If the ergodic action of T_g^* has a Lebesgue character, then the number of orbits consisting of Lebesgue characters, is countable.

In conclusion, the author conveys gratitude to A. M. Stepin for stating the problem and to R. I. Grigorchuk for some useful advice on improving the contents of the text.

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COMPUTATION OF THE OPTIMAL COEFFICIENTS

L. P. Bocharova, V. S. Van'kova, and N. M. Dobrovol'skii

In [1] the notion of the optimal coefficients has been introduced and their significance for the approximate computation of multiple integrals of an arbitrary order s has been indicated. Various algorithms for the computation of the s-dimensional optimal coefficients modulo N, where N is the number of knots of the quadrature formula, have been obtained in [1-4]. To realize these algorithms $O(N^2)$ or $O(N^{1+1/3})$ operations are required. Lowering of the number of operations to O(N) for N equal to a power of two has been accomplished in [4].

By definition [2, p. 96; 7, p. 223], integers $a_1, \ldots, a_s$, reltively prime to a natural number $N \geq 2$, are called <u>the optimal coefficients of index $\beta = \beta(s)$ modulo N with a constant $c = c(s)$</u> if

$$S_N(a_1, \ldots, a_s) = {\sum}'^{N-1}_{m_1, \ldots, m_s=-(N-1)} \frac{\delta_N(a_1m_1 + \ldots + a_sm_s)}{\overline{m}_1 \ldots \overline{m}_s} \leqslant \frac{c \ln^\beta N}{N}, \tag{1}$$

where Σ' means that the zero vector $(m_1, \ldots, m_s) = (0, \ldots, 0)$, has been excluded from the summation, $\overline{m} = \max(1, |m|)$ and

$$\delta_N(a) = \begin{cases} 1, & \text{if} \quad a \equiv 0 \pmod N, \\ 0, & \text{if} \quad a \not\equiv 0 \pmod N). \end{cases}$$

Let the quantity $q = q_N(1, a_1, \ldots, a_{s-1}) = \min \overline{m}_1 \ldots \overline{m}_s$, where the minimum is taken over all the nontrivial solutions of the congruence

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$$m_1 + a_1 m_2 + \ldots + a_{s-1} m_s \equiv 0 \pmod{N}.$$

Then the following estimates are valid [3]:

$$1/q \leqslant S_N(1, a_1, \ldots, a_{s-1}) = 0(\ln^s N/q). \tag{2}$$

We know the relation between the quantities q and

$$Q = Q_N(1, a_1, \ldots, a_{s-1}) = \min |k_0| |k_1| \ldots |k_{s-1}$$

where the minimum is taken over all the nontrivial solutions of the system of congruences

$$\begin{cases} a_1 k_0 \equiv k_1, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{s-1} k_0 \equiv k_{s-1}, \end{cases} \pmod{N} \quad 1 \leqslant |k_\nu| \leqslant N/2 \quad (\nu = 0; \ldots, s-1).$$

By Gel'fond's lemma [3, 5, 6], there exists a positive constant $c_1 = c_1(s)$ such that

$$Q \geqslant c_1 q^{s-1}, \quad q \geqslant c_1 Q^{s-1} / N^{(s-1)^2-1}. \tag{3}$$

The present note gives new variants of the algorithms for the construction of the optimal coefficients modulo 2^n in O(N) elementary operations. The proof of the optimality of the coefficients is carried out in the notation and methodology of [4] and is based on the estimation of the quantity Q.

Let n and s be natural numbers and $x_1, x_2, \ldots, x_{s-1}$ be integers. By Σ_m^* we will denote summation over odd m. For $\nu = 1, 2, \ldots, n$ we define a function $h_\nu(x_1, x_2, \ldots, x_{s-1})$ by means of the eauation

$$h_\nu(x_1, x_2, \ldots, x_{s-1}) = 2^{-\nu} \sum\nolimits_{m=1}^{*2^\nu} (2n - 2\nu + \| m \cdot 2^{-\nu} \|^{-1}) \prod\nolimits_{j=1}^{s-1} (2n - 2\nu + \| m x_j \cdot 2^{-\nu} \|^{-1}),$$

where $\|Y\|$ is the distance of Y from the nearest integer for an arbitrary real number Y.

Let integers $\varepsilon_{j\nu}$ ($j = 1, 2, \ldots, s-1$; $\nu = 1, 2, \ldots, n-1$) take the values ±1 and let natural numbers $a_{j\nu}$ ($j = 1, 2, \ldots, s-1$; $\nu = 1, 2, \ldots, n$) be such that $a_{11} = a_{21} = \ldots = a_{s-11} = 1$ and $a_{j\nu} \equiv \varepsilon_{j\nu} a_{j\nu+1} \pmod{2^\nu}$ ($j = 1, 2, \ldots, s-1$; $\nu = 1, 2, \ldots, n-1$). We adopt the notation

$$h_\nu = h_\nu(a_{1\nu}, \ldots, a_{s-1\nu}) \quad (\nu = 1, 2, \ldots, n).$$

<u>LEMMA 1.</u> If the chain of inequalities

$$h_n \leqslant h_{n-1} \leqslant \ldots \leqslant h_1$$

is fulfilled, then the integers $1, a_{1n}, a_{2n}, \ldots, a_{s-1n}$ are the optimal coefficients modulo 2^n.

<u>Proof.</u> For $\nu = 1, 2, \ldots, n$ we introduce the notation

$$H_\nu = \sum\nolimits_{k=1}^{\nu-1} \sum\nolimits_{m=1}^{*2^k} (\| m \cdot 2^{-k} \| \cdot \| m a_{1k} \cdot 2^{-k} \| \ldots \| m a_{s-1k} \cdot 2^{-k} \|)^{-1} + (2^{n+1} - 2^\nu) h_\nu.$$

Since, by the condition, $a_{11} = a_{21} = \ldots = a_{s-11} = 1$, we have

$$h_1 = \frac{1}{2} \sum\nolimits_{m=1}^{*2} (2n - 2 + \| m \cdot 2^{-1} \|^{-1})^s = 2^{s-1} n^s,$$

$$H_1 = (2^{n+1} - 2) h_1 = (2^n - 1) 2^s \cdot n^s < (2n)^s \cdot 2^n.$$

Let us now estimate H_ν ($\nu = 2, \ldots, n$). Since

$$h_\nu = 2^{-\nu} \sum\nolimits_{m=1}^{*2^\nu} (2n - 2\nu + \| m \cdot 2^{-\nu} \|^{-1}) \prod\nolimits_{j=1}^{s-1} (2n - 2\nu + \| a_{j\nu} m \cdot 2^{-\nu} \|^{-1}),$$

for $\nu \geq 2$ we have

$$H_\nu \leqslant \sum\nolimits_{k=1}^{\nu-2} \sum\nolimits_{m=1}^{*2^k} (\| m \cdot 2^{-k} \| \cdot \| m a_{1k} \cdot 2^{-k} \| \cdot \ldots \cdot \| m a_{s-1k} \cdot 2^{-k} \|)^{-1} +$$

$$+\sum_{m=1}^{*2^{\nu-1}}(2n-2\nu+2+\|m\cdot 2^{-\nu+1}\|^{-1})\cdot\prod_{j=1}^{s-1}(2n-2\nu+2+\|ma_{j\nu-1}\cdot 2^{-\nu+1}\|^{-1})+$$

$$+(2^{n+1}-2^{\nu})h_{\nu}=\sum_{k=1}^{\nu-2}\sum_{m=1}^{*2^{k}}(\|m\cdot 2^{-k}\|\cdot\|ma_{1k}\cdot 2^{-k}\|\cdot\ldots\cdot$$

$$\cdot\|ma_{s-1k}\cdot 2^{-k}\|)^{-1}+2^{\nu-1}h_{\nu-1}+(2^{n+1}-2^{\nu})h_{\nu}\leqslant$$

$$\leqslant\sum_{k=1}^{\nu-2}\sum_{m=1}^{*2^{k}}(\|m\cdot 2^{-k}\|\cdot\|ma_{1k}\cdot 2^{-k}\|\ldots\|ma_{s-1k}\cdot 2^{-k}\|)^{-1}+(2^{n+1}-2^{\nu-1})h_{\nu-1}=H_{\nu-1},$$

and, consequently,

$$H_n\leqslant H_{n-1}\leqslant\ldots\leqslant H_1<(2n)^s\cdot 2^n. \tag{4}$$

By the definition of the quantities $a_{j\nu}$, for k = 1, 2, ..., n − 1

$$a_{1n}\equiv\delta_{1k}\cdot a_{1k},\ \ldots,\ a_{s-1n}\equiv\delta_{s-1k}a_{s-1k}\ (\mathrm{mod}\ 2^k),$$

where $\delta_{jk}=\varepsilon_{jk}\cdot\ldots\cdot\varepsilon_{jn-1}$ (j = 1, ..., s − 1). Since $|\delta_{jk}|=1$ (j = 1, ..., s − 1) and $\|Y\|$ is an even function, the following identities are valid:

$$\|ma_{jn}\cdot 2^{-k}\|=\|ma_{jk}\cdot 2^{-k}\|\quad(j=1,\ldots,s-1);$$

$$\sum_{m=1}^{2^n-1}(\|m\cdot 2^{-n}\|\cdot\|ma_{1n}\cdot 2^{-n}\|\cdot\ldots\cdot\|ma_{s-1n}\cdot 2^{-n}\|)^{-1}=$$

$$=\sum_{k=1}^{n}\sum_{m=1}^{*2^k}(\|m\cdot 2^{-k}\|\cdot\|ma_{1n}\cdot 2^{-k}\|\cdot\ldots\cdot\|ma_{s-1n}\cdot 2^{-k}\|)^{-1}=$$

$$=\sum_{k=1}^{n-1}\sum_{m=1}^{*2^k}(\|m\cdot 2^{-k}\|\cdot\|ma_{1k}\cdot 2^{-k}\|\cdot\ldots\cdot\|ma_{s-1k}\cdot 2^{-k}\|)^{-1}+$$

$$+\sum_{m=1}^{*2^n}(\|m\cdot 2^{-n}\|\cdot\|ma_{1n}\cdot 2^{-n}\|\cdot\ldots\cdot\|ma_{s-1n}\cdot 2^{-n}\|)^{-1}=H_n.$$

Hence it follows from (4) that

$$\sum_{m=1}^{2^n-1}(\|m\cdot 2^{-n}\|\cdot\|ma_{1n}\cdot 2^{-n}\|\cdot\ldots\cdot\|ma_{s-1n}\cdot 2^{-n}\|)^{-1}<(2n)^s\cdot 2^n.$$

Consequently, since

$$\sum_{m=1}^{2^n-1}2^n(m\|ma_{1n}\cdot 2^{-n}\|\cdot\ldots\cdot\|ma_{s-1n}\cdot 2^{-n}\|)^{-1}\leqslant$$

$$\leqslant\sum_{m=1}^{2^n-1}(\|m\cdot 2^{-n}\|\cdot\|ma_{1n}\cdot 2^{-n}\|\cdot\ldots\cdot\|ma_{s-1n}\cdot 2^{-n}\|)^{-1}<(2n)^s\cdot 2^n,$$

it follows that for m = 1, 2, ..., $2^n - 1$

$$m\cdot\left\|\frac{a_{1n}m}{2^n}\right\|\cdot\ldots\cdot\left\|\frac{a_{s-1n}m}{2^n}\right\|>(2n)^{-s}>3^{-s}\ln^{-s}N,$$

since 2n < 3 for N = 2^n. Hence

$$Q=Q_N(1,\ a_1,\ \ldots,\ a_{s-1})>N^{s-1}/(3\ln N)^s.$$

Applying Gel'fond's lemma, we get

$$q=q_N(1,a_1,\ldots,a_{s-1})>\frac{N}{(2s+1)^s\cdot 3^{s(s-1)}\ln^{s(s-1)}N}.$$

Hence it follows from inequality (2) that

$$S_N(1,a_1,\ldots,a_{s-1})=O\left(\frac{\ln^{s^2}N}{N}\right).$$

Thus, the collection 1, a_1, ..., a_{s-1} is a set of optimal coefficients modulo N = 2^n with the index at most s^2. The lemma is proved.

Using Lemma 1, we easily construct a whole class of algorithms to determine the optimal coefficients modulo N = 2^n in O (s^2N) operations, where the constant in the sign O(·) is absolute.

Let integers $\varepsilon_{j\nu} = \pm 1$ ($j = 1, \ldots, s-1$; $\nu = 1, \ldots, n-1$) be fixed and a function $h_{r\nu}(x_1, x_2, \ldots, x_{s-1})$ be defined by the equation

$$h_{r\nu}(x_1, x_2, \ldots, x_{s-1}) = 2^{-\nu} \sum\nolimits_{m=1}^{*2^\nu} (2n - 2\nu + \| m \cdot 2^{-\nu} \|^{-1}) \cdot$$
$$\cdot \prod\nolimits_{j=1}^{r} (2n - 2\nu + \| mx_j \cdot 2^{-\nu} \|^{-1}) \prod\nolimits_{j=r+1}^{s-1} (2n - 2\nu + 2 + \| mx_j \cdot 2^{1-\nu} \|^{-1}). \quad (5)$$

We choose $a_{11} = a_{21} = \ldots = a_{s-11} = 1$. Let $r \geqslant 1$, $\nu \geqslant 2$ and the odd numbers $a_{1\nu-1}, \ldots, a_{s-1\nu-1}$, $a_{1\nu}, \ldots, a_{r-1\nu}$ be known. Then for $2 \le \nu \le n$ we define $a_{r\nu}$ by means of the equation

$$a_{r\nu} = a_{r\nu-1}\ (1 + (\varepsilon_{r\nu-1} - 1)\, z') + z' \cdot 2^{\nu-1}, \quad (6)$$

where z' is the value of z for which the minimum of the function

$$h_{r\nu}(a_{1\nu}, \ldots, a_{r-1\nu}, a_{r\nu-1}(1 + (\varepsilon_{r\nu-1} - 1)\, z) + 2^{\nu-1} z, a_{r+1\nu-1}, \ldots, a_{s-1\nu-1}),$$

is attained when z takes the values 0 and 1.

<u>THEOREM 1.</u> For an arbitrary natural number n, the integers $a_1 = a_{1n}$, $a_2 = a_{2n}$, ..., $a_{s-1} = a_{s-1n}$, obtained by means of function (5) and Eq. (6), are s-dimensional optimal coefficients modulo $N = 2^n$.

<u>Proof.</u> For $\nu = 1$, $r = s - 1$ and for $\nu = 2, \ldots, n$, $r = 1, \ldots, s - 1$ we introduce the notation

$$h_{r\nu} = h_{r\nu}(a_{1\nu}, \ldots, a_{r\nu}, a_{r+1\,\nu-1}, \ldots, a_{s-1\,\nu-1}).$$

Retaining the notation of Lemma 1, we have the equality $h_{s-1\nu} = h_\nu$. For odd a and m the following relations are fulfilled for $\nu \ge 2$:

$$\frac{1}{2} \sum\nolimits_{z=0}^{1} \| m (a (1 + (\varepsilon - 1) z) + z \cdot 2^{\nu-1})/2^\nu \|^{-1} = 2^{-1} (\| ma/2^\nu \|^{-1} + \| m (a\varepsilon + 2^{\nu-1})/2^\nu \|^{-1}) =$$
$$= 2^{-1} \Big(\Big\| \frac{1}{2} \{am/2^{\nu-1}\} + \frac{1}{2} [am/2^{\nu-1}] \Big\|^{-1} +$$
$$+ \Big\| \varepsilon \cdot \frac{1}{2} \{am/2^{\nu-1}\} + \varepsilon \cdot \frac{1}{2} [am/2^{\nu-1}] + \frac{1}{2} \Big\|^{-1} \Big) =$$
$$= 2^{-1} \Big(\Big\| \frac{1}{2} \{am/2^{\nu-1}\} + \frac{1}{2} [am/2^{\nu-1}] \Big\|^{-1} + \Big\| \frac{1}{2} \{am/2^{\nu-1}\} +$$
$$+ \frac{1}{2} [am/2^{\nu-1}] + \frac{1}{2} \varepsilon \Big\|^{-1} \Big) = 2^{-1} \Big(\Big\| \frac{1}{2} \{am/2^{\nu-1}\} \Big\|^{-1} +$$
$$+ \Big\| \frac{1}{2} \{am/2^{\nu-1}\} + \frac{1}{2} \Big\|^{-1} \Big) = 2^{-1} (2 \{am/2^{\nu-1}\}^{-1} +$$
$$+ 2 (1 - \{am/2^{\nu-1}\})^{-1}) = \| am/2^{\nu-1} \|^{-1} + (1 - \| am/2^{\nu-1} \|)^{-1} \leqslant 2 + \| am/2^{\nu-1} \|^{-1}.$$

Hence it follows from the definition of the quantities $a_{r\nu}$ and properties of the minimum that the following relations are valid for $r = 2, \ldots, s - 1$:

$$h_{r\nu} \leqslant 2^{-1} \sum\nolimits_{z=0}^{1} h_{r\nu}(a_{1\nu}, \ldots, a_{r-1\nu}, a_{r\nu-1}(1 + (\varepsilon_{r\nu-1} - 1) z) +$$
$$+ 2^{\nu-1} z, a_{r+1\nu-1}, \ldots, a_{s-1\nu-1}) \leqslant 2^{-\nu} \sum\nolimits_{m=1}^{*2^\nu} (2n - 2\nu + \| m \cdot 2^{-\nu} \|^{-1}) \cdot$$
$$\cdot \prod\nolimits_{j=1}^{r-1} (2n - 2\nu + \| ma_{j\nu} \cdot 2^{-\nu} \|^{-1}) \cdot \prod\nolimits_{j=r}^{s-1} (2n - 2\nu + 2 + \| ma_{j\nu-1} \cdot 2^{1-\nu} \|^{-1}) = h_{r-1\nu}.$$

For $r = 1$ we analogously have

$$h_{1\nu} \leqslant 2^{-\nu} \sum\nolimits_{m=1}^{*2^\nu} (2n - 2\nu + \| m \cdot 2^{-\nu} \|^{-1}) \cdot$$
$$\prod\nolimits_{j=1}^{s-1} (2n - 2\nu + 2 + \| ma_{j\nu-1} \cdot 2^{1-\nu} \|^{-1}) =$$
$$= 2^{1-\nu} \sum\nolimits_{m=1}^{*2^{\nu-1}} (2n - 2\nu + (2 \| m \cdot 2^{-\nu} \|)^{-1} + (2 \| (m + 2^{\nu-1}) \cdot 2^{-\nu} \|)^{-1})$$
$$\cdot \prod\nolimits_{j=1}^{s-1} (2n + 2 - 2\nu + \| ma_{j\nu-1} \cdot 2^{1-\nu} \|^{-1}) \leqslant 2^{1-\nu} \sum\nolimits_{m=1}^{*2^{\nu-1}} (2n - 2\nu + 2 +$$
$$+ \| m \cdot 2^{1-\nu} \|^{-1}) \prod\nolimits_{j=1}^{s-1} (2n + 2 - 2\nu + \| ma_{j\nu-1} \cdot 2^{1-\nu} \|^{-1}) = h_{\nu-1}.$$

Hence $h_\nu = h_{s-1\nu} \le h_{s-2\nu} \le \dots \le h_{1\nu} \le h_{\nu-1}$ and, therefore, $h_n \le h_{n-1} \le \dots \le h_1 = 2^{s-1}n^s$. By the same token, all the conditions of Lemma 1 are fulfilled and the theorem is proved.

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GROWTH OF THE NUMBER OF IMAGES OF A POINT UNDER ITERATES OF A MULTIVALUED MAP

A. P. Veselov

A multivalued map, or a correspondence Φ of the set X into itself, is specified by its graph – a subset Γ_Φ of the Cartesian product $X \times X$. A point y is said to be an image of the point x under the map Φ if $(x, y) \in \Gamma_\Phi$. The images of a point x under the k-th iterate of Φ are defined as the points y for which there exists a sequence $(x_0, x_1, \dots, x_k)$ such that $x_0 = x$, $x_k = y$, $(x_{i-1}, x_i) \in \Gamma_\Phi$, $i = 1, \dots, k$. We are interested in the number $N_x(k)$ of such images as a function of k. In many cases (for instance, in the complex algebraic situation) this number does not depend on a generic point x and is the simplest and a rough characteristic of the corresponding dynamical system.

An important class of correspondences consists of the Lagrangian systems with discrete time (see [1-3]). The main result of our work is that the Liouville-integrability for such and more general symplectic correspondences leads to a "polynomial growth" of N(k) instead of the expected exponential growth.

The problem of describing the correspondences for which N(k) has polynomial growth is important also in the nonsymplectic case, in particular in view of the analogy and connections with Sklyanin's quadratic algebras and the Yang–Baxter equation [4-6]. We give below some partial results in this direction, concerning the simplest case of algebraic correspondences $\mathbf{C} \to \mathbf{C}$, namely those given by a biquadratic equation

$$\sum_{0 \le i,\, j \le 2} a_{ij} x^i y^j = 0.$$

We begin with the symplectic case.

Let M^{2n} be a symplectic manifold with symplectic 2-form ω. We introduce a symplectic structure Ω on the product $M^{2n} \times M^{2n}$ by the formula

$$\Omega := \pi_1^*(\omega) - \pi_2^*(\omega),$$

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