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On the measurability of the quantum phase distribution

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Abstract. We give a prescription for the measurement of the quantum phase distribution of a single-mode radiation field. The phase probability distribution is defined as the squared modulus of the photon wave function in the phase representation. The measurement involves all moments of the signal from the balanced homodyne detection.

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Our definition of the phase probability distribution is based on the notion of the photon wave function in the phase representation. Such a phase representation has already been anticipated a long time ago by London [1], and used extensively by us in the description of intense photon beams [2–5]. The wave function $\psi(\varphi)$ in the phase representation is defined in terms of the expansion coefficients ψ_n of the state vector $|\psi\rangle$,

$$|\psi\rangle = \sum_{n=0}^{\infty} \psi_n |n\rangle, \tag{1}$$

into the photon number state vectors $|k\rangle$ through the formula

$$\psi(\varphi) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \psi_n \exp(in\varphi). \tag{2}$$

It is clear that there is a one-to-one correspondence between the state vectors of a single-mode radiation field and the wave functions (2). The squared modulus of the wave function $|\psi(\varphi)|^2$ gives the phase probability distribution in full analogy with the probability distribution of the x variable being given in wave mechanics of massive particles by the squared modulus $|\psi(x)|^2$ of the wave function in the position representation.

In order to avoid dealing with non-normalizable vectors, we shall restrict ourselves at the beginning to an N-dimensional subspace of the Fock space, and we shall label the corresponding quantities with the superscript N. For all state vectors belonging to this subspace, the wave function in the phase representation can be obtained (up to a normalization factor) by the projection of the state vector $|\psi\rangle$,

$$\psi^N(\varphi) = \langle\varphi^N|\psi\rangle, \tag{3}$$

on a normalized phase-state vector $|\varphi^N\rangle$,

$$|\varphi^N\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \exp(-in\varphi)|n\rangle, \tag{4}$$

that has been introduced by Pegg and Barnett [6].

The aim of this paper is to show how to recover the phase probability distribution directly from experimental data using the method of balanced homodyne detection. A proposal to determine the phase probability distribution with the help of homodyne detection has been recently made by Vogel and Schleich [7]. They have shown that, under certain conditions, one can approximately determine this distribution. We shall show that using the same experimental data one can, in principle, determine the phase probability density exactly. The outline of this scheme has already been sketched in our previous publication [10] in connection with the problem of reconstructing the phase of the photon wave function.

Our method is based on the observation that every operator in Fock space can be expressed as a sum of products of creation and annihilation operators. In particular, we can apply this observation to the projection operator P_φ^N,

$$P_\varphi^N = |\varphi^N\rangle\langle\varphi^N|. \tag{5}$$

This projection operator is directly related to the problem at hand; its expectation value in a given state $|\psi\rangle$ gives the phase probability density

$$\langle\psi|P_\varphi^N|\psi\rangle = |\psi^N(\varphi)|^2. \tag{6}$$

In order to express the operator P_φ^N in terms of creation and annihilation operators, we insert into the definition (5) the explicit form (4) of the phase-state vector to obtain

Dedicated to H.Walter on the occasion of his 60th birthday

$$P_\varphi^N = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} \mathrm{e}^{-\mathrm{i}(k-l)\varphi}\frac{1}{\sqrt{k!l!}} a^{\dagger k}|0\rangle\langle 0|a^l. \tag{7}$$

From this representation, we can see that the projection operator is related by a Fourier transformation to the unitary phase operator U_{PB} of Pegg and Barnett

$$\begin{aligned}\frac{N}{2\pi}\int d\varphi \mathrm{e}^{\mathrm{i}\varphi} P_\varphi^N &= \sum_{k=0}^{N-2} |k\rangle\langle k+1| \\ &= U_{\mathrm{PB}} - |N-1\rangle\langle 0|\mathrm{e}^{\mathrm{i}N\varphi_0},\end{aligned} \tag{8}$$

where φ_0 is an arbitrary phase introduced by Pegg and Barnett.

Using the known representation [11] of the vacuum projection operator in terms of creation and annihilation operators restricted to the N-dimensional subspace,

$$|0\rangle\langle 0| = \sum_{n=0}^{N-1} \frac{(-1)^n}{n!} a^{\dagger n} a^n, \tag{9}$$

we arrive, after some rearrangements, at the following normally ordered expression for the projection operator P_φ^N

$$\begin{aligned}P_\varphi^N = \frac{1}{N}[1 + \sum_{m=1}^{N-1}\sum_{k=0}^{2N-2} d_{mk} \\ \times(\mathrm{e}^{-\mathrm{i}m\varphi} a^{\dagger k+m} a^k + \mathrm{e}^{\mathrm{i}m\varphi} a^{\dagger k} a^{k+m})],\end{aligned} \tag{10}$$

where the numerical coefficients d_{mk} have the values

$$d_{mk} = \sum_{n=0}^{k} \frac{(-1)^n}{n!\sqrt{(k-n)!(k+m-n)!}}. \tag{11}$$

The values of these coefficients decrease rapidly with the increase of the indices, as is shown in Table 1.

We can formally take the limit when $N \to \infty$ in expression (10) for the projection operator provided we change the normalization to account for the transition from the normalized to a non-normalizable state. This limit exists only in a weak sense, i.e., all matrix elements of the operator $(N/2\pi)P_\varphi^N$ tend to the limit given by the matrix elements of the operator P_φ,

$$\begin{aligned}P_\varphi = \frac{1}{2\pi}[1 + \sum_{m=1}^{\infty}\sum_{k=0}^{\infty} d_{mk} \\ \times(\mathrm{e}^{-\mathrm{i}m\varphi} a^{\dagger k+m} a^k + \mathrm{e}^{\mathrm{i}m\varphi} a^{\dagger k} a^{k+m})].\end{aligned} \tag{12}$$

The phase probability distribution $p(\varphi)$ is for a general mixed state given by the trace of the projection operator P_φ with the density operator,

$$p(\varphi) = \mathrm{Tr}(\rho P_\varphi). \tag{13}$$

For pure states, this trace reduces to the diagonal matrix element of the projection operator

$$p(\varphi) = \langle\psi|P_\varphi|\psi\rangle. \tag{14}$$

Thus, on account of the relationships (12) and (13), we have reduced the problem of the measurement of the phase distribution to the problem of the measurement of the expectation values of all products of creation and annihilation operators. These expectation values can, in principle, be measured in the balanced homodyne-detection scheme. Descriptions of the basic experimental setup for the balanced homodyne detection were given in [7, 9, 10], in particular, in connection with squeezing phenomena. Here, we only will state briefly that the signal obtained from the balanced homodyne detection is formed as a difference in photocounts from two detectors. These two detectors record different superpositions of the single-mode radiation field under study and a reference field in a coherent state of the same mode. The measured signal can be characterized by a phase-dependent field operator $E(\phi)$,

$$E(\phi) = \mathrm{i}(a\mathrm{e}^{-\mathrm{i}\phi} - a^\dagger \mathrm{e}^{\mathrm{i}\phi}), \tag{15}$$

where the creation and annihilation operators refer to the quantized radiation field under study. The phase ϕ characterizes the coherent reference field and may be varied continuously during the experiment.

In most experiments on squeezing only the expectation value of $E(\phi)$ and its second moment are determined. In order to determine the phase distribution, we need all moments of $E(\phi)$. We may encode all these moments in the distribution function $p[E(\phi)]$, as has been done by Vogel and Schleich [7], but we found that combining these moments into a generating function $G(\phi, \lambda)$ of an auxiliary parameter λ will lead us to the desired end more quickly. Our generating function is defined as

$$\begin{aligned}G(\phi,\lambda) &= \langle\exp(\lambda a^\dagger \mathrm{e}^{\mathrm{i}\phi})\exp(-\lambda a\mathrm{e}^{-\mathrm{i}\varphi})\rangle \\ &= \sum_{k=0}^{\infty}\frac{(-\lambda^2)^k}{(k!)^2}\langle a^{\dagger k}a^k\rangle \\ &+ \sum_{m=1}^{\infty}[\mathrm{e}^{\mathrm{i}m\phi}\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^{2k+m}}{k!(k+m)!}\langle a^{\dagger k+m}a^k\rangle + \mathrm{c.c.}],\end{aligned} \tag{16}$$

where the angle brackets denote the quantum mechanical expectation value. This function differs only by a factor $\exp(-\lambda^2/2)$ from the characteristic function $\tilde{w}(\eta,\theta)$ used by Vogel and Risken [8] in their study of the phase quasiprobability distributions.

With the help of the Baker-Hausdorff identity, this generating function can be expressed in terms of the moments of $E(\phi)$,

$$\begin{aligned}G(\phi,\lambda) &= \mathrm{e}^{-\lambda^2/2}\langle \mathrm{e}^{\mathrm{i}\lambda E(\phi)}\rangle \\ &= \mathrm{e}^{-\lambda^2/2}\sum_{k=0}^{\infty}\frac{(\mathrm{i}\lambda)^k}{k!}\langle[E(\phi)]^k\rangle.\end{aligned} \tag{17}$$

Table 1. Numerical values of the d_{mk} coefficients

	k=0	1	2	3	4	5	6
m=1	1.00000	-0.29289	0.08157	-0.01846	0.00345	-0.00055	0.00008
2	0.70711	-0.29886	0.08964	-0.02080	0.00393	-0.00063	0.00009
3	0.40825	-0.20412	0.06455	-0.01531	0.00293	-0.00047	0.00006
4	0.20412	-0.11284	0.03713	-0.00898	0.00173	-0.00028	0.00004
5	0.09129	-0.05402	0.01834	-0.00451	0.00088	-0.00014	0.00002
6	0.03727	-0.02318	0.00807	-0.00201	0.00040	-0.00006	0.00000
7	0.01409	-0.00911	0.00324	-0.00082	0.00016	-0.00003	0.00000

Note that the generating function $G(\phi, \lambda)$ and the probability distribution $p_\phi(E) = |\langle E(\phi)|\psi\rangle|^2$ of [7] are related by the Fourier transformation

$$G(\phi, \lambda) = \mathrm{e}^{\lambda^2/2} \int_{-\infty}^{\infty} dE \mathrm{e}^{\mathrm{i}\lambda E} p_\phi(E). \tag{18}$$

The expansion (17) of the generating function into products of creation and annihilation operators leads directly to the following formulas for the expectation values of these products

$$\langle a^{\dagger k+m} a^k \rangle = \frac{(k+m)!k!}{(2k+m)!} \times \frac{d^{2k+m}}{d\lambda^{2k+m}} \frac{(-1)^k}{2\pi} \int_0^{2\pi} d\phi \mathrm{e}^{-\mathrm{i}m\phi} G(\phi, \lambda) \bigg|_{\lambda=0}. \tag{19}$$

The formula with the reversed roles of $a^\dagger$ and a is obtained by Hermitian conjugation.

Our final result is, therefore, the following prescription how to reconstruct the phase probability distribution $p(\varphi)$ from balanced homodyne-detection measurements

$$p(\varphi) = \frac{1}{2\pi} \Bigg\{ 1 + \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} d_{mk} \Bigg[\mathrm{e}^{-im\varphi} \frac{(k+m)!k!}{(2k+m)!} \times \frac{d^{2k+m}}{d\lambda^{2k+m}} \frac{(-1)^k}{2\pi} \int_0^{2\pi} d\phi \mathrm{e}^{-im\phi} G(\phi, \lambda) \bigg|_{\lambda=0} + \mathrm{c.c.} \Bigg] \Bigg\}. \tag{20}$$

The determination of the phase distribution probability from experimental data is thus possible, but the price is high: in principle,one must measure a full distribution (or all moments) of the variable $E(\phi)$ for all values of ϕ. Obviously, in practice, we can measure only a few moments and probe the ϕ dependence only at several points. This should be sufficient if the number of relevant Fock states is not too high. Fortunately, the number of the relevant Fock states might indeed be fairly small (at least for low intensities) due to the fast decrease of the coefficients d_{mk}. Moreover, one may hope to reduce the amount of information needed to recover the quantum phase probability distribution since (12) and (13) show that only certain linear combinations of the expectation values $\langle a^{\dagger n} a^m \rangle$ enter the formulas for $p(\varphi)$. Our prescription is in a sense too wasteful since it enables one to reconstruct not just $p(\varphi)$ but the entire wave function $\psi(\varphi)$.

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References

1. F. London: Z. Phys. **37**, 915 (1926); Z. Phys. **40**, 193 (1927)
2. I. Bialynicki-Birula, Z. Bialynicka-Birula: Phys. Rev. A **14**, 1101 (1976)
3. I. Bialynicki-Birula: Acta Phys. Austriaca Suppl. **18**, 111 (1977)
4. I. Bialynicki-Birula: In *Radiation Theories, Quantum Electrodynamics,* ed. by A. O. Barut (Plenum, New York 1980) p. 119
5. I. Bialynicki-Birula, Cao Long Van: Acta Phys. Polon. A **57**, 599 (1980)
6. D. T. Pegg, S. M. Barnett: Europhys. Lett. **6**, 483 (1988)
7. W. Vogel, W. Schleich: Phys. Rev. A **44**, 7642 (1991)
8. R. Loudon, P. L. Knight: J. Mod. Opt. **34**, 709 (1987)
   See also the first four articles in the topical issue on squeezed states: J. Opt. Soc. Am. B **4**, 1453 (1987)
9. G. M. D'Ariano, M. G. A. Paris: Phys. Rev. A **49**, 3022 (1994)
   This paper contains also a detailed analysis of the sensitivity of phase-detection schemes
10. Z. Bialynicka-Birula, I. Bialynicki-Birula: J. Mod. Opt. (in press)
11. See, for example, I. Bialynicki-Birula, Z. Bialynicka-Birula: *Quantum Electrodynamics* (Pergamon, Oxford 1975) p. 182a and [10]

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