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Observable entities in quantum-optics networks

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Abstract. We consider the quantum analogue of the classical Jones calculus for passive linear optical systems. Those points of the theory where quantum features have to be manifestly included are discussed. The use of different quasidistribution functions and their restrictions to the observable variables only is presented. The consistency of the theory and its usefulness are discussed.

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The present paper is devoted to the discussion of the differences between the classical Jones calculus and its quantum counterpart. In principle, the results are all trivial, but they are not necessarily self-evident, and provide an interesting illustration of the intricate ways in which quantum features enter even extremely simple systems like those treated in the present work.

1 Passive networks in quantum optics

Quantum optics has proved one of the most fertile and rapidly progressing fields of modern physics. It has produced devices where both the atomic constituents and the interacting fields contribute at the level of one single quantum unit or less. Such systems have made it possible to investigate fundamental questions related to quantum coherence and measurement theory in a regime never before reached. On the other hand, it has also produced new technology revolutionizing optical communications, superhigh-resolution measurements and optical data processing. In all these fields, the inclusion of quantum fluctuations has been an essential feature. In short, quantum optics has been both challenging, rewarding and productive; it has played a central role in the development of modern atomic and molecular physics.

Dedicated to H. Walther on the occasion of his 60th birthday

Classical optics has long served a similar function in technology and fundamental physical research. Here, the theory is well developed, the technology is mature and the industrial applications have a well-established position. Optical imaging, information processing and data transfer are often based on passive linear devices that transform and distribute the incoming signals to various outputs. In all these cases, there exists a linear relationship between the input and output signals, which has been the basis of an optical design procedure called the Jones calculus [1]. This enables the scientist to combine networks by defining the transfer functions of the components used and their interconnections. For the classification of optical instruments and the design and analysis of complex systems, the Jones calculus is an invaluable tool.

With the advent of high-precision quantum-optics systems, even the passive linear networks have been discussed as quantum devices [2–4]. The quantum noise at the input ports becomes redistributed and mixed, and using the technology of squeezed states one can attempt to optimize the performance of measuring devices and data communication systems. In order to systematize this activity and bring it to the level of the classical Jones calculus, the present author formulated a general theory for linear passive networks in [5]; see also [6] for an earlier use of a similar formalism. In our later work we have applied this tool to the analysis of the operation of an interferometer [7] and an optical-field measurement [8].

The idea of a linear network implies that the output is a linear function of the input fields. If the system is genuinely passive, there is no mixing between annihilation and creation operators, and the transfer function is unitary; no energy is gained or lost in the transmission. A phase conjugator violates the first condition and amplifiers and attenuators violate the second. In this paper, we exclude such devices.

Formally, the theory appears to be very similar to the classical one. Thus, I consider it to be of some interest to pinpoint those features where quantum effects are manifest. It is the purpose of this paper to carry out such an investigation in some detail.

The first quantum feature enters the theory in the choice of initial state of the incoming fields. These cannot be described by probability distributions but only with the aid of so called quasiprobability distributions [9, 10]. These allow nonclassical features like negative values for regions of phase space; such distributions have no classical counterparts. On the other hand, all classically allowed distributions cannot be realized in the quantum theory; e.g., the uncertainty relations must be obeyed.

Quantum measurements are aimed at operators that do not always commute. Thus, the ordering of the observables becomes essential. The literature defines various quasidistribution functions corresponding to different orderings; the ordinary photodetector gives normal order (Glauber-Sudarshan distribution) but various mixing detection schemes can record other orderings, e.g., the symmetric one (Wigner distribution). Ekert and Knight [11] have pointed out that the use of such functions may create confusion even in the linear regime of a quantum device. In Sect. 2, I show that, with the present definition of a linear passive network, these problems do not arise, and we can apply the techniques developed in [5] for any of the ordinary quasidistribution functions.

Another quantum feature that is unavoidable is the non-commutativity between the two quadrature components of an optical output. This implies that we can access only half of the potential information at a quantum output in contrast to the classical situation. This is the quantum optics manifestation of the complementarity aspect between position and momentum in quantum theory.

As a consequence, we have to select which component we measure at each output of an optical device, and only the marginal distribution reduced to the corresponding phase-space variables carries observational meaning. In our case, this applies to the quasiprobability distribution functions chosen to represent the outputs. For a general input state of the fields, the ensuing reduced quasidistribution function may become rather unwieldy. In order to simplify the treatment, we look at input Gaussian distributions in Sect. 3. Of course, these form an extremely restricted subset of all allowed distributions [12] but, as long as we are mainly interested in the transfer of the first two moments, they provide adequate illustrations of the main ideas. They also include the important coherent and squeezed states as special cases. In Sect. 3, the reduced quasidistribution function depending only on the accessible information is derived in a general case using the formalism developed in [5]. The noise characteristics of the system are obtained from the second moments of such a distribution. Another equivalent method to obtain the same information is to Fourier transform the full quasidistribution, which gives the moment-generating characteristic function. Its derivatives with respect to the Fourier variables gives the moments as in classical probability theory. The reduction to the accessible information can be achieved directly from this by setting the Fourier variables corresponding to the unobserved components equal to zero. This gives another expression for the correlation functions between the observed outputs.

In Sect. 4, I go through a formal proof that the two methods discussed do indeed give exactly the same results. Physical considerations demand this to be the case, and the proof is, in fact, but a trivial verification of the consistency of the formalism.

In Sect. 5, I conclude the paper by a brief summary of the theory and make a few comments on its use and significance.

2 The quasidistribution functions

The linear networks we are going to discuss have a set of M inputs described by the photon annihilation operators $\{a_i | i = 1, \dots, M\}$ and M outputs, respectively, $\{b_i | i = 1, \dots, M\}$. These are taken to be column vectors according to

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_M \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_M \end{bmatrix}. \tag{1}$$

The main characteristic of a linear passive network is that it does not mix creation and annihilation variables, and hence its transfer function between inputs and outputs can be written

$$\mathbf{b} = \mathscr{L}\mathbf{a}, \tag{2}$$

where $\mathscr{L}$ is a unitary transformation.

In quantum optics, the use of generating functions for operator moments have found widespread use. If we define a complex column vector ξ, we can introduce the generating function labelled by σ in the following way

$$Z_\sigma(\xi) = \mathrm{Tr}\,\{\rho \exp[\mathrm{i}(\mathbf{a}^\dagger \cdot \xi + \xi^\dagger \cdot \mathbf{a}) + \tfrac{\sigma}{2}\xi^\dagger \cdot \xi]\}. \tag{3}$$

For $\sigma = 1$, this gives the normally ordered expectation values, and for $\sigma = 0$ the symmetrically ordered ones. The Hermitian conjugation sign $\dagger$ changes column vectors to row vectors and takes the Hermitian conjugation (for $\mathbf{a}$) and the complex conjugation (for ξ).

Fourier transforming the moment generating functions, we obtain the quasidistribution functions according to

$$\begin{aligned} W_\sigma(\mathbf{z}) &= \tfrac{1}{(2\pi)^{2M}} \int d^{2M}\xi\, \mathrm{e}^{-\mathrm{i}(\xi^\dagger \cdot \mathbf{z} + \mathbf{z}^\dagger \cdot \xi)} Z_\sigma(\xi) \\ &= \tfrac{1}{(2\pi)^{2M}} \int d^{2M}\xi\, \mathrm{e}^{-\mathrm{i}(\xi^\dagger \cdot \mathbf{z} + \mathbf{z}^\dagger \cdot \xi)} \\ &\quad \times \mathrm{Tr}\{\rho \exp[\mathrm{i}(\mathbf{a}^\dagger \cdot \xi + \xi^\dagger \cdot \mathbf{a}) + \tfrac{\sigma}{2}\xi^\dagger \cdot \xi]\}. \end{aligned} \tag{4}$$

For $\sigma = 1$ this gives the Glauber-Sudarshan function, and for $\sigma = 0$ we obtain the Wigner function; $\sigma = -1$ gives the Q-function, a special case of the Husimi function [13]. Ekert and Knight [11] point out the special advantages of using the Wigner function for an arbitrary linear transformation. For our passive networks, we show that no complications arise.

We first introduce a new integration variable in (4) according to

$$\eta = \mathscr{L}\xi. \tag{5}$$

Because $\mathscr{L}$ is unitary, the integration domain is not changed; the other terms in the exponent are transformed as follows

$$\begin{aligned} \xi^\dagger \cdot \xi &= (\mathscr{L}^\dagger \eta)^\dagger \cdot \mathscr{L}^\dagger \eta = \eta^\dagger \cdot \eta \\ \xi^\dagger \cdot \mathbf{a} &= (\mathscr{L}^\dagger \eta)^\dagger \cdot \mathbf{a} = \eta^\dagger \cdot \mathscr{L}\mathbf{a} \\ \xi^\dagger \cdot \mathbf{z} &= (\mathscr{L}^\dagger \eta)^\dagger \cdot \mathbf{z} = \eta^\dagger \cdot \mathscr{L}\mathbf{z}, \end{aligned} \tag{6}$$

and their conjugate relations.

If we now introduce the relation (2) between the input and output variables into the relation (4) we obtain the result

$$W_\sigma(\mathbf{z}) = \frac{1}{(2\pi)^{2M}} \int d^{2M}\eta \, e^{-i(\eta^\dagger \cdot \mathscr{L} \cdot z + (\mathscr{L}z)^\dagger \cdot \eta)} \times \mathrm{Tr}\{\rho \exp[i(\mathbf{b}^\dagger \cdot \eta + \eta^\dagger \cdot \mathbf{b}) + \tfrac{\sigma}{2}\eta^\dagger \cdot \eta]\}. \tag{7}$$

The function given by the trace expression clearly generates the moments of the output variables $\mathbf{b}$ in the σ-ordered form. If we introduce the new complex variables

$$w = \mathscr{L} z, \tag{8}$$

the function in (7) is the quasidistribution function of the outgoing variables in terms of the density matrix of the incoming states. We can thus write the general input–output relationship

$$W_\sigma^{\text{out}}(w) = W_\sigma^{\text{in}}(\mathscr{L}^\dagger w). \tag{9}$$

This result corresponds to the result (7) in our earlier publication [5], which has now been shown to be valid for an arbitrarily ordered quasidistribution function. Thus, the problems mentioned by Ekert and Knight do not affect the passive optical network.

3 Extraction of observable entities

Following our work [5], we now separate real and imaginary parts of the input and output operators according to

$$\begin{aligned} a_i &= \tfrac{1}{\sqrt{2}}(x_{2i-1} + i x_{2i}) \\ b_i &= \tfrac{1}{\sqrt{2}}(y_{2i-1} + i y_{2i}) \end{aligned} \quad (i = 1, 2, \ldots, M). \tag{10}$$

From these we construct new $2M$-dimensional column vectors $\mathbf{x}$ and $\mathbf{y}$. The linear relationship (2) induces a linear relationship L between these input and output quantities; we write

$$\mathbf{y} = L\mathbf{x}, \tag{11}$$

where L now is a real orthogonal transformation of dimension $2M \times 2M$.

We are prevented from measuring all the output observables by their quantum-mechanical character. In fact, only half of the output variables can be observed simultaneously because we have

$$\begin{aligned} [x_{2i-1}, x_{2i}] &= i, \\ [y_{2i-1}, y_{2i}] &= i. \end{aligned} \tag{12}$$

We now assume that we observe the odd components of the column vector $\mathbf{y}$ and rearrange the vector according to this

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_o \\ \mathbf{y}_e \end{bmatrix}, \tag{13}$$

where the components $\mathbf{y}_o, \mathbf{y}_e$ are M-dimensional vectors. If we choose to observe some combination of odd and even components, we rearrange the vector accordingly; the index o may then mean "observed" and e "eliminated".

If we, for simplicity, assume a Gaussian input distribution of the form

$$W^{\text{in}}(\mathbf{x}) = N \exp(-\tfrac{1}{2}\mathbf{x}^T \mathbf{A} \mathbf{x}), \tag{14}$$

we find the output to be of the form

$$W^{\text{out}}(\mathbf{y}) = N \exp(-\tfrac{1}{2}\mathbf{y}^T \mathbf{K} \mathbf{y}), \tag{15}$$

where according to (9)

$$\mathbf{K} = LAL^T. \tag{16}$$

All variables are chosen real here, and the superscript T denotes the transpose. In (14, 15), the averages are subtracted off the variables to simplify the notation.

Because we can access only the components $\mathbf{y}_o$, we derive a reduced distribution function for these variables only by integrating over the unobserved conjugate variables $\mathbf{y}_e$

$$W_{\text{red}}^{\text{out}}(\mathbf{y}_o) = N \int d^M y_e \exp\left(-\tfrac{1}{2}[\mathbf{y}_o^T \mathbf{y}_e^T] \begin{bmatrix} K_{oo} & K_{oe} \\ K_{eo} & K_{ee} \end{bmatrix} \begin{bmatrix} \mathbf{y}_o \\ \mathbf{y}_e \end{bmatrix}\right). \tag{17}$$

The quadratic form in the exponent can be expanded as

$$\begin{aligned} &\mathbf{y}_o^T K_{oo} \mathbf{y}_o + \mathbf{y}_o^T K_{oe} \mathbf{y}_e + \mathbf{y}_e^T K_{eo} \mathbf{y}_o + \mathbf{y}_e^T K_{ee} \mathbf{y}_e \\ &= \mathbf{y}_o^T (K_{oo} - K_{oe} K_{ee}^{-1} K_{eo}) \mathbf{y}_o \\ &\quad + (\mathbf{y}_e^T + \mathbf{y}_o^T K_{oe} K_{ee}^{-1}) K_{ee} (\mathbf{y}_e + K_{ee}^{-1} K_{eo} \mathbf{y}_o). \end{aligned} \tag{18}$$

The second term is integrated out in (17), and calculating the output correlation functions with the reduced distribution we obtain the result

$$\langle y_o^a y_o^b \rangle = [(K_{oo} - K_{oe} K_{ee}^{-1} K_{eo})^{-1}]^{ab}, \tag{19}$$

according to the algebra of Gaussian distribution functions.

There is, however, another way to obtain these same correlations. We calculate the moment generating function by inverting the Fourier transform (7)

$$Z(\lambda) = \int d^{2M} y \, e^{i\lambda^T \cdot \mathbf{y}} W^{\text{out}}(\mathbf{y}). \tag{20}$$

When we set to zero all variables λ_e corresponding to the unobserved quantities, we obtain the simple function

$$Z(\lambda) = N \exp(-\tfrac{1}{2}\lambda_o^T C_{oo} \lambda_o). \tag{21}$$

The correlation functions are now directly the components of the observed correlation matrix C_{oo}. This can be calculated directly from

$$C = \begin{bmatrix} C_{oo} & C_{oe} \\ C_{eo} & C_{ee} \end{bmatrix} = K^{-1} = LA^{-1}L^T, \tag{22}$$

as follows from (16). This is, of course, the simplest way to obtain the correlations, but the physical process of parameter elimination was elucidated by the previous method. In the following section, I will show that these two methods lead to identical results, as they necessarily must.

4 Proof of consistency

The result (22) is shown to be equivalent with the result (19) by inversion of the matrix (16) in partitioned form

$$C = \begin{bmatrix} K_{oo} & K_{oe} \\ K_{eo} & K_{ee} \end{bmatrix}^{-1}. \tag{23}$$

Without going into the calculations, I give the result as follows

$$C = K^{-1} = \begin{bmatrix} (K_{oo} - K_{oe}K_{ee}^{-1}K_{eo})^{-1} & (K_{eo} - K_{ee}K_{oe}^{-1}K_{oo})^{-1} \\ (K_{oe} - K_{oo}K_{eo}^{-1}K_{ee})^{-1} & (K_{ee} - K_{eo}K_{oo}^{-1}K_{oe})^{-1} \end{bmatrix}. \tag{24}$$

The inverses of the off-diagonal square matrices are defined as

$$K_{eo}^{-1}K_{eo} = 1; \quad K_{oe}^{-1}K_{oe} = 1. \tag{25}$$

Using the easily proved matrix identities

$$\frac{1}{K_{oe} - K_{oo}K_{eo}^{-1}K_{ee}} = -\frac{1}{K_{ee} - K_{eo}K_{oo}^{-1}K_{oe}} K_{eo}K_{oo}^{-1},$$
$$\frac{1}{K_{eo} - K_{ee}K_{oe}^{-1}K_{oo}} = -\frac{1}{K_{oo} - K_{oe}K_{ee}^{-1}K_{eo}} K_{oe}K_{ee}^{-1}, \tag{26}$$

one verifies directly that the result (24) is correct.

From (24), we can directly see that the correlation matrix between the observed quantities in (27) is given by

$$C_{oo} = \frac{1}{K_{oo} - K_{oe}K_{ee}^{-1}K_{eo}}, \tag{27}$$

as derived differently in (19). The equivalence between the two methods has hence been proved conclusively. The result (24) also gives expressions for the correlations between variables not subject to measurements in the chosen scheme.

5 Conclusions

In this paper, I have discussed the quantum equivalent of the optical Jones calculus for passive linear systems. I have considered especially those points where quantum effects have to be taken explicitly into account. They affect the formulation of the theory in ways discussed in this paper.

There are mainly three points where quantum considerations are inevitable. Firstly, the initial states of the incoming fields must conform to the requirements of quantum theory. This excludes certain classically possible distributions but allows cases without any classical counterparts. Secondly, the measurement of operator quantities assumes certain ordering rules because of the non-commutability of the observables. These features enter the theory in the forms of quasidistribution functions defined on the quantum analogue of classical phase space. For the linear passive networks considered here, it has been shown that we can choose to use any distribution function we like. All standard orderings are thus allowed in the formalism. Finally quantum requirements restrict the possible observables to a set of commuting variables; only half of the classically accessible information can be extracted from a quantum system.

This third point motivates the introduction of reduced quasidistribution functions defined only on the observationally accessible part of the full phase space. These functions contain all the information obtainable about the averages (which we have not discussed in this paper) and the noise and correlation properties between the observed outputs. These reduced functions have been derived in two different ways, and in Sect. 4, I have shown that these two methods are equivalent. The proof is, in fact, only a verification of consistency, but gives an explicit expression (27) for the observable correlations.

The result (27) gives the correlations in terms of the parameters of the Gaussian describing the output of the system. If, e.g., K_{oe} is zero, the elimination of the unobserved degrees-of-freedom introduces no additional correlations; all observed ones are given simply by K_{oo}^{-1}, in other cases the influence of K_{oe} can be determined directly. This may be useful if one wants to manipulate or modify the correlation and noise properties of the system. We are not discussing the detailed applications of the formalism in this paper, but we will return to this in connection with considerations of physically interesting cases.

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