DISJUNCTIVE PROPERTY OF SUPERINTUITIONIST AND MODAL LOGICS

M. V. Zakhar'yashchev

As shown in [1], the formula X ∨ Y is derivable in the intuitionist propositional calculus Int if and only if at least one of its disjunctive terms is derivable in Int. This is the fundamental difference between the intuitionist disjunction and the classical disjunction, since any formula X ∨ ¬Y is derivable in the classical calculus Cl. Attempts to clarify why the disjunctive property occurs in Int but not in Cl have led to the investigation of this property in superintuitionist logics, intermediate between Int and Cl.

The study of simple superintuitionist logics gave rise to the conjecture [2] that no proper consistent extension of Int possesses the disjunctive property. Soon, however, examples were devised of superintuitionist logics both with and without the disjunctive property (see, e.g., [3, 4]). Maksimova [4] presented an algebraic equivalent to the disjunctive property. As yet, however, no syntactic characterization of logics having the property has been proposed.

Our aim is to establish syntactic necessary conditions for a logic to possess the disjunctive property (DP). Specifically: we shall show that if a consistent superintuitionist logic L has the DP, and X is a formula derivable in L and containing no occurrences of ∨, then X is derivable in Int. This result solves problem 19 of [3].

Along with the class $\mathcal{J}$ of superintuitionist logics, we shall consider the class $\mathcal{M}$ of normal modal logics that contain Lewis' system S4. We shall say that a logic $L \in \mathcal{M}$ possesses the disjunctive property if $\Box X \vee \Box Y \in L$ implies $\Box X \in L$ or $\Box Y \in L$. S4 has the disjunctive property [1]. In addition, the largest modal twin of every logic in $\mathcal{J}$, having the property will also have the DP [5].

1. We shall use the relational semantics of logics in $\mathcal{M}$. Let K be a preordered set, or, briefly, a <u>scale</u>. The preorder relation on K will always be denoted by the letter R; if the letter K has some index, the same index will be attached to the letter R. K^c will denote the <u>skeleton</u> of a scale K, i.e., the quotient set K/~ modulo the equivalence $\alpha \sim \beta \rightleftharpoons (\alpha R\beta) \,\&\, (\beta R\alpha)$ with the partial order relation R^c: $\alpha^c R^c \beta^c \rightleftharpoons \alpha R\beta$, where α^c and β^c are the equivalence classes generated by elements α and β, called <u>clots</u>. An element $\alpha \in K$ is said to be R-<u>least</u> if α^c is a least element of K^c. For every $G \subseteq K$, we put

$$IG = \{\alpha \in G \mid (\forall\beta \in K)\,(\alpha R\beta \Rightarrow \beta \in G)\},$$
$$[G)_R = \{\alpha \in K \mid (\exists\beta \in G)\,(\alpha R\beta)\},$$
$$(G]_R = \{\alpha \in K \mid (\exists\beta \in G)\,(\beta R\alpha)\}.$$

A <u>model structure</u> (MS) for S4 is a pair μ = ⟨K, S⟩, where K is a scale, S a system of subsets of K containing ∅ and K and closed under union, intersection and the operation I. Under these conditions the algebra $A_\mu = \langle S;\, \cap, \cup, -, I, \varnothing, K\rangle$ is a topological Boolean algebra with an interior operation I (∅ and K are the zero and identity elements of this algebra), and any topological Boolean algebra is isomorphic to the algebra A_μ of some MS μ [6]. It is known [7] that every finite topological Boolean algebra is isomorphic to the algebra A_μ of some finite MS μ = ⟨K, P(K)⟩, where $P(K) = \{G \mid G \subseteq K\}$. We may therefore assume that every finite MS is of the form ⟨K, P(K)⟩.

Let Var denote a set of propositional variables $\{p_0, p_1, \ldots\}$. Formulas are constructed from these variables by means of connectives ⊃, &, ∨, □ and the constant f – "falsehood." The valuation function F: Var × $K \to \{0, 1\}$ is defined on the MS μ = ⟨K, S⟩ in the usual way; it is only required that $\{\alpha \in K \mid F(p_i, \alpha) = 1\} \in S$ for any $i \in \omega$. Validity of a formula U on μ is denoted by $\mu \models U$. Rational semantics is clearly adequate for all logics in $\mathcal{M}$.

M. V. Keldysh Institute of Applied Mathematics, Academy of Sciences of the USSR, Moscow.
Translated from Matematicheskie Zametki, Vol. 42, No. 5, pp. 729-738, November, 1987. Original article submitted November 15, 1985.

Let $\mu = \langle K_1, S_1\rangle$ be a MS and K a scale. A partial mapping φ from K_1 onto K is called a partial p-morphism if φ satisfies the conditions:

$$(\forall\alpha,\ \beta \in \varphi^{-1}(K))\ (\alpha R_1\beta \Rightarrow \varphi(\alpha)\ R\varphi(\beta));$$

$$(\forall\delta,\ \gamma \in K)\ (\delta R\gamma \Rightarrow (\forall\alpha \in \varphi^{-1}(\delta))\ (\exists\beta \in \varphi^{-1}(\gamma))\ (\alpha R_1\beta)).$$

The mapping φ is said to be regular if $\varphi^{-1}(\alpha) \in S_1$ for any $\alpha \in K$. If μ is a finite MS, then, obviously, every partial p-morphism from K_1 onto K is regular.

2. In [8] we defined canonical formulas of class $\mathscr{M}$ and showed that any logic in $\mathscr{M}$ is axiomatizable by means of canonical formulas. As these formulas will play an important role in the sequel, we shall recall all the necessary definitions and propositions of [8]. Formulas of the type

$$W_1 \supset (W_2 \supset \ldots \supset (W_n \supset X)\ldots)$$

will be written in the abbreviated form

$$W_1 \supset . W_2 \supset . \ldots \supset . W_n \supset X$$

or $\Gamma \supset X$, where $\Gamma = \{W_1, W_2, \ldots, W_n\}$.

Let K be a finite scale with R-least element. Let us assume that a certain element is designated in each clot $\alpha^c \subseteq K$, called a representative of α^c. Sequences of elements of K^c will be denoted by $\bar\alpha^c$ and $\bar\beta^c$.

Let us assume that the sequences $\bar\alpha^c$ and $\bar\beta^c$ are nonempty and consist of elements no two of which are comparable in K^c; $|\bar\alpha^c| \geqslant 2$, the relation $\alpha^c R^c \beta^c$ does not hold for any $\alpha^c \in \bar\alpha^c$, $\beta^c \in \bar\beta^c$ and $(\forall\gamma^c \in K^c)\ ((\forall\alpha^c \in \bar\alpha^c)\ (\gamma^c R^c \alpha^c) \Rightarrow (\exists\beta^c \in \bar\beta^c)\ (\gamma^c R^c \beta^c))$. In that case, a pair of sets $(\bar\alpha^c, \bar\beta^c)$ will be called a d-region of K. Let D be some (possibly empty) set of d-regions of K, and $\alpha_0, \ldots, \alpha_n$ all different elements of K. Given K and D, a formula Y(K, D, f) is constructed as follows.

If α_i^c is a least element of K^c and α_i the representative of α_i, then the conclusion of Y(K, D, f) will be the variable p_i. With every two distinct elements $\alpha_i,\ \alpha_j \in K$, such that either $\alpha_i^c = \alpha_j^c$ or α_i^c is an immediate predecessor of α_j^c in K^c and α_i, α_j are the representatives of α_i^c, α_j^c, respectively, we associate the formula $W_{ij} \rightleftharpoons \Box\ (\Box p_j \supset p_i)$. In addition, with every $\alpha_i \in K$ we associate the formula $W_i \rightleftharpoons \Box\ ((\Gamma_i \supset . \Delta_i \supset p_i) \supset p_i)$, where $\Gamma_i = \{\Box p_l \mid \neg\, \alpha_i R \alpha_l\}$, $\Delta_i = \{p_0, \ldots, p_n\} \setminus \{p_i\}$. Finally, with every d-region $d = (\bar\alpha^c, \bar\beta^c) \in D$ we associate the formula

$$V_d \rightleftharpoons \Box(\Box p_{j_1} \supset . \ldots \supset . \Box p_{j_l} \supset (\Box p_{i_1} \vee \ldots \vee \Box p_{i_m})),$$

where $\bar\alpha^c = \{\alpha^c_{i_1}, \ldots, \alpha^c_{i_m}\}$, $\bar\beta^c = \{\alpha^c_{j_1}, \ldots, \alpha^c_{j_l}\}$ and the elements α_s are the representatives of the clots $s \in \{i_1, \ldots, i_m, j_1, \ldots, j_l\}$.

The premises of Y(D, K, f) will be the formulas W_{ij}, W_i, and V_d for all possible values of the parameters i, j, and d, as well as the formula $Z \rightleftharpoons \Box(\Box p_0 \supset . \ldots \supset . \Box p_n \supset f)$. Formulas of type Y(K, D, f) are called canonical formulas of class $\mathscr{M}$. As shown in [8], every logic in $\mathscr{M}$ can be obtained by adding to S4 a certain set of canonical formulas $\{Y(K_i, D_i, f)\}_{i\in T}$ as new axioms; the logic $S4 + \{Y(K_i, D_i, f)\}_{i\in T}$ will henceforth be denoted by $[Y(K_i, D_i, f)]_{i\in T}$.

We shall say that a MS $\mu = \langle K_1, S_1\rangle$ is admissible for a formula Y(K, D, f) if there exists a regular partial p-morphism φ from K_1 onto K, satisfying the following conditions:

1) If $(\bar\alpha^c, \bar\beta^c) \in D$, $\gamma \in (\varphi^{-1}(K)]_{R_1}$ and $\alpha \in \varphi((\gamma]_{R_1})$ for all $\alpha^c \in \bar\alpha^c$, then $\beta \in \varphi((\gamma]_{R_1})$ for some $\beta^c \in \bar\beta^c$.

2) If $\gamma \in (\varphi^{-1}(K)]_{R_1}$, then $\gamma \in [\varphi^{-1}(K))_{R_1}$.

THEOREM 1 [8]. $\mu \not\models Y(K, D, f)$ if and only if the MS μ is admissible for the formula Y(K, D, f).

Canonical formulas for class $\mathscr{Y}$ are constructed in [8, 9] along lines analogous to the construction of formulas Y(K, D, f). They are denoted by X(K, D, f) and are based on a partially ordered scale K. Every logic $L_1 \in \mathscr{Y}$ may be expressed in the form

$$L_1 = \mathrm{Int} + \{X(K_i, D_i, f)\}_{i\in T} \rightleftharpoons [X(K_i, D_i, f)]_{i\in T}.$$

A logic $L \in \mathscr{M}$ is called a modal twin of a logic $L_1 \in \mathscr{Y}$ if L_1 consists of all formulas whose McKinsey–Tarski translations [1] are derivable in L.

THEOREM 2 [8]. A logic $L \in \mathcal{M}$ is a twin of a logic $L_1 = [X(K_i, D_i, f)]_{i \in T}$ if and only if it may be expressed in the form $L = [Y(K_i, D_i, f), Y(K_j, D_j, f)]_{i \in T, j \in Q}$, where each of the scales K_j, $j \in Q$, has at least one nontrivial clot (i.e., a clot containing at least two elements).

3. As follows from Theorem 1, the simplest structure is that of countermodels of formulas $Y(K, \varnothing, f)$. Let $\mathcal{M}_1$ denote the class of logics in $\mathcal{M}$ that are axiomatizable by means of formulas of this type only. Note that the class $\mathcal{J}(\supset, f)$ of logics in $\mathcal{J}$ that are axiomatizable by means of formlas $X(K, \varnothing, f)$ consists precisely of those superintuitionist logics that can be obtained by adding axioms not containing occurrences of [9].

THEOREM 3. Let $L = [Y(K_i, \varnothing, f)]_{i \in T}$. Then $Y(K, D, f) \in L$ if and only if the MS $\mu = \langle K, P(K)\rangle$ is admissible for at least one of the formulas $Y(K_i, \varnothing, f)$, $i \in T$.

Proof. Assume that μ is admissible for a formula $Y(K_i, \varnothing, f)$ for some $i \in T$. Then there exists a partial p-morphism φ: $K \to K_i$ such that $\alpha \in (\varphi^{-1}(K_i)]_R$ implies $\alpha \in [\varphi^{-1}(K_i))_R$. Now suppose that $\mu_1 = \langle K_1, S_1\rangle$ and $\mu_1 \nvDash Y(K, D, f)$. We then have a regular partial p-morphism ψ: $K_1 \to K$, for which $\alpha \in (\psi^{-1}(K)]_{R_1}$ implies $\alpha \in [\psi^{-1}(K))_{R_1}$. Since K is finite, the partial mapping $\varphi\psi$: $K_1 \to K_i$ is a regular partial p-morphism and, in addition, it is easy to see that $\alpha \in (\psi^{-1}\varphi^{-1}(K_i)]_{R_1}$ implies $\alpha \in [\psi^{-1}\varphi^{-1}(K_i))_{R_1}$. Hence $\mu_1 \nvDash Y(K_i, \varnothing, f)$ and $Y(K, D, f) \in [Y(K_i, \varnothing, f)]$.

The converse follows from Theorem 1.

COROLLARY 1. All logics of class $\mathcal{M}_1$ are finitely approximable.

Proof. Let $L \in \mathcal{M}_1$, $U \notin L$ and $[U] = [Y(K_1, D_1, f), \ldots, Y(K_n, D_n, f)]$. By Theorem 3, this is possible only if the MS $\mu = \langle K_i, P(K_i)\rangle$ is a model of L. But since $\mu \nvDash Y(K_i, D_i, f)$, it follows that $\mu \nvDash U$.

In exactly the same way it can be proved that logics of class $\mathcal{J}(\supset, f)$ are finitely approximable – a well-known result of McKay [10].

4. We shall call a formula $Y(K, \varnothing, f)$ singular if the scale K satisfies the following conditions:

3) A clot generating an R-least element of K is trivial.

4) There exists an element $\alpha \in K$ such that $(\alpha]_R = \{\alpha\}$.

THEOREM 4. Let $L \in \mathcal{M}_1$ and assume that L is axiomatizable by means of nonsingular formulas alone. Then L possesses the DP.

Proof. Suppose the contrary. Then there is a formula $\Box X \vee \Box Y \in L$ such that $\Box X \notin L$ and $\Box Y \notin L$. The logic L is finitely approximable, and so there exist finite MS's $\mu_1 = \langle K_1, P(K_1)\rangle$ and $\mu_2 = \langle K_2, P(K_2)\rangle$ such that $\mu_1 \vDash L$, $\mu_2 \vDash L$ but $\mu_1 \nvDash \Box X$, $\mu_2 \nvDash \Box Y$. We may clearly assume that the scale K_1 has an R_1-least element α_1 and K_2 an R_2-least element α_2. Now let $K' = \{\alpha_0, \beta\} \cup K_1 \cup K_2$; let R' be the transitive closure of the relation $\{(\alpha_0, \alpha_0), (\alpha_0, \beta), (\beta, \beta), (\alpha_0, \alpha_1), (\alpha_0, \alpha_2)\} \cup R_1 \cup R_2$ and let $\mu = \langle K', P(K')\rangle$. It is readily seen that $\mu \nvDash \Box X \vee \Box Y$. To complete the proof, we must show that μ is a model of L.

If not, then μ refutes at least one of the axioms of L, say a nonsingular formula $Y(K, \varnothing, f)$. We thus have a partial p-morphism φ: $K' \to K$ such that $\alpha \in (\varphi^{-1}(K)]_{R'}$ implies $\alpha \in [\varphi^{-1}(K))_{R'}$. Note that $\alpha_0 \in \varphi^{-1}(K)$, for otherwise, by Theorem 1, either $\mu_1 \nvDash Y(K, \varnothing, f)$ or $\mu_2 \nvDash Y(K, \varnothing, f)$, or the formula $Y(K, \varnothing, f)$ is refuted on the one-element scale, which is impossible (the third case would imply that L is contradictory). Hence, in particular, it follows that $K' = [\varphi^{-1}(K))_{R'}$ and $\beta \in \varphi^{-1}(K)$.

The scale K cannot satisfy both conditions 3 and 4. If a least clot α^c in K^c is nontrivial, then there exists $\gamma \in K$, $\alpha \neq \gamma$, $\alpha R\gamma$, and $\gamma R\alpha$. Obviously, $\varphi(\alpha_0) \in \alpha^c$; to fix ideas, suppose that $\varphi(\alpha_0) = \alpha$. But then $(\exists\delta \in K')(\varphi(\delta) = \gamma)$ and $(\exists\varepsilon \in K')(\delta R'\varepsilon \;\&\; \varphi(\varepsilon) = \alpha)$, whence it follows that $\varepsilon \in K_1$ or $\varepsilon \in K_2$. Consider the scale $K_3 = (\varepsilon]_{R'}$ and the mapping φ', the restriction of φ to K_3. It is readily verified by a direct check that φ' is a partial p-morphism from K_3 onto K, and so one of the MS's μ_1 or μ_2 is admissible for $Y(K, \varnothing, f)$, which is impossible.

Now suppose that condition 4 is violated, and consider the element $\varphi(\beta) \in K$. By the definition of a p-morphism, $(\forall\gamma \in K)$ $(\varphi(\beta) R\gamma \Rightarrow$, whence it follows that $\varphi(\beta) = \gamma)$ $(\varphi(\beta)]_R = \{\varphi(\beta)\}$. Contradiction.

5. Given a scale K and one of its elements α, let us define a new scale $K^{\alpha} = (\alpha]_R$ with a relation R^{α}, the restriction of R to K_{α}. A singular formula $Y(K, \emptyset, f)$ will be called a minimal formula of a logic $L \in \mathcal{M}$ if it is derivable in L, but $Y(K^{\alpha}, \emptyset, f) \notin L$ if $\alpha \in K$ and $K^{\alpha} \neq K$.

It is by no means true that every logic has a minimal formula. For example, such formulas do not exist for logics in $\mathcal{M}_1$ that are axiomatizable by means of nonsingular formulas (this follows from Theorem 5 below). However, all other logics in $\mathcal{M}_1$ have minimal formulas. In fact, let L be one of these logics. For each of its axioms $Y(K, \emptyset, f)$, choose an element $\alpha \in K$, such that $Y(K^{\alpha}, \emptyset, f) \in L$ but $Y(K^{\beta}, \emptyset, f) \notin L$ if $\beta \in K^{\alpha}$ and $K^{\alpha} \neq K^{\beta}$. Since $Y(K, \emptyset, f) \in [Y(K^{\alpha}, \emptyset, f)]$, it follows that L can be defined by means of formulas $Y(K^{\alpha}, \emptyset, f)$, and at least one of these is singular. There are also minimal formulas for any logic $L = [Y(K, \emptyset, f), Y(K_i, D_i, f)]_{i \in T}$ with a partially ordered scale K.

THEOREM 5. If a consistent logic $L \in \mathcal{M}$ has a minimal formula, then L does not possess the DP.

Proof. Let $Y(K, \emptyset, f)$ be a minimal formula of L, $\alpha_0, \alpha_1, \ldots, \alpha_m, \alpha_{m+1}, \ldots, \alpha_n$ all elements of K, where α_0 is the unique R-least element of K, $(\alpha_n]_R = \{\alpha_n\}$, and the elements $\alpha_1, \ldots, \alpha_m$ are the representatives of clots $\alpha_1^c, \ldots, \alpha_m^c$ which are immediate successors of α_0^c in K^c. Starting from K, we construct a formula U(K). Put

$$\Sigma = \{W_{ij} \mid i, j \in \{1, \ldots, n\} \,\&\, \alpha_i R \alpha_j\} \cup \{W_1, \ldots, W_n\}$$

and $V_i \rightleftharpoons \Gamma_i \supset . \Delta_i \supset . \Sigma \supset . Z \supset p_i$ for every $i \in \{1, \ldots, m\}$ [the formulas W_{ij}, W_i and sets Γ_i and Δ_i were defined in Sec. 2, and $Z \rightleftharpoons \Box(\Box p_1 \supset . \ldots \supset . \Box p_n \supset f)$]. If m = 1, we let $U(K) \rightleftharpoons \Box p_0 \vee \Box V_1$; otherwise, $U(K) \rightleftharpoons \Box V_1 \vee \ldots \vee \Box V_m$.

LEMMA. $U(K) \in [Y(K, \emptyset, f)]$.

Proof. It will suffice to show that if $\mu = \langle K_1, P(K_1) \rangle$ is a finite MS and $\mu \not\models U(K)$, then $\mu \not\models Y(K, \emptyset, f)$. Fix a valuation function F on μ such that $F(U(K), \alpha) = 0$ for some $\alpha \in K_1$, and mark certain elements of K_1 with letters $p_0, \ldots, p_n$. Specifically: the letter p_0 will mark all elements $\beta \in K_1$ for which $F(U(K), \beta) = 0$; but if $F(X, \beta) = 1$ for any formula $X \in \Gamma_i \cup \Delta_i \cup \Sigma \cup \{Z\}$, $F(p_i, \beta) = 0$ $(i > 0)$, then β is marked with the letter p_i. Put $\varphi(\beta) = \alpha_i$ if β is marked with p_i. We shall show that φ is a partial p-morphism from K_1 onto K.

Let $\alpha \in K_1$ be marked with p_i and $\alpha_i R \alpha_j$. We claim that there is an element $\beta \in (\alpha]_{R_1}$ marked with the letter p_j. If $i \neq 0$, then $j \neq 0$ and $W_{ij} \in \Sigma$. Since $F(W_{ij}, \alpha) = 1$, while $F(p_i, \alpha) = 0$, it follows that $F(\Box p_j, \alpha) = 0$. Consequently, there exists $\beta \in (\alpha]_{R_1}$, $F(p_j, \beta) = 0$. Obviously, $F(X, \beta) = 1$ for any formula $X \in \Sigma \cup \{Z\}$, whence, in particular, we obtain $F(W_j, \beta) = 1$. Together with $F(p_j, \beta) = 0$, this last equality yields $F(X, \beta) = 1$ for every formula $X \in \Gamma_j \cup \Delta_j$. Now let $i = 0$. There exists $l \in \{1, \ldots, m\}$ such that $\alpha_l R \alpha_j$. Hence, in view of what has been proved, we need only find an element $\gamma \in (\alpha]_{R_1}$ marked with the letter p_l. But this follows immediately from $F(\Box V_l, \alpha) = 0$.

Let $\alpha, \beta \in K_1$ be elements marked with letters p_i and p_j, respectively, such that $\alpha R_1 \beta$. We claim that $R_i R \alpha_j$. If $i = 0$, then $\alpha_0 R \alpha_j$ for any $j \in \{0, \ldots, n\}$. Suppose now that $i \neq 0$ and $\neg \alpha_i R \alpha_j$. Then $\Box p_j \in \Gamma_i$, and so $F(\Box p_j, \alpha) = 1$. If $j \neq 0$, then $F(p_j, \beta) = 0$, whence $F(\Box p_j, \alpha) = 0$, which is impossible. Let $j = 0$. If m = 1, then $F(\Box p_0, \beta) = 0$, contradicting $\Box p_0 \in \Gamma_i$ (since $\alpha_0^c = \{\alpha_0\}$) and $F(\Box p_0, \alpha) = 1$. But if m > 1, then there exists an element α_l such that $\neg \alpha_i R \alpha_l$ and $1 \leqslant l \leqslant m$, and since $\alpha_0 R \alpha_l$, there is an element $\gamma \in (\beta]_{R_1}$ marked with the letter p_l, so we have again arrived at a contradiction. Hence, in particular, it follows that each element of K_1 can be marked with at most one letter.

We have thus shown that φ is a partial p-morphism from K_1 onto K. However, it need not satisfy condition 2. We must therefore "adjust" φ slightly. Namely: let $\beta \in K_1$ be an element such that

5) β^c is a maximal element of the scale K_1^c;

6) there exists an element $\alpha \in [\beta)_{R_1}$ marked with the letter p_0;

7) if an element $\gamma \in [\beta)_{R_1}$ is marked with a letter p_i, then $i = 0$. We now put $\varphi(\beta) = \alpha_n$. It is obvious that this "adjusted" mapping φ is still a partial p-morphism from K_1 onto K.

Let $\gamma \in (\varphi^{-1}(K)]_{R_1}$. Choose an element $\beta \in (\gamma]_{R_1}$ satisfying condition 5. If there is an element $\delta \in [\beta)_{R_1}$ marked with the letter p_i, $i \neq 0$, then $F(Z, \delta) = F(Z, \beta) = 1$. Consequently, there exist an element $\varepsilon \in (\beta]_{R_1}$ and a number $j \in \{1, \ldots, n\}$, such that $F(p_j, \varepsilon) = 0$ and, as already shown, ε is marked with the letter p_j. But if there is no such element δ, then there exists an element $\alpha \in [\beta)_{R_1}$ marked with p_0, whence it follows that β is marked with the letter p_n. Thus, $\gamma \in [\varphi^{-1}(K))_{R_1}$. We have shown that the MS μ is admissible for the formula $Y(K, \varnothing, f)$, and so $\mu \not\models Y(K, \varnothing, f)$. This completes the proof of the lemma.

We can now complete the proof of Theorem 5 by showing that $Y(K^{\alpha_i}, \varnothing, f) \in [V_i]$, $1 \leqslant i \leqslant m$. Since $Y(K^{\alpha_i}, \varnothing, f) \notin L$ [because the formula $Y(K, \varnothing, f)$ is minimal for L], it will follow from these inclusions that $\Box V_i \notin L$; and since L is consistent, it is also true that $\Box p_0 \notin L$. This implies that L does not possess the DP.

Let $\mu = \langle K', S\rangle$, $\mu \not\models Y(K^{\alpha_i}, \varnothing, f)$ and let F be a valuation function on μ such that $F(Y \times (K^{\alpha}i, \varnothing, f), \alpha) = 0$ for some $\alpha \in K'$. If $\alpha_j \notin (\alpha_i]_R$, then the variable p_j has no occurrences in $Y(K^{\alpha_i}, \varnothing, f)$. We may therefore assume that $F(p_j, \beta) = 1$ for all $\beta \in K'$. It follows from this assumption that $F(X, \alpha) = 1$ for any premise X of the formula V_i. Moreover, $F(p_i, \alpha) = 0$. Hence $F(V_i, \alpha) = 0$. This completes the proof of Theorem 5.

COROLLARY 2. If a logic $L_1 \in \mathcal{J}$ is consistent, $U \in L_1$, $U \notin \mathrm{Int}$ and the formula U contains no occurrences of $\vee$, then L_1 does not possess the DP.

Proof. Let $[U] = [X(K_1, \varnothing, f), \ldots, X(K_n, \varnothing, f)]$, $n \geqslant 1$ [9]. Then L_1 can be represented in the form $L_1 = [X(K_1, \varnothing, f), X(K_i, D_i, f)]_{i \in T}$, and its largest twin [8] is $L = [Y(K_1, \varnothing, f), Y(K_j, D_j, f)]_{j \in Q}$. As marked above, L has a minimal formula (K_1 is partially ordered). Therefore L does not have the DP. But since the disjunctive property is preserved by transition to the modal twin [5], the same conclusion holds true for L_1.

One consequence of this corollary is that no logic in the class $\mathcal{J}(\supset, f)$, other than Int and the contradictory logic, possesses the DP.

Applying the law of contraposition to this last statement, we obtain

COROLLARY 3. Let $L \in \mathcal{J}$ be a consistent logic possessing the DP and let U be a formula with no occurrences of $\vee$. Then $U \in L$ if and only if $U \in \mathrm{Int}$.

LITERATURE CITED

1. I. C. C. McKinsey and A. Tarski, "Some theorems about the sentential calculi of Lewis and Heyting," J. Symb. Log., 13, No. 1, 1-15 (1948).
2. I. Lukasiewicz, "On the intuitionistic theory of deduction," Indag. Math., 14, No. 2, 202-212 (1952).
3. T. Hosoi and H. Ono, "Intermediate propositional logics (a survey)," J. Tsuda Coll., No. 1, 67-82 (1973).
4. L. L. Maksimova, "Remarks on the disjunctive property in superintuitionist logics," in: Semiotic Aspects of the Formalization of Intellectual Activity. School-Seminar. Telavi, 29 October-6 November, 1983. Abstracts of Lectures [in Russian], VINITI, Moscow (1983), pp. 83-85.
5. V. L. Gudovshchikov and V. V. Rybakov, "The disjunctive property in modal logics," in: Materials for the Seventh All-Union Conference on Logic and the Methodology of Science, Vilnius, 1982 [in Russian], IFAN, Moscow (1982), pp. 35-36.
6. B. Jonsson and A. Tarski, "Boolean algebras with operators," Am. J. Math., 73, No. 4, 891-939 (1951).
7. M. A. E. Dummett and E. J. Lemmon, "Modal logics between S4 and S5," Z. Math. Logik und Grundl. Math., No. 3/4, 250-264 (1959).
8. M. V. Zakhar'yashchev, "Normal modal logics containing S4," Dokl. Akad. Nauk SSSR, 275, No. 3, 537-540 (1984).
9. M. V. Zakhar'yashchev, "On intermediate logics," Dokl. Akad. Nauk SSSR, 269, No. 1, 18-22 (1983).
10. C. G. McKay, "The decidability of certain intermediate logics," J. Symb. Logic, 33, No. 2, 258-264 (1968).