITIES

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1. We consider the following problem of optimal control:

$$\varphi_0(x_0) \to \min, \quad p(x_0) = 0, \tag{1}$$

$$\varphi_i(x_0) \leqslant 0, \quad i = 1, \ldots, \nu, \tag{2}$$

$$\dot{x} = f(x, u, t), \quad x(1) = x_1, \quad g(x, u, t) = 0. \tag{3}$$

Here $t \in [0, 1]$, $x_0 = x(0)$, x is an absolutely continuous, u is a measurable bounded function, the dimensions of the functions x, u, p, g are equal to n, r, d, s, respectively, x_1 is a fixed vector in $\mathbf{R}^n$, i.e., the right-hand end of the trajectory is fixed.

This is the classical Lagrange—Mayer—Bolza problem of the calculus of variations [1, Sec. 69] with additional constraints in the form of the end inequalities (2). (The problem with constraints in the form of integral equalities and inequalities reduces to the considered one after introducing additional phase variables.) Here we consider conditions of the Jacobi type of weak minimum, i.e., the minimum in the norm $\|x\|_\infty + \|u\|_\infty$.

We introduce the required notations (see details in [2, Sec. 13]). Let (x^0, u^0) be the required trajectory. As usual, we assume that the matrix $g'_u(x^0(t), u^0(t), t)$ has a bounded right inverse. We denote by Λ_0 the set of all collections of Lagrange multipliers $\lambda = (\alpha, c, \psi, m)$ (where $\alpha = (\alpha_0, \ldots, \alpha_\nu) \geqslant 0$, $c \in \mathbf{R}^d$, the function ψ is Lipschitz, $m \in L_1^{(s)}$), ensuring that the Euler equation holds for the trajectory (x^0, u^0) with some normalization $\|\lambda\| = 1$. The set Λ_0 is a finite-dimensional compactum. We shall assume that it is nonempty since otherwise there is no weak minimum in (x^0, u^0). The fact that Λ_0 contains, in general, more than one element, is explained basically by the presence of the inequalities (2). While in the case of their absence we can still make more or less reasonable assumptions regarding the problem of the joint nondegeneracy of the equality constraints (when Λ_0 consists of a single element), for the problem with inequalities this cannot be done anymore. For each λ we set

$$l[\lambda](x_0) = \sum_{i=0}^{\nu} \alpha_i \varphi_i(x_0) + (c, p(x_0)),$$

$$\overline{H}[\lambda](x, u, t) = (\psi(t), f(x, u, t)) - (m(t), g(x, u, t)).$$

We introduce the Lagrange function

$$\Phi[\lambda](x, u) = l[\lambda](x_0) + \int ((\psi, \dot{x}) - \overline{H}[\lambda](x, u, t))\, dt,$$

and we denote its second variation with respect to (x, u) at the point (x^0, u^0) for a fixed λ by

$$\Omega[\lambda](\bar{x}, \bar{u}) = (l''[\lambda]\bar{x}_0, \bar{x}_0) - \int_0^1 ((\overline{H}_{xx}[\lambda]\bar{x}, \bar{x}) + 2(\overline{H}_{ux}[\lambda]\bar{x}, \bar{u}) + (\overline{H}_{uu}[\lambda]\bar{u}, \bar{u}))\, dt. \tag{4}$$

We define the functional $J(\bar{x}, \bar{u}) = \max_{\lambda \in \Lambda_0} \Omega[\lambda](\bar{x}, \bar{u})$. (Since $\Omega[\lambda]$ depends linearly on λ, in the last formula the maximum can be taken over the convex hull of Λ_0.)

We denote by $\mathcal{K}$ the cone of the critical variations. It is defined by the linearization of the constraints (1)-(3) at the point (x^0, u^0):

$$\varphi_0' \bar{x}_0 \leqslant 0, \quad p'\bar{x}_0 = 0,$$

$$\varphi_i' \bar{x}_0 \leqslant 0 \quad \text{if} \quad \varphi_i(x_0^0) = 0,$$

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$$\dot{\bar{x}} = f_x\bar{x} + f_u\bar{u}, \quad \bar{x}(1) = 0,$$
$$g_x\bar{x} + g_u\bar{u} = 0. \tag{5}$$

In [2, Sec. 13], for problem (1)-(3), under certain general assumptions (regarding the smoothness of the functions φ_i, p, f, g) one gives the following conditions for a weak minimum of the second order (for the problems without the inequalities (2) and Λ_0 consisting of a single point they can be found in [1]):

necessary condition:

$$J(\bar{x}, \bar{u}) \geqslant 0 \quad \text{for all} \quad (\bar{x}, \bar{u}) \in \mathscr{K}, \tag{6}$$

sufficient condition: there exists $\delta > 0$ such that

$$J(\bar{x}, \bar{u}) \geqslant \delta \int (\bar{u}, \bar{u})\, dt \quad \text{for all} \quad (\bar{x}, \bar{u}) \in \mathscr{K}. \tag{7}$$

Thus, there arises the problem of the investigation of the functional J for nonnegativity and for positive definiteness on a certain cone $\mathscr{K}$ of variations, in the same way as in the classical calculus of variations there arises the problem of the investigation of the quadratic functional on some subspace of variations. The conditions (6), (7) differ from the classical ones by: a) the functional J is not quadratic but it is the maximum of quadratic ones; b) $\mathscr{K}$ is a cone and not a subspace.*

In the present paper we give a Jacobi-type condition, equivalent to inequality (7), and a necessary Jacobi-type condition for inequality (6). As in the classical calculus of variations, we shall vary the upper integration limit in (4) and we shall watch when inequalities (6), (7) are satisfied.

2. The problem of the investigation of inequalities (6), (7) will be considered at once in the following abstract formulation. Let $\mathscr{H}$ be a Hilbert space, let $\{K_t, a \leqslant t \leqslant b\}$ be a family of convex closed cones in it, increasing and continuous in the sense that $K_t \subset K_s$ for $t < s$, $K_t = \bigcap_{s>t} K_s$, $K_s = \overline{\bigcup_{t<s} K_t}$. The cone K_b will be denoted by $\mathscr{K}$. In the space $\mathscr{H}$ there is defined the functional

$$J(x) = \max_{\alpha \in Z} \sum_{i=1}^{m} \alpha_i \Omega_i(x), \tag{8}$$

where Z is an arbitrary convex compactum in $\mathbf{R}^m$ and $\Omega_i(x) = (Q_i x, x)$ are quadratic functionals, defined by arbitrary bounded symmetric operators Q_i.

One has to determine for which t is $J \geqslant 0$ on K_t and for which ones is positive on K_t, i.e., J > 0 on any nonzero element of K_t. (For an important class of functionals, from positivity there follows positive-definiteness; this will be considered later.)

Remark 1. Usually, the family $\{K_t\}$ is generated by one convex closed cone $\mathscr{K}$, namely: $K_t = \mathscr{K} \cap E_t$, where $\{E_t, a \leqslant t \leqslant b\}$ is a family of subspaces in $\mathscr{H}$, increasing and continuous in the above-mentioned sense, and such that $E_b = \mathscr{H}$. Moreover, if, for example, the initial cone is finite-faced (i.e., it is the intersection of a finite number of closed halfspaces), then the generated family $\{K_t\}$ is always continuous. (Obviously, in general, this is not so.)

Remark 2. For problem (1)-(3) the generating cone $\mathscr{K}$ is the cone of critical variations, $E_T = \{\bar{u} \in L^{(r)}[0, T]\colon g'_u(x^0(t), u^0(t), t)\,\bar{u}(t) = 0$ almost everywhere$\}$, $T \in [0, 1]$. In the space $\mathscr{H} = E_1$ the cone $\mathscr{K}$ is finite-faced. The component $\bar{u}$, orthogonal to Ker g'_u, can be expressed linearly in terms of $\bar{x}$ from the equality (5).

Remark 3. The scheme for the investigation of (6), (7), presented here, can be applied also to the problem with unfastened endpoints:

$$\varphi_0(x_0, x_1) \to \min, \quad \varphi_i(x_0, x_1) \leqslant 0,$$
$$p(x_0, x_1) = 0, \quad \dot{x} = f(x, u, t), \quad g(x, u, t) = 0,$$

for which one does not have the equality $\bar{x}(1) = 0$. In this case, instead of varying the upper integration limit in (4), one has to vary only the support $[0, T]$, $T \leqslant 1$, the components

*We note that there exist also other classes of problems leading to the conditions (6), (7), for example, strange as it may seem at the first view, the problem (1)-(3) in which f, g are linear with respect to u (see [3]).

of $\bar{u}$ belonging to Ker g'_u, and to consider $\bar{x}$ and the integral itself all the time on [0, 1].

3. In order to solve the formulated problem we shall follow [4], where one has considered the case when the set Z consists of a single point. (In this case we shall write $Z = \{\cdot\}$.) In that paper we introduced the concepts of table and focal segment.* We recall their definitions. We shall assume that on K_a we have $J \geqslant 0$ (otherwise the investigation is concluded since then on any K_t, J takes negative values too).

If J is not positive on all $\mathscr{K}$, then by a table we mean a segment $[t_0, t_1] \subset [a, b]$ defined descriptively: t_0 is the sup of all t for which J is positive on K_t (the sup over the empty set is assumed to be equal to a), while t_1 is the max of all t for which $J \geqslant 0$ on K_t. If, however, J is positive on $\mathscr{K}$, then we shall say that the table is absent. Obviously, in order to solve the above-formulated problem it is necessary to find a table or to establish its absence. In [4] there are examples which show that there exist tables which do not degenerate to a point. Here we give only the following (isoperimetric).

Example 1. $E_T = L_2[0, T]$, $T \in [0, b]$, $b > 1$, $Z = \{\cdot\}$, $J(u) = -x^2(0) + \int_0^b u^2\,dt$, $\dot{x} = u$, $x(b) = 0$. The cone $\mathscr{K}$ is given by the equality

$$\int_0^b \chi_{[1, S]}(t)\, u(t)\, dt = 0,$$

$S \in [1, b]$. Here the table is equal to [1, S].

The focal segment $[\tau_0, \tau_1] \subset [a, b]$ is defined with the aid of the Euler–Jacobi equation (so as to say, constructively) in the following manner. Assume that for some $t \in [a, b]$, the element $\hat{x} \in K_t$ is a nontrivial solution of the Euler–Jacobi equation for J on K_t, i.e., $\hat{x}$ is a nonzero stationary point in the problem $J(x) \to \min$, $x \in K_t$. (The condition of stationarity of the point $\hat{x}$ in this problem consists in the fact that there exists $\varphi \in K_t^*$ such that $(\varphi, \hat{x}) = 0$ and $\varphi \in \partial J'(\hat{x}, \cdot)$, i.e., $\varphi = \sum \hat{\alpha}_i \Omega_i'(\hat{x})$, where $\hat{\alpha} \in Z$, $\sum \hat{\alpha}_i \Omega_i(\hat{x}) = J(\hat{x}) = 0$.) We set $\xi = \min \{s\colon \hat{x} \in K_s\}$, and η is equal to the maximum of all $s \geqslant t$ for which $\hat{x}$ is a solution of the Euler–Jacobi equation for J on K_s. (That ξ, η are well defined follows from the continuity of the family $\{K_s\}$.) The segment $[\xi, \eta]$ obtained in this manner is said to be a prefocal segment. (For $\xi = \eta$ it degenerates into a point.) Thus, each nontrivial solution of the Euler–Jacobi equation generates some prefocal segment. Considering all $t \in [a, b]$ and for each t considering the set of all nontrivial solutions of the Euler–Jacobi equation for J on K_t, we obtain the set of all prefocal segments $\{[\xi, \eta]\}$. Let $\{\xi\}$ and $\{\eta\}$ be the sets of their left-hand and right-hand endpoints, respectively. Then $\tau_0 = \inf\{\xi\}$ and $\tau_1 = \inf\{\eta\}$. The natural character of this definition is explained in [4].

If there exists no prefocal interval at all (i.e., the Euler–Jacobi equation does not have nontrivial solutions for any t), then we shall say that the focal segment is absent.

We intend to find the position of the table, knowing the position of the focal segment. To this end we study their mutual position.

LEMMA 1. If $x \in K_t$ is a solution of the Euler–Jacobi equation for J on K_t, then $J(x) = 0$.

Proof. For $r \in \mathbf{R}$ we set $f(r) = J(rx)$. Obviously, for a stationary x we have $f'(1) = 0$. By Euler's formula for homogeneous functionals, $J(x) = f(1) = 1/2 f'(1) = 0$.

With the aid of this lemma it is easy to show that if there exists a focal segment, then there exists also a table and for all ξ, η, $t_0 \leqslant \xi$, $t_1 \leqslant \eta$, and, therefore, $t_0 \leqslant \tau_0$, $t_1 \leqslant \tau_1$. Both these inequalities are entirely trivial and, basically, they give little information on the position of the table. (For example, it may happen that a table exists and consists of a single point $t_0 = t_1 \in (a, b)$, but we do not detect it since the focal segment is absent.)

The consideration of the focal segment becomes more meaningful if the following property holds. First we note that, by virtue of the continuity of the family $\{K_t\}$, from the definition of a table there follows that $J \geqslant 0$ on K_{t_0}.

Definition [4]. Under the presence of a table, we shall say that J passes through the value zero if J is not positive on K_{t_0} (i.e., if there exists a nonzero element $x_0 \in K_{t_0}$ such that $J(x_0) = 0$).

*In the case when $Z = \{\cdot\}$ and $\mathscr{K}$ is a subspace, the concept of focal segment has been basically used in [5, 6].

THEOREM 1. If a table exists and J passes through zero, then the set of prefocal segments is not empty, $\inf\{\xi\}$ is attained, and $t_0 = \tau_0$.

Proof. We take any nonzero $x_0 \in K_{t_0}$ such that $J(x_0) = 0$. Since $J \geqslant 0$ on K_{t_0}, it follows that x_0 is a point of (absolute) minimum of J on K_{t_0} and, consequently, satisfies the necessary condition for a minimum, namely Euler's equation. Then x_0 generates some prefocal segment $[\xi_0, \eta_0]$, $\xi_0 \leqslant t_0$. Thus, the set of prefocal segments is not empty. We assert that for any of them one has $\xi \geqslant t_0$. Indeed, otherwise one has $\xi < t_0$, and the element $x \in K_\xi$, generating the segment $[\xi, \eta]$. But then, by Lemma 1, $J(x) = 0$, which contradicts the positivity of J on K_ξ, existing by the definition of t_0. Thus, any $\xi \geqslant t_0$ and, recalling that $\xi_0 \leqslant t_0$, we obtain $t_0 = \tau_0$.

4. In order to make use of Theorem 1, we indicate a condition which ensures that J passes through zero. We recall that a quadratic functional $\omega(x)$ is said to be Legendre [6] if it is weakly lower semicontinuous and from $x_n \xrightarrow{w} x_0$, $\omega(x_n) \to \omega(x_0)$ there follows that $x_n \Rightarrow x_0$. (The arrow $\Rightarrow$ denotes convergence in norm.) Now we give the following

Definition. The functional J is said to be 0-Legendre if it is weakly lower semicontinuous and from $x_n \xrightarrow{w} 0$, $J(x_n) \to 0$ there follows that $x_n \Rightarrow 0$.

For a quadratic functional (i.e., in the case $Z = \{\cdot\}$) the Legendre and the 0-Legendre properties are equivalent.

THEOREM 2. Under the presence of a table, every 0-Legendre functional passes through zero.

Proof. Assume that there exists a table. The case $t_0 = b$ is trivial. If $t_0 < b$, then, by the definition of t_0, for any n there exists $x_n \in K_{t_0+1/n}$ such that $\| x_n \| = 1$ and $J(x_n) \leqslant 0$. Taking, if necessary, a subsequence and making use of the weak closedness of the cones $K_{t_0+1/n}$ and of the continuity of the family $\{K_t\}$, we assume that $x_n \xrightarrow{w} x_0 \in K_{t_0}$. Since J is weakly lower semicontinuous and $J \geqslant 0$ on K_{t_0}, we have $J(x_0) = \lim J(x_n) = 0$. If $x_0 = 0$, then by virtue of the 0-Legendre property of J, we obtain that $x_n \Rightarrow 0$, which contradicts the equality $\| x_n \| = 1$. Therefore, $x_0 \neq 0$, i.e., J passes through zero.

Remark 4. From Theorems 1, 2 and Lemma 1 it follows easily that for a 0-Legendre functional J, the absence of the focal segment is equivalent to the absence of a table, i.e., to the positivity of J on all $\mathscr{K}$.

THEOREM 3. Every 0-Legendre functional J, positive on a closed convex cone K, is positive-definite on it (i.e., $J(x) \geqslant \delta \| x \|^2$ on K for some $\delta > 0$).

Proof. We assume the opposite. Then there exists a sequence $x_n \in K$ such that $\| x_n \| = 1$, $x_n \xrightarrow{w} x_0 \in K$ and $J(x_n) \to 0$. Since $J(x_0) \leqslant \lim J(x_n) = 0$, and J is positive on K, we have $x_0 = 0$. From here, by virtue of the 0-Legendre property of J we have $x_n \Rightarrow 0$ and this contradicts the equality $\| x_n \| = 1$.

Now we give a criterion for the 0-Legendre property. We introduce the notations: $\Omega = (\Omega_1, \ldots, \Omega_m)$, $(\alpha, \Omega) = \sum \alpha_i \Omega_i$.

THEOREM 4. Assume that for each $\alpha \in Z$ the functional (α, Ω) is lower weakly semicontinuous. Then the 0-Legendre property of J is equivalent to the fact that there exists $\alpha^0 \in Z$ such that the quadratic functional (α^0, Ω) is Legendre.

Proof. The weak semicontinuity of J is obvious. Now, let $x_n \xrightarrow{w} 0$ and $\underline{\lim} \| x_n \| > 0$. Then $\underline{\lim}\, (\alpha^0, \Omega(x_n)) > 0$ and thus, so much more, $\underline{\lim}\, J(x_n) > 0$.

For α^0 we take any point from the relative interior of Z. Obviously, for all $\alpha \in Z$ one has the representation

$$\alpha^0 = p\alpha + (1 - p)\alpha', \tag{9}$$

where $\alpha' \in Z$ and $p \in [p_0, 1]$, $p_0 > 0$.

We consider now an arbitrary sequence $x_n \xrightarrow{w} 0$, $\| x_n \| = 1$. For each n let $\alpha_n \in Z$ be such that $J(x_n) = (\alpha_n, \Omega(x_n))$. From the 0-Legendre property of J it follows that for large n one has

$$(\alpha_n, \Omega(x_n)) \geqslant \delta > 0. \tag{10}$$

By virtue of (9), $\alpha^0 = p_n\alpha_n + (1 - p_n)\alpha_n'$, where $\alpha_n' \in Z$ and $p_n \in [p_0, 1]$ and, therefore

$$(\alpha^0, \Omega(x_n)) = p_n(\alpha_n, \Omega(x_n)) + (1 - p_n)(\alpha_n', \Omega(x_n)). \tag{11}$$

From the weak lower semicontinuity of the functionals (α, Ω) there follows easily that $\underline{\lim}\, (\alpha_n', \Omega(x_n)) \geqslant 0$ and then from (11), taking into account (10), we obtain that $\underline{\lim}\, (\alpha^0, \Omega(x_n)) \geqslant p_0\delta > 0$. Thus, the functional (α^0, Ω) is 0-Legendre.

5. The following theorem shows that if the cone K on which one considers J is finite-faced, then one can always consider that the conditions of Theorem 4 is satisfied.

We denote by Z_0 the set of all $\alpha \in Z$ for which the functional (α, Ω) is weakly lower semicontinuous. Clearly, Z_0 is convex and closed (the latter follows from the fact that for lower semicontinuous functions, the uniform limit on every compactum is a lower semicontinuous function).

THEOREM 5. The nonnegativity (positive-definiteness) of J on a finite-faced cone K is equivalent to the fact that Z_0 is nonempty and the functional

$$J_0(x) = \max_{\alpha \in Z_0} (\alpha, \Omega(x))$$

is nonnegative (positive definite) on K.

Proof.* First we show that from the nonnegativity of J on K there follows the nonemptiness of Z_0 and the nonnegativity of J_0 on K (the converse statement is obvious).

a) Let $\mathfrak{S} \subset \mathbf{R}^m$ be the set of all vectors σ for which there exists a sequence $x_n \xrightarrow{w} 0$ such that $\Omega(x_n) \to \sigma$. It is easy to see that $\mathfrak{S}$ is a cone, moreover, convex. Indeed, let $x_n \xrightarrow{w} 0$, $\Omega(x_n) \to \sigma$, $y_n \xrightarrow{w} 0$, $\Omega(y_n) \to \rho$. Then, for some sequence of indices $k_n \to \infty$, for all i = 1, . . ., m we have $(Q_i x_n, y_{k_n}) \to 0$ and, therefore,

$$\Omega(x_n + y_{k_n}) = \Omega(x_n) + \Omega(y_{k_n}) + o(1) \to \sigma + \rho.$$

b) We show that the conjugate cone $\mathfrak{S}^*$ consists of all those and only those $\alpha \in \mathbf{R}^m$ for which the functional (α, Ω) is weakly lower semicontinuous. Indeed, if $\alpha \in \mathfrak{S}^*$, then for any sequence $x_n \xrightarrow{w} 0$ we have $\underline{\lim}\, (\alpha, \Omega(x_n)) \geqslant 0$ (since, otherwise, for some subsequence $x_{n_k} \xrightarrow{w} 0$ one has $\Omega(x_{n_k}) \to \sigma \in \mathfrak{S}$ and $(\alpha, \sigma) < 0$, contradiction) and, therefore, the functional (α, Ω) is weakly lower semicontinuous. Conversely, if (α, Ω) is weakly lower semicontinuous, then for any $\sigma \in \mathfrak{S}$ and a corresponding sequence $x_n \xrightarrow{w} 0$, $\Omega(x_n) \to \sigma$ we have $(\alpha, \sigma) = \lim (\alpha, \Omega(x_n)) \geqslant 0$, i.e., $\alpha \in \mathfrak{S}^*$. Thus, the assertion b) is proved.

From it we obtain, in particular, that $\mathfrak{S}^* \cap Z = Z_0$.

c) We show that for all $\bar{x} \in K$, for all $\sigma \in \mathfrak{S}$, there exists a sequence $x_n' \in K$ such that

$$\Omega(x_n') \to \Omega(\bar{x}) + \sigma.$$

Let $x_n \xrightarrow{w} 0$, $\Omega(x_n) \to \sigma$. Then, obviously, $\Omega(\bar{x} + x_n) \to \Omega(\bar{x}) + \sigma$, but in this case, in general, the sequence $\bar{x} + x_n$ does not lie in K. By assumption, the cone K is finite-faced, i.e., it is defined by inequalities $(l_i, x) \leqslant 0$, $i = 1, \ldots, s$, where $l_i \in \mathscr{H}$. Since for all i we have $(l_i, \bar{x}) \leqslant 0$ and $(l_i, x_n) \to 0$, it follows that $(l_i, \bar{x} + x_n) \leqslant o(1)$ and then, by Hoffman's lemma (see [2, Sec. 1], [7]), there exists a sequence $x_n' \in K$ such that $\| x_n' - (\bar{x} + x_n) \| \to 0$. By virtue of the Lipschitz property of Ω, on any bounded set we have

$$\Omega(x_n') = \Omega(\bar{x} + x_n) + o(1) \to \Omega(\bar{x}) + \sigma.$$

d) From the proved assertion c) and from the nonnegativity of J on K it follows that for all $\bar{x} \in K$ and for all $\sigma \in \mathfrak{S}$ one has $\max_{\alpha \in Z} (\alpha, \Omega(\bar{x}) + \sigma) \geqslant 0$ and, therefore, for all $\bar{x} \in K$

$$\inf_{\alpha \in \mathfrak{S}} \max_{\alpha \in Z} (\alpha, \Omega(\bar{x}) + \sigma) \geqslant 0.$$

*For functionals of form (15) in the space $L_2 \times \mathbf{R}^\mu$ the proof is given in [3].

Since $\mathfrak{S}$ and Z are convex and Z is compact, applying the well-known Neumann–Kneser theorem on the minimax (see, for example, [8]), we obtain that the set $\mathfrak{S}^* \cap Z = Z_0$ is nonempty and for all $x \in K$ one has

$$\max_{\alpha \in \mathfrak{S}^* \cap Z} (\alpha, \Omega(x)) \geqslant 0;$$

this means that $J_0 \geqslant 0$ on K.

Assume now that for some $\delta > 0$ we have

$$J(x) \geqslant \delta(x, x) \quad \text{for all} \quad x \in K. \tag{12}$$

We show that the same inequality is satisfied also for J_0. To this end, we introduce the set $\mathscr{Z} = Z \times \{-1\} \subset \mathbf{R}^{m+1}$ with elements $\tilde{\alpha} = (\alpha, \alpha_{m+1})$, $\alpha \in Z$, $\alpha_{m+1} = -1$ and the extended collection of functionals

$$\tilde{\Omega}(x) = (\Omega_1(x), \ldots, \Omega_m(x), \delta I(x)),$$
$$I(x) = (x, x).$$

From (12) there follows that for all $x \in K$ we have

$$\mathscr{G}(x) = \max_{\tilde{\alpha} \in \mathscr{Z}} (\tilde{\alpha}, \tilde{\Omega}(x)) \geqslant 0,$$

and then, according to what has been already proved,

$$\mathscr{G}_0(x) = \max_{\tilde{\alpha} \in \mathscr{Z}_0} (\tilde{\alpha}, \tilde{\Omega}(x)) \geqslant 0. \tag{13}$$

Assume that Z_δ is the set of all those $\alpha \in Z$ for which the functional $(\alpha, \Omega(x)) - \delta I(x)$ is weakly lower semicontinuous. Then $\mathscr{G}_0 = Z_\delta \times \{-1\}$ and (13) is equivalent to the fact that

$$J_\delta(x) = \max_{\alpha \in Z_\delta} (\alpha, \Omega(x)) \geqslant \delta I(x) \text{ on } K;$$

from here, by virtue of the obvious inclusion $Z_\delta \subset Z_0$, we obtain that $J_0(x) \geqslant \delta I(x)$ on K. The theorem is proved.

We note that Theorem 5 can be also obtained from the following stronger theorem, due to A. A. Milyutin. We denote by Z_+ the set of all $\alpha \in Z$ for which the functional $(\alpha, \Omega(x)) \geqslant 0$ on some subspace $\Gamma_\alpha \subset \mathscr{H}$ of finite codimension.

THEOREM 6 (Milyutin [9]). Let $J \geqslant 0$ on a finite-faced cone K. Then Z_+ is nonempty and

$$J_+(x) = \max_{\alpha \in Z_+} (\alpha, \Omega(x)) \geqslant 0 \text{ on } K.$$

Since any quadratic functional, nonnegative on some subspace of finite codimension, is weakly lower semicontinuous, we have $Z_+ \subset Z_0$ and, therefore, from Theorem 6 there follows Theorem 5.

For the application of Theorems 4, 5, the following theorem is useful [6].

THEOREM 7. The quadratic functional $\omega(x) = (Qx, x)$ is weakly lower semicontinuous (Legendre) if and only if the operator Q admits a representation Q = S + R, where the operator S is completely continuous and R is nonnegative definite (resp. positive definite). (The operators Q, S, R are symmetric and bounded.) In particular, for the quadratic functionals of the calculus of variations

$$\omega(u) = \int_0^T ((P(t)x, x) + 2(C(t)x, u) + (R(t)u, u))\, dt\colon \; L_2 \to \mathbf{R},$$
$$\dot{x} = A(t)x + B(t)u, \quad x(T) = 0,$$

weak lower semicontinuity is equivalent to the Legendre necessary condition: $R(t) \geqslant 0$ and the Legendre property is equivalent to the strong Legendre condition: $R(t) \geqslant \text{const} > 0$.

Thus, Theorem 5 is a generalization of the classical necessary Legendre condition and the requirement of the existence of $\alpha^0 \in Z$ for which the functional (α^0, Ω) is Legendre is the generalization of the classical strong Legendre condition.

6. We write down the Euler–Jacobi equation for the case which covers the problems of form (1)-(3) and the problems that are linear with respect to control, from [3].

Assume that on the segment [0, T], x, u are connected by the equations

$$\dot{x} = Ax + Bu, \quad x(T) = 0, \quad Fx + Gu = 0, \tag{14}$$

where G(t) is a measurable bounded s × r matrix having a bounded right inverse. We denote $L(t) = \{u \in \mathbf{R}^r: G(t)u = 0\}$ and we shall assume that u = u' + u'', where $u'(t) \in L(t)$ and $u''(t) \perp$ L(t). Let $\mathscr{L}_T = \{u' \in L_2^{(r)}[0, T]: u'(t) \in L(t)$ almost everywhere$\}$. Clearly, if $u' \in \mathscr{L}_T$ is given, then u'', x are uniquely and completely continuously expressed in terms of u' from (14). On the space $E_T = \mathscr{L}_T \times \mathbf{R}^\mu$ with elements (u', h) we consider the functionals

$$\Omega_i(u', h) = q_i(x_0, h) + \int_0^T ((P_i x, x) + 2(C_i x, u) + (R_i u, u))\, dt, \tag{15}$$

where q_i are quadratic forms of the $(n + \mu)$-dimensional vector (x_0, h); all the matrices in (14), (15) are measurable and bounded, P_i, R_i are symmetric, i = 1, . . ., m. According to Theorems 4, 5, 7, a functional J of the form (8) will be 0-Legendre if for all $\alpha \in Z$ the functional (α, Ω) satisfies the weak Legendre condition, i.e., almost everywhere $\sum \alpha_i R_i(t) \geqslant 0$ on L(t) and there exists $\alpha^0 \in Z$, for which (α^0, Ω) satisfies the strong Legendre condition, i.e., almost everywhere $\sum \alpha_i R_i(t) \geqslant \mathrm{const} > 0$ on L(t).

In the case when $Z = \{\cdot\}$, i.e., the functional J is quadratic, we obtain that, for the applicability of the above presented results, J has to satisfy the strong Legendre condition on L(t) which is always assumed in the classical calculus of variations.

Assume that a finite-faced cone $K_T \subset E_T$ is defined by the constraints:

$$(v_j, x_0) + (w_j, h) \begin{cases} \leqslant 0, & j = 1, \ldots, \nu; \\ = 0, & j = \nu + 1, \ldots, \nu + d. \end{cases}$$

The point $(u', h) \in K_T$ is a solution of the Euler–Jacobi equation for J on K_T if and only if [10] there exist: an element $\hat{\alpha} \in Z$, an absolutely continuous function $\psi(t)$, a function $\varphi \in L_1^{(s)}[0, T]$ and numbers λ_j, $j = 1, \ldots, \nu + d$, from which $\lambda_j \geqslant 0$ for $j = 1, \ldots, \nu$, such that

a) $\max_{\alpha \in Z} (\alpha, \Omega(u', h)) = (\hat{\alpha}, \Omega(u', h)) = 0$,

b) $\lambda_j((v_j, x_0) + (w_j, h)) = 0$, $j = 1, \ldots, \nu$ (the complementing nonrigidity conditions),

c) $\sum \hat{\alpha}_i (C_i x + R_i u) = B^* \psi - G^* \varphi$,

d) $\sum \hat{\alpha}_i (P_i x + C_i^* u) = \dot{\psi} + A^* \psi - F^* \varphi$,

e) $\psi(0) = \sum \lambda_j v_j + (1/2)(\hat{\alpha}, q'_{x_0})$ (the transversality condition),

f) $\sum \lambda_j w_j + (1/2)(\hat{\alpha}, q'_h) = 0$.

7. It has been established above that for a 0-Legendre functional one has $t_0 = \tau_0$ and $t_1 \leqslant \tau_1$. We given an example showing that, in general, the equality $t_1 = \tau_1$ does not hold.

Example 2. $K_T = E_T = L_2[0, T]$, $J = \max(\Omega_1, \Omega_2)$,

$$\Omega_1(u) = \int_0^T (-x^2 + u^2)\, dt,$$

$$\Omega_2(u) = -\int_0^T \chi_{[T^*, \infty)} x^2\, dt,$$

$$T^* > \pi/2, \quad \dot{x} = u, \quad x(T) = 0,$$

the endpoint x(0) is free.

Since a table for Ω_1 is equal to $\{\pi/2\}$, and $\Omega_2(K_{\pi/2}) = 0$, it follows that the left-hand endpoint of the table is $t_0 = \pi/2$.

LEMMA 2. $t_1 = T^*$.

Proof. Inequality $t_1 \geqslant T^*$ follows from the fact that $J(K_{T^*}) \geqslant \Omega_2(K_{T^*}) = 0$. Let $T > T^*$. We select an arbitrary $u \in E_T$ for which $\Omega_1(u) < 0$ (such a u exists since $T > \pi/2$). For t < 0 we set u(t) = 0. Assume that to x there corresponds u and that S = max supp x. Shifting x, u to the right by T − S, we obtain x', u'. Obviously, $\Omega_1(u') \leqslant \Omega_1(u) < 0$ and, since $x' \not\equiv 0$ on $[T^*, T]$, we have $\Omega_2(u') < 0$ and, consequently, J(u') < 0.

Thus, the table is $[\pi/2, T^*]$. Now we find all the prefocal segments. The Euler–Jacobi equation for J on K_T has the form (in our case $A = 0, B = E, F = G = 0, L(t) \equiv R$, $q_{1,2} = 0$, $C_{1,2} = 0$, $R_1 = 1$, $R_2 = 0$, $P_1 = -1$, $P_2 = -\chi_{[T^*,\infty)}$, $v_j = 0$, $w_j = 0$):

$$\left.\begin{array}{ll} \alpha_1 \dot{x} = \psi, & x(T) = 0, \\ \dot{\psi} = -(\alpha_1 + \alpha_2 \chi_{[T^*, \infty]}) x, & \psi(0) = 0, \end{array}\right\} \tag{16}$$

$$\alpha_1 \geqslant 0, \quad \alpha_2 \geqslant 0, \quad \alpha_1 + \alpha_2 = 1, \quad \alpha_1 \Omega_1(u) = 0,$$
$$\alpha_2 \Omega_2(u) = 0.$$

Let $u \in K_{T^*}$ and J(u) = 0. Two cases are possible: a) $\Omega_1(u) < 0, \Omega_2(u) = 0$ and b) $\Omega_1(u) = \Omega_2(u) = 0$. We note that in both cases system (16) is obviously satisfied for $\hat{\alpha}_1 = 0$, $\hat{\alpha}_2 = 1$, $\hat{\psi} \equiv 0$ on $[0, T^*]$.

We select now any $T > T^*$ and we extend u, x, $\hat{\psi}$ by zero on $[T^*, T]$. We can see that, as before, the system (16) is satisfied (with the same $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\psi}$). Thus, any stationary point J on K_{T^*} generates a prefocal semiinterval $[T', \infty)$, where $T' = \max \operatorname{supp} x \leqslant T^*$ (all $T' \in [\pi/2, T^*]$ are realized).

Assume now that $T > T^*$ and that u is a stationary point of J on K_T. By Lemma 1 we have J(u) = 0. If $\Omega_2(u) = 0$, then $x \equiv 0$ on $[T^*, T]$, and we return to the case $u \in K_{T^*}$, which has just been considered; therefore, we shall assume that $\Omega_2(u) < 0$, $\Omega_1(u) = 0$. Then $\alpha_2 = 0$, $\alpha_1 = 1$ and x satisfies the equation

$$\ddot{x} = -x, \quad x(T) = 0, \quad \dot{x}(0) = 0. \tag{17}$$

From here $x = a \cos t$ and, consequently, T can take only the discrete values $T_k = \pi/2 + \pi k$. It is easy to see that here, extending x by zero to the right of T, the equations (17) are not satisfied and, therefore, to each of these values of T_k there corresponds a degenerate prefocal segment $\{T_k\}$.

Thus, there exist in all two series of prefocal segments: the continual series $[T', \infty]$, $T' \in [\pi/2, T^*]$, and the discrete series $\{T_k\}$, $T_k = \pi/2 + \pi k > T^*$ (k is an integer). Clearly,

$$\tau_1 = \min\{T_k\colon T_k > T^*\} > T^* = t_1.$$

In the given example, the fact that the table and the focal segment do not coincide is due to the fact that the functional J is not quadratic but it is the maximum of two quadratic functionals. Another example, in which J is a Legendre quadratic functional, but, on the other hand, it is generating a cone K which is not finite-faced, is given in [4].

8. Assume now that $Z = \{\cdot\}$, J is a Legendre quadratic functional, and the family $\{K_t\}$ is generated by a finite-faced cone $\mathscr{K}$.

THEOREM 8 [7]. In the indicated case we have $t_1 = \tau_1$, i.e., the table coincides with the focal segment.

Thus, in this case, as in the classical calculus of variations, the position of the table is entirely determined by the solutions of the Euler–Jacobi equation.

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EXTENDABLE AND NONEXTENDABLE SOLUTIONS OF A NONLINEAR DIFFERENTIAL EQUATION OF ARBITRARY ORDER

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We will consider the equation

$$u^{(n)} = p(t)\,|u|^{\lambda}\operatorname{sgn} u,\ \lambda > 1,\quad t \geqslant 0, \tag{1}$$

of n-th order with piecewise-continuous* functions $p(t) \geqslant 0$. The main results in the investigation of the asymptotic properties of the solutions of this equation are due to Kiguradze [1-4]. In colloboration with Kvinikadze, he [1, 4] has established that under the condition

$$\int_0^{+\infty} p(\tau)\,\tau^{(n-1)\lambda}\,d\tau < +\infty \tag{2}$$

Eq. (1) has an n-parameter family of regular (infinitely extensible to the right) solutions $u(t) \not\equiv 0$ with the initial conditions†

$$u^{(i)}(t_0) \geqslant 0 \quad (\text{ or } \quad u^{(i)}(t_0) \leqslant 0),$$
$$t_0 \geqslant 0, \quad i = 0, 1, \ldots, n-1, \tag{3}$$

and has posed the problem [3, Problem 2.4; see also the detailed bibliography there] about the elucidation of the necessity of the condition (2) for the existence of regular solutions of Eq. (1).

In the present note, we investigate the problem of existence or absence of an n-parameter family of regular solutions (including also those with an additional differential property) of Eq. (1) under the condition

$$J_\mu[\varphi(\tau)] \equiv \int_1^{+\infty} p^{\mu}(\tau)\,\varphi(\tau)\,d\tau = +\infty \tag{4}$$

with different positive constants μ and positive functions $\varphi(t) > 0$. In particular, we prove that Eq. (1) does not have regular solutions under the condition

$$J_\mu[\tau^{(n-1)\lambda\nu+\nu-1}] = +\infty, \quad \nu < \mu \in (0,\ 1/n) \tag{5}$$

(and therefore the convergence of the integral $J_\mu[\tau^{(n-1)\lambda\nu+\nu-1}]$ for $\nu < \mu \in (0,\ 1/n)$ is a necessary condition for the existence of regular solutions of Eq. (1) and establish the sharpness of this condition: For arbitrary number $\mu \geqslant 1/n$ and function $\varphi(t) > 0$ a function $p(t) \geqslant 0$, satisfying the condition (4), can be constructed such that Eq. (1) has an n-parameter family

*All the functions used in the sequel will be assumed to be piecewise-continuous for $t \geqslant 0$ and this will not be stated further.
†We will consider solutions only with these initial conditions, and therefore, as a rule, we will not indicate this in the sequel.

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