Journal of Risk and Uncertainty, 11:5–16 (1995)

A Note on Deriving Rank-Dependent Utility Using Additive Joint Receipts

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Abstract

Luce and Fishburn (1991) derived a general rank-dependent utility model using an operation $\oplus$ of joint receipt. Their argument rested on an empirically supported property (now) called *segregation* and on the assumption that utility is additive over $\oplus$. This note generalizes that conclusion to the case where utility need not be additive over $\oplus$, but rather is of a more general form, which they derived but did not use in their article. Tversky and Kahneman (1992), conjecturing that the joint receipt of two sums of money is simply their sum, criticized that original model because $\oplus = +$ together with additive utility implies the unacceptable conclusion that the utility of money is proportional to money. In the present generalized theory, if $\oplus = +$, utility is a negative exponential function of money rather than proportional. Similar results hold for losses. The case of mixed gains and losses is less well understood.

Key words: rank-dependent utility, joint receipt, bounded exponential utility

1. Introduction

A theory of preferences was proposed for binary gambles by Luce (1991), and a closely related one for general finite gambles that led to a general rank- and sign-dependent utility representation was proposed by Luce and Fishburn (1991). This representation is essentially the same as that proposed, independently, by Tversky and Kahneman (1992) and axiomatized by Wakker and Tversky (1993). Ours was novel primarily because it included an additional primitive of receiving two things at once, called *joint receipt*. This made the derivation of the rank- and sign-dependent form rather natural and easy without, however, imposing on the axioms properties that only make sense if one already knows the intended representation (as appears to be true of the axiomatization of Wakker and Tversky (1993), which rests upon the not very intuitive concept of co-monotonicity).

The difficulty with our derivation is that it was carried out under the restrictive assumption that the utility U is additive over joint receipt, i.e., if $x \oplus y$ denotes the joint receipt of both x and y,

$$U(x \oplus y) = U(x) + U(y). \tag{1}$$

As Tversky and Kahneman (1992) remarked, if the joint receipt of money is simply the sum, i.e.,

$$x \oplus y \sim x + y, \tag{2}$$

then, necessarily, the utility for money is proportional to money, i.e., for some constant $k > 0$,

$$U(x) = kx. \tag{3}$$

Since there is ample evidence against (3)—e.g., evidence for diminishing marginal utility—one is forced to reject either (1), (2), or both.

Thaler (1985) (see also Thaler and Johnson, 1990; Linville and Fischer, 1991) studied $x \oplus y$ in a context of somewhat naturalistic scenarios and concluded that Eq. (2) held for losses, but not for gains or for some cases of mixed gains and losses. In the framework of a gambling experiment using certainty equivalents, Cho and Luce (in press) studied (2) and concluded that it was sustained, confirming Tversky and Kahneman's conjecture. If so, then we know that Eq. (1) cannot be correct.

Luce and Fishburn (1991) actually provided an argument for a more general form than (1). Assuming commutativity and associativity of $\oplus$,

$$x \oplus y \sim y \oplus x, \tag{4a}$$

$$x \oplus (y \oplus z) \sim (x \oplus y) \oplus z, \tag{4b}$$

which, of course, follow from (2), and monotonicity of the operation, that is,

$$U(x) \geq U(y) \Leftrightarrow U(x \oplus z) \geq U(y \oplus z), \tag{5}$$

the more general form becomes:

$$U(x \oplus y) = U(x) + U(y) - \frac{U(x)U(y)}{C}, \quad C > 0. \tag{6}$$

In deriving the rank-dependent (or cumulative) utility representation, we elected to use the special case of (1) rather than the full form of (6), because we believed, without actually verifying it, that the effect of the added multiplicative term in (6) would lead to an enormous increase in the complexity of the resulting representation. Our intuitions were wrong, and the purpose of this note is to establish, to the contrary, that the final rank-dependent form is unchanged. We believe that this conclusion undercuts to a considerable degree the Tversky and Kahneman objection to our approach to rank-dependent utility.

The note is organized as follows. The next section states more fully the conditions that lead to (6). Section 3 explores the implications of (2) and (6) for the form of $U(x)$ and how (6) generalizes to the joint receipt of three or more things. Section 4 rederives from (6) the general rank-dependent representation. Section 5 points out some of the unresolved issues when mixed gains and losses are involved. Finally, section 6 provides a summary.

2. Segregation and the form of $U(x \oplus y)$

Let $(x,p;y)$ denote a gamble that pays x if an event with probability p obtains, and pays y otherwise. We consider only gains $(x, y > 0)$ and 0 for the moment. Several years before publication of the preceding references, Luce and Narens (1985) demonstrated that the most general form for the interval scale (cardinal) measurement of utility U of binary gambles is:

$$U(x,p;y) = \begin{cases} W_>(p)U(x) + [1 - W_>(p)]U(y), & x \geq y \\ W_<(p)U(x) + [1 - W_<(p)]U(y), & x < y, \end{cases} \tag{7}$$

where U is strictly increasing in the outcomes and the two weighting functions $W_>$ and $W_<$ map [0, 1] onto [0, 1]. Assuming

$$(x,p;y) \sim (y, 1 - p;x), \tag{8}$$

where $\sim$ denotes indifference, it follows that there is but one function, because $W_>(p) + W_<(1 - p) = 1$.

We suppose, as did Tversky and Kahneman (1979), that the status quo 0 is a singular point,[1] so that

$$U(0) = 0. \tag{9}$$

For a fairly general theory of singular points, see Luce (1992).

We generalize the concept of joint receipt $\oplus$ to gambles, so that $g \oplus h$, where g and h are gambles, means that one receives both of the gambles, g and h. An example is the purchase, at the same time, of tickets to two different lotteries. This generalized operator includes, as a special case, the joint receipt of sums of money. Suppose further that the following property holds, which was called *distribution* in the original publication, but was subsequently renamed *(binary) segregation:* for $x, y > 0$,

$$(x,p;0) \oplus y = (x \oplus y, p; 0 \oplus y), \tag{10a}$$

$$y \oplus (0,p;x) = (y \oplus 0, p; y \oplus x). \tag{10b}$$

This has the very simple interpretation that receipt of a gamble whose outcomes are a gain and the status quo jointly with a positive sum y is seen as indifferent to the gamble formed from the original one by substituting for each outcome the joint receipt of that outcome and y. It is a highly rational property. Binary segregation has been studied by Cho and Luce (in press) in a lottery experiment and appears to be sustained.

Under the assumptions of (7)–(10) with U strictly increasing and certain density requirements that are fulfilled when U is onto a real interval and W is a function from [0, 1] onto [0, 1], Luce and Fishburn (1991, Theorem 1) showed that U over money must satisfy

$$U(x \oplus y) = AU(x) + BU(y) + DU(x)U(y). \tag{11}$$

As was noted above, in developing our generalization of rank-dependent theory to arbitrary finite gambles, we did not use the full generality of (11), but rather assumed the special case of additive utility, namely, $A = B = 1$ and $D = 0$ in (11) resulting in (1). It is easily verified that $A = B = 1$ (with or without $D = 0$) corresponds to $\oplus$ being commutative and associative, (4). But we had little justification for setting $D = 0$ beyond mathematical simplicity, because we presumed that $D \neq 0$ would make the representation very complex. As noted above, a major purpose of this note is to establish that, to the contrary, exactly the same representation arises with $D \neq 0$.

3. An extensive model for joint receipts

Suppose that $\oplus$ is commutative and associative and (11) holds. Then, as noted, $A = B = 1$. We suppose also that $U(x + y) < U(x) + U(y)$ for positive x and y, as required by diminishing marginal utility. It follows that $D < 0$ in (11), which we rewrite as (6) above. Note that this means U is bounded unless $C = \infty$.

The first two theorems characterize a familiar negative exponential form for U and develop the equation for the utility of n-fold joint receipt for gains.

Theorem 1. Suppose $U(x)$ is strictly increasing at a decreasing rate in $x \geq 0$ and (2) and (6) hold with $C < \infty$. Then

$$U(x) = C(1 - e^{-kx}), \quad k > 0 \tag{12}$$

Proof. Assume the hypotheses of the theorem. Because $U(x + y) = U(x \oplus y) > U(x)$ for $x, y > 0$, (6) implies $U(y)\left[1 - \frac{U(x)}{C}\right] > 0$, hence that $1 - \frac{U(x)}{C} > 0$. Let

$$V(x) = \ln\left(1 - \frac{U(x)}{C}\right).$$

Then it is easy to verify from (2) and (6) that

$$V(x + y) = V(x \oplus y) = V(x) + V(y),$$

and so with the monotonicity of V, which follows immediately from that of U, we have $V(x) = -kx$ for some $k > 0$, from which (12) follows. □

Theorem 2. Suppose $U(x)$ is strictly increasing at a decreasing rate in $x \geq 0$, (6) holds, and $S_j^{(n)}$ denotes a typical subset of j elements from $\{1, 2, \ldots, n\}$. Then

$$U(x_1 \oplus x_2 \oplus \cdots \oplus x_n) = \sum_{j=1}^{n} (-1/C)^{j-1} \sum_{S_j^{(n)}} \prod_{i \in S_j^{(n)}} U(x_i). \tag{13}$$

Proof. (13) is simply (6) for $n = 2$, and (6) implies associativity, i.e., (4b). We proceed by induction:

$$\begin{aligned}
U(x_1 \oplus x_2 \oplus \cdots \oplus x_n) &= U[(x_1 \oplus x_2 \oplus \cdots \oplus x_{n-1}) \oplus x_n] \quad \text{(4b)} \\
&= U(x_1 \oplus x_2 \oplus \cdots \oplus x_{n-1})\left(1 - \frac{U(x_n)}{C}\right) + U(x_n) \quad \text{(6)} \\
&= \sum_{j=1}^{n-1} (-1/C)^{j-1} \sum_{S_j^{(n-1)}} \prod_{i \in S_j^{(n-1)}} U(x_i) \quad \text{(Induction hypothesis)} \\
&\quad + \sum_{j=1}^{n-1} (-1/C)^{j} U(x_n) \sum_{S_j^{(n-1)}} \prod_{i \in S_j^{(n-1)}} U(x_i) + U(x_n) \\
&= \sum_{j=2}^{n-1} (-1/C)^{j-1} \Bigg(\sum_{S_j^{(n-1)}} \prod_{i \in S_j^{(n-1)}} U(x_i) \quad \text{(Rearrangement)} \\
&\quad + U(x_n) \sum_{S_{j-1}^{(n-1)}} \prod_{i \in S_{j-1}^{(n-1)}} U(x_i) \Bigg) \\
&\quad + (-1/C)^{n-1} U(x_n) \prod_{i \leq n-1} U(x_i) + \sum_{i=1}^{n-1} U(x_i) + U(x_n) \\
&= \sum_{j=1}^{n} (-1/C)^{j-1} \sum_{S_j^{(n)}} \prod_{i \in S_j^{(n)}} U(x_i). \quad \text{(Rearrangement)}
\end{aligned}$$

□

4. The general form for rank dependence

Suppose g is a gamble of m nonnegative outcomes x_i, ordered $x_1 > x_2 > \cdots > x_m \geq 0$, with probability $p_i > 0$, for $i = 1, \ldots, m$, and $\sum p_i = 1$. This gamble we denote symbolically as

$$g = \sum_{i=1}^{m} (x_i, p_i).$$

Note that if $m = 1$, then $p_1 = 1$ and $g = x_1$, and if $m = 2$, we use the abbreviated notation $(x_1, p_1; x_2)$. We say that g is *strictly positive* if $x_m > 0$.

Generalizing (7), we say that *preferences are rank-dependent of order n* if for each $m \leq n$ and every strictly positive $g = \sum_{i=1}^{m}(x_i, p_i)$

$$U(g) = \sum_{i=1}^{m} U(x_i)[W(P_i) - W(P_{i-1})], \tag{14}$$

where $P_i = \sum_{j=1}^{i} p_j, P_0 = 0, P_m = 1, W(0) = 0$, and $W(1) = 1$. For a general survey of this class of models, see Quiggin (1993).

Our goal is to show that this form arises under fairly reasonable assumptions. We let $CE(g)$ denote the so-called certainty equivalent of gamble g, defined as the monetary amount indifferent to g:

$$CE(g) \sim g, \text{ where } CE(g) \text{ is an amount of money.} \tag{15}$$

Theorem 3. Suppose the following are true for gambles composed of monetary gains and their joint receipt. There exists a constant C and a function U: $\mathfrak{R}^+ \cup \{0\}$ onto $[0, C)$ and W: $[0, 1]$ onto $[0, 1]$ such that:

1. U is strictly increasing over gambles, money, and joint receipts, $U(0) = 0$, and for x, $y > 0$, $U(x \oplus y) > U(x), U(y)$ and $U(x + y) < U(x) + U(y)$.
2. Gambles are monotonic in the consequences.
3. $\oplus$ is associative.
4. $\oplus$ is a monotonic operation in the following sense:

$$g \oplus x \sim CE(g) \oplus x. \tag{16}$$

5. Binary rank dependence, (7), holds.
6. Binary segregation, (10), holds.
7. Suppose $m \geq 2, g = \sum_{i=1}^{m}(x_i, p_i), x_m > 0$, and $g' = \sum_{i=1}^{m-1}\left(x_i, \frac{p_i}{1-p_m}\right)$. Then

$$g \sim (g', 1 - p_m; x_m). \tag{17}$$

Then (6) holds and preferences are rank-dependent of order n, as in (14).

Before giving a proof, a few words of comment are in order. Assumption 2 has been brought into question by Birnbaum (1992) and Mellers, Weiss, and Birnbaum (1992), but as von Winterfeldt, Chung, Luce, and Cho (submitted) have argued, it may be valid if CE is determined by an appropriate choice procedure, although possibly not when judged CEs are used. As noted earlier, Assumption 3 appears to be correct. Assumption 4 was shown in Luce (1995) to be equivalent in the presence of Assumption 1 to monotonicity of $\oplus$ in the usual sense. Although plausible, Assumption 4 may be incorrect, as Cho and Luce (in press) have argued. They studied it empirically, using money lotteries, and found that it was not sustained for gains, although it appears to be for losses. So Assumption 4 is in some doubt. Assumptions 5 and 6 were discussed earlier. Note that, by Luce and Fishburn (1991, Theorem 1), one may replace either Assumption 5 or 6 by (6), if that seems more natural. Assumption 7, which was made in Luce and Fishburn (1991), simply says that a gamble of order m can be thought of as a binary gamble, the first outcome of which is the renormalized gamble in the first $m - 1$ elements.

Five lemmas are useful in proving Theorem 3.

Lemma 1. Suppose assumptions 1, 3, 4, 5, and 6 hold. Then (6) holds, as well as the following generalization, where g is a gamble and x is a sum of money:

$$U(g \oplus x) = U(g) + U(x) - \frac{U(g)U(x)}{C}, \quad C > 0. \tag{18}$$

Proof. Luce and Fishburn (1991, Theorem 1) prove (11), and the argument given earlier establishes (6). Using this and the other assumptions,

$$\begin{aligned} U(g \oplus x) &= U[CE(g) \oplus x] = U[CE(g)] + U(x) - \frac{U[CE(g)]U(x)}{C} \\ &= U(g) + U(x) - \frac{U(g)U(x)}{C}. \end{aligned}$$

□

The following concept of "subtraction" is useful:

$$g \ominus h = f \text{ iff } f \oplus h = g. \tag{19}$$

Lemma 2. Suppose Assumptions 1, 3, 5, and 6 hold. Then:

(i) For $x \geq y$, $U(x \ominus y) = \dfrac{U(x) - U(y)}{1 - U(y)/C}$. (20)

(ii) $x \ominus x \sim 0$

(iii) $(f \ominus h) \ominus (g \ominus h) = f \ominus g$.

Proof.

(i) By Lemma 1, apply (6) to $x = (x \ominus y) \oplus y$ and solve.

(ii) By Assumption 1, U is monotonic and $U(0) = 0$. So from (20) we have

$$U(x \ominus x) = 0 = U(0) \Leftrightarrow x \ominus x \sim 0.$$

(iii) Set $u = f \ominus h$, $v = g \ominus h$, $w = u \ominus v$. Thus, by definition, $u = w \oplus v$ and

$$f = (f \ominus h) \oplus h = u \oplus h = (w \oplus v) \oplus h = w \oplus (v \oplus h) = w \oplus g,$$

and so by definition $w = f \ominus g$. □

Lemma 3. Suppose Assumptions 1 and 5 hold. Then:

$$U(x,p;0) = U(x)W(p). \tag{21}$$

Proof. Trivial. □

Lemma 4. Suppose Assumptions 1, 2, 4, 6, and 7 hold. Then the following generalization of segregation holds: If $g = \sum_{i=1}^{m}(x_i, p_i)$ is strictly positive, then

$$g \sim \left[\sum_{i=1}^{m}(x_i \ominus x_m, p_i)\right] \oplus x_m. \tag{22}$$

Proof. Using the notation of Assumption 7,

$$\begin{aligned}
g &\sim (g', 1 - p_m; x_m) && \text{(Assumption 7)}\\
&\sim (CE(g'), 1 - p_m; x_m) && \text{(Assumption 2 and (15))}\\
&\sim (CE(g') \ominus x_m, 1 - p_m; 0) \oplus x_m && \text{(Assumption 6)}\\
&\sim (g' \ominus x_m, 1 - p_m; 0) \oplus x_m && \text{(Assumptions 2 and 4)}\\
&\sim \left[\sum_{i=1}^{m-1}\left(x_i \ominus x_m, \frac{p_i}{1-p_m}\right), 1 - p_m; 0\right] \oplus x_m && \text{(Assumption 2, Induction)}\\
&\sim \left[\sum_{i=1}^{m}(x_i \ominus x_m, p_i)\right] \oplus x_m. && \text{(Assumptions 6 and 7)}
\end{aligned}$$
□

Lemma 5. Suppose Assumptions 1, 2, 5, and 7 hold. Then, for $0 < p \le q \le 1$,

$$W\left(\frac{p}{q}\right)W(q) = W(p). \tag{23}$$

Proof.

$$\begin{aligned}
U(x)W(p) &= U(x,p;0,1-p) && \text{(Lemma 3)}\\
&= U\left[\left(x, \frac{p}{q}; 0, 1 - \frac{p}{q}\right), q; 0\right] && \text{(Assumption 7)}\\
&= U\left[CE\left(x, \frac{p}{q}; 0, 1 - \frac{p}{q}\right), q; 0\right] && \text{(Assumption 2)}\\
&= U\left[CE\left(x, \frac{p}{q}; 0\right)\right]W(q) && \text{(Lemma 3)}\\
&= U\left(x, \frac{p}{q}; 0\right)W(q) && \text{((15) and Assumption 2)}
\end{aligned}$$

$$= U(x)W\left(\frac{p}{q}\right)W(q). \qquad \text{(Lemma 3)}$$

Dividing by $U(x) > 0$ yields (23). □

Proof of Theorem 3. Lemma 1 establishes that (6) holds. To prove that preferences are rank dependent, we proceed by induction. By Assumption 5, binary rank dependence holds. Suppose preferences are rank dependent of order $n - 1$. Let $g = \sum_{i=1}^{m}(x_i, p_i)$ be strictly positive, i.e., $x_m > 0$, and $g' = \sum_{i=1}^{m-1}\left(x_i, \frac{p_i}{1-p_m}\right)$. Then

$$
\begin{aligned}
U(g) &= U\left[\left[\sum_{i=1}^{m}(x_i \ominus x_m, p_i)\right] \oplus x_m\right] && \text{(Lemma 4)}\\
&= [1 - U(x_m)/C]U\left[\sum_{i=1}^{m}(x_i \ominus x_m, p_i)\right] + U(x_m) && \text{(Lemma 1, (6))}\\
&= [1 - U(x_m)/C]U\left[\sum_{i=1}^{m-1}\left(x_i \ominus x_m, \frac{p_i}{1-p_m}\right), 1 - p_m; 0\right]\\
&\quad + U(x_m) && \text{(Assumption 7)}\\
&= [1 - U(x_m)/C]U\left[\sum_{i=1}^{m-1}\left(x_i \ominus x_m, \frac{p_i}{1-p_m}\right)\right]W(1 - p_m) + U(x_m) && \text{(Lemma 3)}\\
&= [1 - U(x_m)/C]\sum_{i=1}^{m-1}U(x_i \ominus x_m)\left[W\left(\frac{P_i}{1-p_m}\right)\right.\\
&\quad \left. - W\left(\frac{P_{i-1}}{1-p_m}\right)\right]W(1 - p_m) + U(x_m) && \text{(Induction hypothesis)}\\
&= [1 - U(x_m)/C]\sum_{i=1}^{m-1}U(x_i \ominus x_m)[W(P_i) - W(P_{i-1})] + U(x_m) && \text{(Lemma 5)}\\
&= \sum_{i=1}^{m-1}[U(x_i) - U(x_m)][W(P_i) - W(P_{i-1})] + U(x_m) && \text{(Lemma 2)}\\
&= \sum_{i=1}^{m}U(x_i)[W(P_i) - W(P_{i-1})].
\end{aligned}
$$

□

5. Issues concerning mixed gains and losses

An entirely parallel development holds for the domain of losses with (6) as before, except that C for losses is negative. However, the issue of mixed gains and losses is much more problematic. The problem is as follows. Does (2), $x \oplus y = x + y$, continue to hold for $x > 0 > y$? The only data bearing on the issue are those of Thaler (1985), Thaler and Johnson (1990), and Linville and Fischer (1991), which suggest that it is not true in general.[2] However, data from these same experiments suggested that (2) was not true for gains, whereas the Cho and Luce (in press) experiment using simple lotteries and sums of money concluded otherwise. Unfortunately, they did not study the mixed case. If we do suppose that (2) continues to hold, then the relation of $U(x \oplus y) = U(x + y)$ to $U(x)$ and $U(y)$ becomes very problematical, as we shall see.

Luce and Fishburn (1991) made the assumption that if g^+ is a gamble of gains and so is preferred to 0, which in turn is preferred to g^- a gamble of losses, then

$$U(g^+ \oplus g^-) = U(g^+) + U(g^-). \tag{24}$$

They were not able to give any compelling argument for this form aside from the fact that if (6) holds, with appropriately signed constants $C(+)$ and $C(-)$ for gains and losses, then (24) retains the additive terms, and it is sufficient to maintain the monotonicity of $\oplus$. Moreover, this assumption, when coupled with the only nonrational assumption of the theory yields the same formula for the utility of a mixed gamble as does Kahneman and Tversky's (1979) prospect theory. Our nonrational assumption was that any gamble of mixed outcomes is perceived as indifferent to the following *duplex decomposition:* receiving the gamble that is obtained by replacing all losses by 0 jointly with the independently run gamble that is obtained by replacing all gains by 0. Empirical data, such as those of Slovic and Lichtenstein (1968) and Cho, Luce, and von Winterfeldt (1994), sustain that postulate.

But as we now show, one cannot have all of (2), (6), and (24). From (2) and (6), Theorem 1 establishes

$$U(x) = C(+)(1 - e^{-k(+)x}), \quad C(+) > 0, k(+) > 0, x > 0,$$
$$U(x) = C(-)(1 - e^{k(-)x}), \quad C(-) < 0, k(-) > 0, x < 0.$$

Suppose $x > x + y > 0 > y$. Then

$$U(x + y) = C(+)(1 - e^{-k(+)(x+y)}) = C(+)(1 - XY'),$$

where $X = e^{-k(+)x}$ and $Y' = e^{-k(+)y}$. Moreover,

$$U(x) + U(y) = C(+)(1 - X) + C(-)(1 - Y),$$

where $Y = e^{k(-)y}$. Thus, for (24) to hold we have

$$C(+)X(1 - Y') = C(-)(1 - Y).$$

But, since for any y, we can make x any value such that $x + y > 0$, this cannot be satisfied except for $Y = Y' = 1$, which is impossible. So the three equations are incompatible.

There seems to be a major issue here. We need to check (2) in the mixed case and to check the implications on $\oplus$ if (24) is correct.

6. Conclusions

The major point of this note is to establish that one can derive the rank-dependent form of utility, (14), from binary rank dependence (7), segregation (10), and a commutative and associative (4) binary operation of joint receipt using the fully general form for $U(x \oplus y)$ of (6), rather than the additive specialization, (1), used by Luce and Fishburn (1991). Somewhat incidental to that, we derived the negative exponential form for utility of money when $\oplus$ is simply addition for money, (2). Although this theory deals with rank-dependence for gains and losses separately, it does not tell us about the mixed case when $x > 0 > y$. The form for $U(x \oplus y)$ in that case is an open problem.

Acknowledgments

This research was supported in part, by National Science Foundation grant SBR-9308959 to the University of California, Irvine. We appreciate the comments of Mr. Liping Liu, especially for pointing out a correction needed in Lemma 5.

Notes

1. Defined formally, a point is singular if it is invariant under all automorphisms of the structure, which in this case means those mappings that correspond to multiplication by a positive constant in the representation.
2. Indeed, they argued that utility should satisfy the *hedonic* rule: $U(x \oplus y) = \max\{U(x + y), U(x) + U(y)\}$. With U concave for gains and concave for losses, the second term holds for gains and the first for losses. The mixed case is much more complex, and Thaler's (1985) discussion of where the boundary lies at which $U(x + y) = U(x) + U(y)$ was informal. Fishburn and Luce (1995) work it out in detail as well as axiomatizing the hedonic rule.

Note added in proof: Mr. Liping Liu of the University of Kansas has improved Theorem 3 greatly by weakening some assumptions and eliminating others entirely.

References

Birnbaum, M. H. (1992). "Violations of Monotonicity and Contextual Effects in Choice-based Certainty Equivalents," *Psychological Science* 3, 310-314.

Cho, Y., and R. D. Luce. (in press). "Tests of Hypotheses about Certainty Equivalents and Joint Receipt of Gambles." Organizational Behavior and Human Decision Processes.

Cho, Y., R. D. Luce and D. von Winterfeldt. (1994). "Tests of Assumptions about the Joint Receipt of Gambles in Rank- and Sign-Dependent Utility Theory," *Journal of Experimental Psychology: Human Perception and Performance* 20, 931-943.

Fishburn, P. C., and R. D. Luce. (1995). "Joint Receipt and Thaler's Hedonic Editing Rule," *Mathematical Social Sciences* 29, 33-76.

Kahneman, D., and A. Tversky. (1979). "Prospect Theory: An Analysis of Decision under Risk," *Econometrica* 47, 263-291.

Linville, P. W., and G. W. Fischer. (1991). "Preferences for Separating or Combining Events," *Journal of Personality and Social Psychology* 60, 5-23.

Luce, R. D. (1991). "Rank- and Sign-Dependent Linear Utility Models for Binary Gambles," *Journal of Economic Theory* 53, 75-100.

Luce, R. D. (1992). "Singular Points in Generalized Concatenation Structures that Otherwise Are Homogeneous," *Mathematical Social Sciences* 24, 79-103.

Luce, R. D. (1995). "Joint Receipt and Certainty Equivalents of Gambles," *Journal of Mathematical Psychology* 39, 73-81.

Luce, R. D., and P. C. Fishburn. (1991). "Rank- and Sign-Dependent Linear Utility Models for Finite First-Order Gambles," *Journal of Risk and Uncertainty* 4, 29-59.

Luce, R. D., and L. Narens. (1985). "Classification of Concatenation Measurement Structures According to Scale Type," *Journal of Mathematical Psychology* 29, 1-72.

Mellers, B., R. Weiss, and M. H. Birnbaum. (1992). "Violation of Dominance in Pricing Judgments," *Journal of Risk and Uncertainty* 5, 73-90.

Quiggin, J. (1993). *Generalized Expected Utility Theory: The Rank-Dependent Model.* Boston: Kluwer Academic Publishers.

Slovic, P., and S. Lichtenstein. (1968). "Importance of Variance Preferences in Gambling Decisions," *Journal of Experimental Psychology* 78, 646-654.

Thaler, R. H. (1985). "Mental Accounting and Consumer Choice," *Marketing Science* 4, 199-214.

Thaler, R. H., and E. Johnson. (1990). "Gambling with the House Money or Trying to Break Even: The Effects of Prior Outcomes on Risky Choice," *Management Science* 36, 643-660.

Tversky, A., and D. Kahneman. (1992). "Advances in Prospect Theory: Cumulative Representation of Uncertainty," *Journal of Risk and Uncertainty* 5, 204-217.

von Winterfeldt, D., N.-K. Chung, R. D. Luce, and Y. Cho. (submitted). "Tests of Consequence Monotonicity in Decision Making under Uncertainty," *Journal of Experimental Psychology: Learning, Memory, and Cognition.*

Wakker, P. P., and A. Tversky. (1993). "An Axiomatization of Cumulative Prospect Theory," *Journal of Risk and Uncertainty* 7, 147-176.