NONLINEAR MATHEMATICAL MODELS FOR THE ORIGIN OF ASYMMETRY IN BIOLOGICAL MOLECULES*

ADOLF R. HOCHSTIM

*Research Institute for Engineering Sciences,** College of Engineering, Wayne State University, Detroit, Mich. 48202, U.S.A.*

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Abstract. The origin of chemical chirality is probably associated with a difference in the initial concentrations of two separate populations of primeval organic molecules and possibly even two types of primeval organisms and amplified by nonlinear kinetic processes leading to the death of one population. This theory, as originally developed by F. C. Frank of the University of Bristol, is reviewed in this paper with additional derivations, discussions and generalizations.

The possible effect of asymmetry in the rate constant is compared to the role of statistical fluctuations, and it is shown, that within the simple model investigated here that the role of statistical fluctuations is much more important for the death of one isomer. In the unlikely absence of any fluctuations, the nonlinear kinetic processes amplify the asymmetry in the rate constant and lead to the death of one enanthiomorph.

The role of spatial diffusion is discussed, and it is shown that in the presence of a local excess of one enanthiomorph this excess would have spread in space and grown, destroying the opposite enanthiomorph. If the total population of both enanthiomorphs was exactly composed of equal parts of both types, but local fluctuation increased one type at one place and decreased the same type at a different location, the diffusion and growth rate would have caused spatial separation in the population of both enanthiomorphs.

For general *n*th order nonlinear symmetric rate processes (incorporating multitudes of reactions and general diffusion), it is shown that if initially two populations of enanthiomorphs were exactly the same at all locations, then for all times both populations would have increased and remained equal to each other.

Mathematical model was constructed for stereoscopic autocatalysis suggested by Calvin. It was found that under certain special local conditions in the presence of large fluctuation it is possible indeed to have growth of only one type of isomer.

Various approximate methods and numerical solutions are presented in order to facilitate the handling of nonlinear rate equations.

1. Introduction

In this section we will attempt to organize and catalog various hypotheses for the possible mechanisms for the origin of optical activity in living organisms, or more precisely chirality, i.e., configurationally left (*L*) or right (*D*) handed enanthiomorphs present in living organisms, e.g., *L*- or *D*-amino acids. Most or all of these mechanisms have been previously postulated, and for the appropriate references we point to the latest systematic review of the origin of molecular chirality by Bonner (1972).

1.1. FORMATION OF ORGANIC MOLECULES ON THE EARTH

Currently it is assumed that the Earth condensed from stellar matter, became very hot from compression and radioactive decay, and that only after the surface and atmo-

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** On leave at Physical Dynamics, Inc., and the University of California, San Diego, La Jolla, California.

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sphere were sufficiently cooled, stable organic molecules were produced from the primeval atmosphere. A multitude of processes have been suggested for this production: lightning (Beutner, 1938; Dauvillier, 1938; Bernal, 1949; Urey, 1952; Miller and Urey, 1959; Oparin, 1924); ultraviolet radiation (Oparin, 1924); radioactive decay (Calvin, 1969); quenching reactions from shock waves due to falling meteorites (Hochstim, 1963, 1971); lightning on water surfaces (Park *et al.*, 1971, 1974; Park, 1973); shock waves from thunder (Bar-Nun *et al.*, 1970); etc.

It is assumed that all of these processes produced an equal amount of L and D amino acids. Up to now, no processes have been found to indicate definite asymmetric synthesis on the earth, i.e., no definite preference for the formation of L amino acids (Bonner, 1972).

Even if there were some unexplained, yet asymmetric, production of one type of amino acids, the normal racemization, i.e., slow conversion of L to D and D to L, would have taken place in the relatively short time of a few hundred thousand years (e.g., Bada, 1972; Hochstim *et al.*, 1975).

Recently, much interest has been aroused by the continuous discoveries by radio astronomers of more and more complex organic molecules in interstellar space, e.g., CN, C_2N_2, HCN, H_2CO, HC_3N, CH_3OH, CH_2O_2, CH_3CN, H_2CNH, etc. (e.g., Gordon and Snyder, 1973; Metz, 1973). These molecules were probably formed by atoms condensing on dust particles. These discoveries have led many to suggest that because this material was condensed to form the Earth, it was also the carrier and source of the organic molecules found on the Earth. First, if the Earth were formed by such material, subsequent heating of the Earth due to gravitational pull and radioactive decay would have destroyed the original molecules. Second, if the primeval Earth contained any atmosphere, the frictional heating and ablation of any small particle (meteor) entering the Earth's atmosphere at high velocities ($v > 11$ km s^{-1}) would have evaporated and destroyed the original organic molecules. Large meteorites, however, with simple organic molecules contained inside could have partially survived (Hochstim, 1971). An analysis of the Murchison meteorite indicates equal amounts of L and D amino acids (Ponnamperuma, 1972). On the other hand, meteorites with associated shock waves would have produced much more organic material in the primeval atmosphere (Hochstim, 1963).

Thus, one can assume that in the early stages (before life started) amino acids were formed in exactly equal mixtures of D and L amino acids, with only small local fluctuations from the mean.

1.2. FORMATION OF L AND D MOLECULAR POLYMERS

In this phase, the peptide chains were probably formed in a primitive 'soup' of an equal mixture of D and L amino acids. The most abundant peptides would have been small mixed ones with D and L amino acids joined together, with a very small concentration of pure chains of L and D amino acids linked together. It is known that D and L mixed chain polymers are unstable (Blout *et al.*, 1957, see reviews by Wald (1957) and Bonner (1972)). Therefore, after a long time there should have been high concentrations of configurationally homogeneous pure *D* and pure *L* peptides (e.g. Wald, 1957).

Simultaneously, with the formation of proteins from peptides, there were probably processes leading to the formation of sugar polymers (polysaccharides), lipids and nucleic acids (e.g., see Calvin, 1969). Then, somehow (the big step!) processes led to living organisms containing *L* amino acids and *D*-sugars. It may be impossible to give a meaningful answer to the question of why such symmetry was chosen, but we can catalog various hypotheses and try at least to speculate on possible mechanisms and simple models.

Most of the arguments presented in the rest of the paper with respect to organisms can also apply to the growth and destruction of large molecules.

1.3. FORMATION OF THE FIRST LIVING ORGANISM

Below, we group various hypotheses into separate categories.

1.3.1. *Hypothesis: The Germination of the First Living Organism is an Extremely Rare Event*

When a living organism, capable of reproducing, is formed from all the required molecules, there is an equal chance for this event to create an *L* or a *D* organism (defined here as consisting only of *L* or *D* amino acids*). Even if the event were extremely improbable, once the organism had formed at a certain location and survived, it would not be surprising to find that it was only *L*-type and that there was no other life in the universe (see Figure 1, case 1). The consequence of such a very rare phenomenon, had it occurred, would be very predictable.

In all literature on the origin of life, this theory is rejected on the basis of an estimation of the extremely small probability of a purely random event. Small probability, however, does not necessarily imply that it would be a purely random event. It is doubtful that this question can ever be settled unless in the future: (a) we can reject it by experiment (e.g., by detection of *L* or *D* life on Mars or elsewhere in the Universe), or (b) we find out that once all the required molecules and special conditions were given, the formation of a reproducing primitive organism was a very highly probable event, leading to a virtual explosion of life in the primitive soup (e.g., laboratory synthesis of *D* organisms). Then we will be able to understand that within a relatively short time in the history of the Earth, i.e., within about 500 million years of accumulation of organic molecules on the surface of the Earth, living primitive organisms of *L*-type and/or *D*-type evolved.

1.3.2. *Hypothesis of Likely Origin*

Let us assume that the occurrence of the first living organism in the primeval soup was highly probable. The following subcases are then possible:

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* It is not clear if the presence in the proteins of only *L* amino acids leads to another stereoselectivity: the incorporation of *D* sugars in nucleic acids (e.g., *D*-ribose and deoxy-*D*-ribose) and vice-versa. For discussion, see Eyring *et al.* (1962).

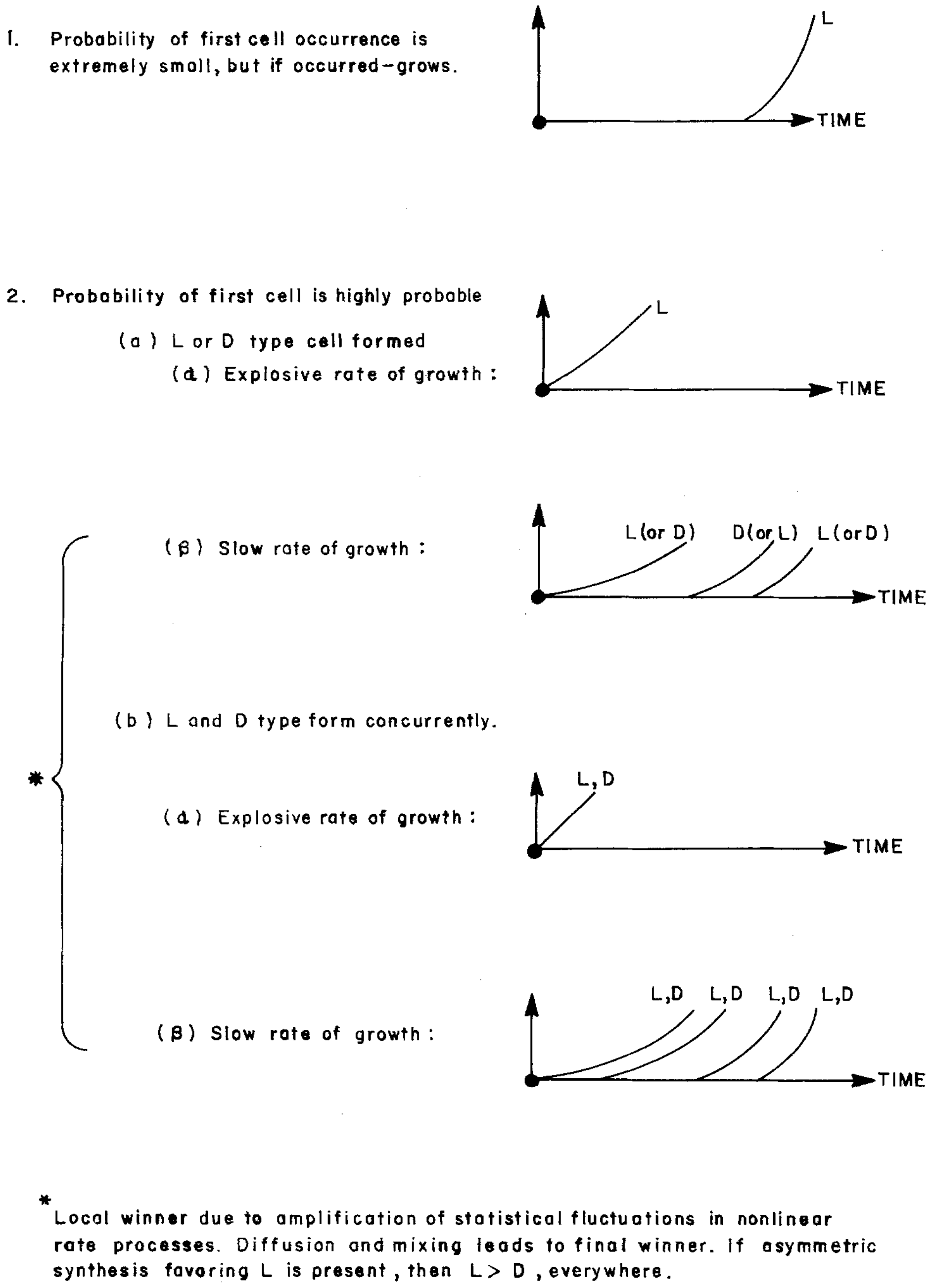

Fig. 1. Simplified catalogue of various models for the origin of optical activity.

(a) *L and D type organisms formed separately*, with
  (α) *Explosive Rate of Growth* (Spontaneous asymmetric synthesis). The first organism formed and quickly multiplied before any other organism had time to be formed. If the first organism happened to be *L*-type, all life on Earth is descended from this organism (see Figure 1, 1.3.2 (aα)). or
  (β) *Slow Rate of Growth.* The organisms formed at various times at different locations with slow rate of growth. *L* and *D* type living organisms appeared. Diffusion and various mixing (e.g., rain) must have facilitated wars between *D* and *L* populations (see Figure 1, 1.3.2 (aβ)).

(b) *L and D type organisms formed concurrently*, with
  (α) *Explosive Rate of Growth* (Spontaneous asymmetric synthesis). The first organisms formed at the same place and time and were of *L* and *D* type, with subsequent fights between species (see Figure 1, 1.3.2 (bα)). or
  (β) *Slow Rate of Growth.* Same as (aβ), but due to certain mechanisms equal amounts of *L* and *D* organisms formed (e.g., Wald, 1957) at the same time and the same place and occurred at different times and different places. Diffusion and mixing were probably important and led to wars between *D* and *L* populations (see Figure 1, 1.3.2 (bβ)).

1.3.3. *Application of Generalized Frank's Model*

In cases 1.3.2 (aβ), 1.3.2 (bα) and 1.3.2 (bβ), both self-reproducing *L* and *D* populations coexist initially. We assume that mutual antagonism was either developed or became noticeable with higher concentrations of *L* and *D* organisms. The generalized Frank's model, with which we will deal at length in this paper, is applicable to these cases. Then if there was some small excess of *L* (or *D*) population over *D* (or *L*) population due to statistical fluctuations, the *L* (or *D*) population would have increased and eventually won because of the amplification through nonlinear chemical rate processes. The multiplying winner was assisted by the death of the opposite enanthiomorph. Diffusion and mixing would have brought winners from various areas into contact with each other, leading to the final *L* winner. Similarly, if there were some asymmetric rate process present favoring the *L* population, the *L* enanthiomorph would have grown and won quicker because of amplification through nonlinear chemical rate processes.

Statistical fluctuations in general (e.g. via spontaneous asymmetric synthesis (Havinga, 1954; Calvin, 1969) would have lead to the formation of two spatially separated *L* and *D* colonies of protocells. Ritchie (1947) and also Bonner (1972) were critical of the concept of both colonies for the origin of chirality. In our opinion the discrepancy is completely resolved by adding mutual antagonism of the two populations at least with diffusion (obviously also rain, flow and mixing would have also facilitated bringing together the two populations into a contact). For detail see p. 365.

The remainder of this paper will deal with a model of two separate populations of primeval organisms (or polymers), one based on *L*- and one on *D*-amino acids. Possible interactions between those two separate, self-reproducing, and mutually antagonistic populations will be discussed.

2. The Local Growth and Death of *L* and *D* Cells*

2.1. INTRODUCTION

Let us consider a population of primeval cells composed exclusively of L or D amino acids and let us designate them by the letters L and D, respectively. If one L-type cell absorbs L-amino acids (from an infinite reservoir (open system) of organic materials containing L and D amino acids) and divides itself, we may write for the self-reproductive reaction:

$$L \xrightarrow{k_1} L + L, \tag{1}$$

where k_1 is the rate constant for this first order process. A similar expression can be written for the D cells,

$$D \xrightarrow{k_1} D + D, \tag{2}$$

where the fundamental principle was used that the rate constant k_1 in first reaction is the same as in second (in Section 2.4, we discuss the case in which these might not be equal). This follows from the conviction that molecular collisions, and all statistical processes, have the same dynamic time sequence as mirror events *under identical externally symmetric conditions.* If we assume that the L and D organisms die at the same rate and let X denote the concentration of dead organisms resulting from the interaction of L and D organisms, i.e., assuming simultaneous processes

$$L \xrightarrow{k_1} L + L; \qquad L \xrightarrow{k_1'} X$$

$$D \xrightarrow{k_1} D + D; \qquad D \xrightarrow{k_1'} X,$$

we obtain the description of simple linear rate processes in the absence of diffusion $(k_1 > k_1')$,

$$\left.\begin{aligned} \frac{\mathrm{d}L}{\mathrm{d}t} &= (k_1 - k_1')\, L \\ \frac{\mathrm{d}D}{\mathrm{d}t} &= (k_1 - k_1')\, D, \end{aligned}\right\} \tag{3}$$

with the solutions**

$$L = L_0\, e^{(k_1 - k_1')\,t}, \qquad D = D_0\, e^{(k_1 - k_1')\,t}$$

and

$$L - D = (L_0 - D_0)\, e^{(k_1 - k_1')\,t}, \qquad \frac{L}{D} = \frac{L_0}{D_0}$$

where L_0 and D_0 are the concentrations of L and D at time $t = 0$. If $L_0 = D_0$, then in

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* We will be discussing here primitive cells or organisms, but the same model can be applied to organic polymers.

** On molecular level the rate constants are temperature dependent and temperature may vary with time; however, we are assuming throughout this paper that all rate constants are time independent.

this model, $L=D$ at all times. If due to initial conditions (e.g., statistical fluctuations), $L_0>D_0$, then the difference $L-D$ will grow exponentially in time. Many similar schemes have been proposed, which lead essentially only to the growth of $|L-D|$, if $L_0\neq D_0$ (see Bonner, 1972). In this simple linear system *there are no instabilities, no destruction of one species.* In the rest of the paper, we will use k_1 for both k_1 and k_1', i.e., for an effective rate of population increase.

Let us consider the case of an asymmetric rate constant, i.e. $k_1\neq k_1^*$ (for further discussion see Section 2.4).

$$L\xrightarrow{k_1}L+L$$

$$D\xrightarrow{k_1^*}D+D,$$

where k_1 and k_1^* are effective rate constants for the L and D population growths, respectively. Thus,

$$\frac{dL}{dt}=k_1L; \qquad \frac{dD}{dt}=k_1^*D$$

$$L=L_0\,e^{k_1t}; \qquad D=D_0\,e^{k_1^*t} \tag{4}$$

$$L-D=L_0\,e^{k_1t}-D_0\,e^{k_1^*t}=L_0\,e^{k_1t}-D_0\,e^{k_1t}+D_0\,e^{k_1t}-D_0\,e^{k_1^*t}=$$

$$=(L_0-D_0)\,e^{k_1t}+D_0\,e^{k_1t}(1-e^{-\varepsilon k_1t}), \tag{5}$$

where

$$\varepsilon=\frac{k_1-k_1^*}{k_1}$$

Here again both population grow exponentially, with the difference increasing, but there is no mechanism of destruction of one population.

We see that if $L_0>D_0$ and $k_1>k_1^*$, the difference $L-D$ will grow due to an initial disproportion from fluctuation (the L_0-D_0 term) and due to asymmetry in rate constants. Initially

$$L-D\simeq(L_0-D_0)+D_0\left[\left(\frac{L_0-D_0}{D_0}\right)+\varepsilon\right]k_1t. \tag{6}$$

If $L_0<D_0$ and $k_1>k_1^*$ the two processes would counteract and result will be depending on the value of ε vs D_0-L_0.

2.2. STATISTICAL FLUCTUATIONS

The chemical rate constant can be deduced by averaging over a statistical process, i.e., over the collision probability (cross-section) and the relative velocity during collision (or duration of collision). Strictly speaking the cross-sections for the specific process (and not the rate constants) must be equal for chiral reactions under identical symmetric (mirror) conditions. Because of the statistical variables in the description of the rate constant (e.g., yielding temperature dependence) we may expect at a given time and location, slightly different amounts of L and D, i.e., we may expect that at some location $L-D$ is fluctuating in time around zero value. For living organisms reaction rates are even more complicated and represent overall rates of many chemical processes. They are thus affected by internal statistical variables and external variables (e.g., accidental deaths). Further cause of the fluctuations can be due to statistical averaging in the transport motion (diffusion), to non-uniform flows (e.g., turbulence), to non-uniform mixing, etc.

In the following we will review statistical estimates of the magnitude of the fluctuations in number of L and D cells (or polymers). If $N_L=\bar{N}_L\pm\Delta N_L$ and $N_D=\bar{N}_D\pm\Delta N_D$, where N_L and N_D are total numbers and ΔN_L and ΔN_D are the fluctuations, then the average fluctuation are given by (e.g., see Landau and Lifshitz, 1958)

$$\overline{(\Delta N_L)^2}=\bar{N}_L, \quad \overline{(\Delta N_D)^2}=\bar{N}_D, \quad \overline{\Delta N_L}=\overline{\Delta N_D}=0,$$

where bar over the quantity signifies averaging, and expectations of N_L can be given by

$$N_L=\bar{N}_L\left(1\pm\frac{1}{\sqrt{\bar{N}_L}}\right), \quad N_D=\bar{N}_D\left(1\pm\frac{1}{\sqrt{\bar{N}_D}}\right) \tag{7}$$

and one can show that*

$$\overline{(N_L-N_D)^2}=(\bar{N}_L-\bar{N}_D)^2+\bar{N}_L+\bar{N}_D \tag{8}$$

and if $\bar{N}_L=\bar{N}_D$

$$\overline{(N_L-N_D)^2}=2\bar{N}_L,$$

so that on the average (rms)

$$N_L-N_D=\sqrt{2\bar{N}_L}$$

or

$$\frac{L-D}{L}=\frac{N_L-N_D}{\bar{N}_L}=\sqrt{\frac{2}{\bar{N}_L}}. \tag{9}$$

It is interesting to note that once in a while fluctuations will reach an extremely high value, and that these extreme values, although purely random in magnitude and in time of arrival, satisfy some regular properties. If one plots the largest amplitude reached up *to a given time* vs logarithm of time, one gets usually a nearly straight line (e.g., see Gumbel, 1954, 1958). The slope depends on a particular law of distribution of amplitudes. It can be shown that under the assumption of independance and stationary property of extreme random variables a distribution function of extrema can be derived for general class of exponential functions (e.g., Gumbel, 1958; Montroll, 1974). The straight line relationship vs $\ln t$ has been verified on various data obtained from largest floods, largest snowfalls, greatest earthquake, oldest ages of men, etc., wherever there were large statistical sources of information (e.g., Gumbel, 1958).

$$\begin{aligned} * \ \overline{(N_L-N_D)^2} &= \overline{(\bar{N}_L\pm\Delta N_L-\bar{N}_D\mp\Delta N_D)^2} \\ &=(\bar{N}_L-\bar{N}_D)^2+\overline{(\Delta N_L-\Delta N_D)^2}\mp2(\bar{N}_L-\bar{N}_D)\,\overline{(\Delta N_L-\Delta N_D)} \\ &=(\bar{N}_L-\bar{N}_D)^2+\overline{(\Delta N_L-\Delta N_D)^2} \\ &=(\bar{N}_L-\bar{N}_D)^2+\overline{(\Delta N_L)^2}+\overline{(\Delta N_D)^2}-2\overline{\Delta N_L\Delta N_D}, \end{aligned}$$

since the N_L's are independent from ΔN_D's, $\overline{\Delta N_L\Delta N_D}=0$ and

$$\overline{(N_L-N_D)^2}=(\bar{N}_L-\bar{N}_D)^2+\bar{N}_L+\bar{N}_D.$$

Thus, we see that although on the average the fluctuations can be expected to oscillate around rms values, if we wait a sufficiently long time fluctuation of much larger size can be available.

The possible role of fluctuations in the origin of optical asymmetry was first discussed by Pearson (1898a, b). For other references, see Mills (1932) and an excellent review by Bonner (1972). The chemical instabilities due to fluctuations in general were discussed by Prigogine (1967). The model for the role of fluctuations in the origin of optical asymmetry *with growth and destruction* was first expressed in a simple mathematical model by F. C. Frank (1953), which we will review first and extend in the next section. In the Section 5 we will analyze another model (stereoscopic autocatalysis) suggested by Calvin (1969).

2.3. FRANK'S MODEL FOR MUTUALLY ANTAGONISTIC, SELF-REPRODUCING SYSTEMS

Now let us consider the simple pairing of L and D cells ('eating', fighting between) which we assume leads to death*, i.e., let

$$L+D\xrightarrow{k_2'}X.$$

Combining the previous processes with this as the only death mechanism, i.e., assuming simultaneous processes (k_1 includes the natural death k_1')

$$L\xrightarrow{k_1}L+L; \qquad D\xrightarrow{k_1}D+D; \qquad L+D\xrightarrow{k_2'}X,$$

one obtains two coupled nonlinear** differential equations:

$$\frac{\mathrm{d}L}{\mathrm{d}t}=k_1L-k_2'LD \tag{10}$$

$$\frac{\mathrm{d}D}{\mathrm{d}t}=k_1D-k_2'LD. \tag{11}$$

This system of equations was first suggested as a model for the origin of optical activity by F. C. Frank (1953). These equations are of the Lotka-Volterra type (Lotka, 1920, 1924, 1932; Volterra, 1930; Leigh, 1966; Goel *et al.*, 1971), except for the signs in front of the k's and therefore have no oscillatory solutions as are present in the normal Lotka-Volterra equations.

The behavior of the solutions can be studied by plotting $\ln L$ vs $\ln D$. To do this one combines Equations (10) and (11) (eliminating time) to obtain ($L\neq\alpha$),

$$\frac{\mathrm{d}\ln L}{\mathrm{d}\ln D}=\frac{D-\alpha}{L-\alpha},$$

where $\alpha=k_1/k_2$. The slope is positive when $D>\alpha$ and $L>\alpha$ (or when $D<\alpha$ and $L<\alpha$) and negative when $D<\alpha$ and $L>\alpha$ (or when $D>\alpha$ and $L<\alpha$). Furthermore, the slope

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* In polymer formation: instability of DL chain, e.g. DL helices.

** Because of LD product in equations.

is zero when $D=\alpha$ and is infinite when $L=\alpha$. Thus, the sketch below of $\ln L$ vs $\ln D$ can be traced simply to satisfy the preceding conditions.

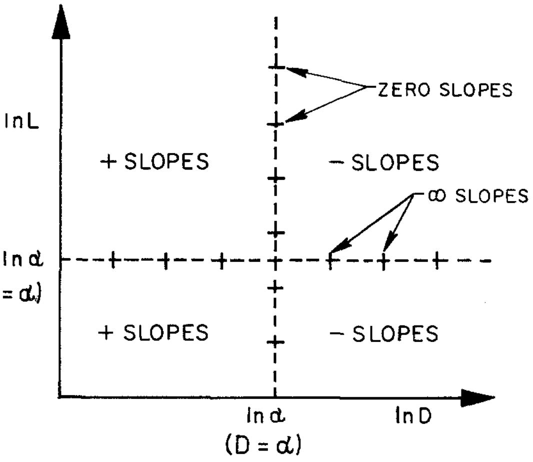

Figure 2 shows dependence of L vs D. Through any point in this plane (except at the stationary (singular) point, $L=\alpha$, $D=\alpha$) there passes a curve which indicates the path followed in time by the representative point (L, D) in the direction indicated by the arrow.

If at some time* (e.g., $t=0$) $L_0<D_0$ (indicated by a dot in Figure 2 on curve a), as time progresses (indicated by an arrow in Figure 2 on curve a), D will decrease initially, L will decrease and the ratios D/L will increase. The variation of the same case is seen in Figure 2, curve b, with L initially undergoing a maximum and again with D/L increasing all the time.

If $L_0>D_0$, the opposite occurs.

If $L_0=D_0$ and if we assume no fluctuations, $L=D$ at all times and either both decrease or both increase as they approach steady state (Figure 2g). The slope at $t=0$ of $\ln L$ vs time is $k_1(1-L_0/\alpha)$, and if $L_0>\alpha=k_1/k_2'$ the slope is negative, $L=D$ and both will be decreasing in time, while asymptotically approaching $L=D=\alpha$. Similarly, if $L_0<\alpha$ the slope is positive, $L=D$, and both will be increasing in time while asymptotically approaching the same steady state solution $L=D=\alpha$ (see Figure 2h).

We note from Equations (10) and (11) that if $L_0<D$, D passes through minimum when $L=\alpha$ (see Figure 2a), and L has a maximum when $D=\alpha$ (see Figure 2b). If $L_0>D_0$, L has a minimum when $D=\alpha$ and D passes through maximum when $L=\alpha$. Subtracting Equation (11) from Equation (10) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(L-D)=k_1(L-D) \tag{11a}$$

$$L-D=(L_0-D_0)\,e^{k_1t}, \tag{11b}$$

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* We assume in this section that we have only initial disproportion, i.e., that at subsequent times when L or D grow, the fluctuations are no longer significant, which is the case when the time between occurrence of consecutive fluctuations in number of L and D is greater than the time of reproduction $1/k_1$ (see Section 2.3.6, p. 331).

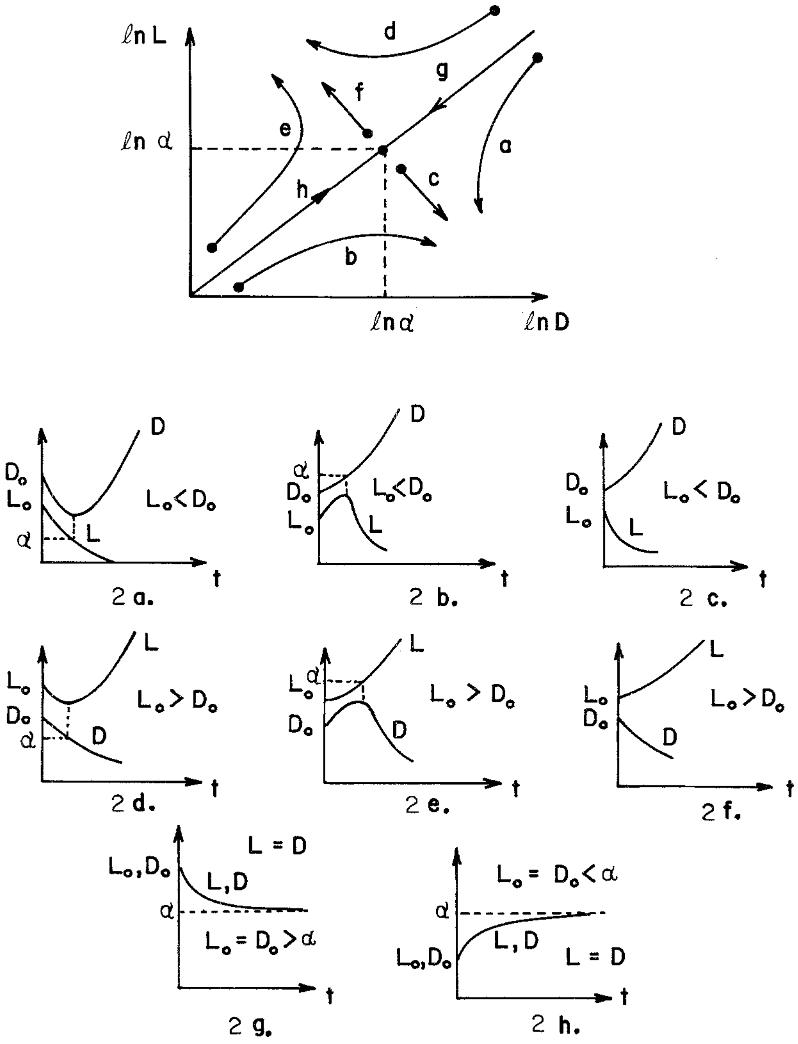

Fig. 2. ln L vs ln D and L and D vs times for various initial conditions.

where L_0 and D_0 are values of L and D at $t=0$. Dividing Equation (10) by L and Equation (11) by D, and subtracting Equation (11) from Equation (10), one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln(L/D)=k_2'(L-D)=k_2'(L_0-D_0)\,e^{k_1t}.$$

Integrating again gives

$$\ln(L/D)=K(L_0-D_0)\,e^{k_1t}+A, \tag{11c}$$

where $K=k_2'/k_1=1/\alpha$. At $t=0$,

$$\ln(L_0/D_0)=K(L_0-D_0)+A. \tag{11d}$$

Subtracting Equation (11c) from (11b) yields

$$\ln(L/D)=\ln(L_0/D_0)+K(L_0-D_0)\,(e^{k_1t}-1)$$

and

$$\frac{L}{D}=\left(\frac{L_0}{D_0}\right)e^{K(L_0-D_0)\,(e^{k_1t}-1)}. \tag{12}$$

Because of the identity

$$D=\frac{L-D}{\dfrac{L}{D}-1}; \qquad L=\frac{L}{D}\cdot D=\frac{L}{D}\,\frac{L-D}{\dfrac{L}{D}-1}, \tag{13}$$

we can explicitly write these equations in a symmetric form:

2.3.1. $L_0>D_0$ (see Figure 3)

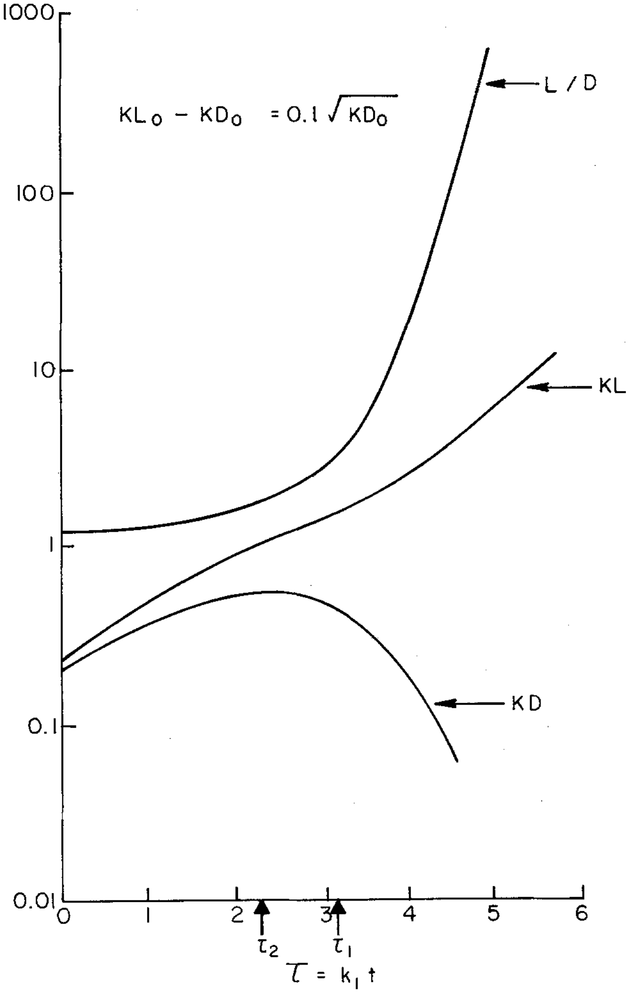

Fig. 3. KL, KD and L/D vs normalized time τ for a particular initial condition. Note the growth of L and L/D and very quick decay of D.

$$L=\frac{(L_0-D_0)\,e^{k_1t}}{\frac{L_0}{D_0}\,e^{K(L_0-D_0)\,(e^{k_1t}-1)}-1}\cdot\frac{L_0}{D_0}\,e^{K(L_0-D_0)\,(e^{k_1t}-1)} \tag{14}$$

$$D=\frac{(L_0-D_0)\,e^{k_1t}}{\frac{L_0}{D_0}\,e^{K(L_0-D_0)\,(e^{k_1t}-1)}-1}. \tag{15}$$

As $t\to\infty$

$$L\to(L_0-D_0)\,e^{k_1t} \tag{16}$$

$$D\to\frac{D_0}{L_0}(L_0-D_0)\,e^{-k(L_0-D_0)\,e^{k_1t}}, \tag{17}$$

i.e., L will grow exponentially but D *will disappear* (the 'death' rate) *much faster than exponentially.* This means that if a small fluctuation from $D=L$ exists at any time, the nonlinear processes perform strong amplification of the fluctuation, leading to the survival of one population and the rapid destruction of the other.* *Since fluctuation in concentration must occur in nature because of the randomness of all collisions rate and flow phenomena, this makes, after a while, the state of equal concentration of D and L cells very improbable, and occurrence of only L (or D) type organisms very probable.*

The opposite case, when $D_0>L_0$ can be obtained from Equations (14) and (15) by changing L_0-D_0 to $-(D_0-L_0)$ and rearranging the terms. D will be found to grow exponentially but L will disappear much faster than exponentially.

It is possible that in isolated places on Earth, a D colony won in one place, and an L colony in another. Subsequently, wars between these colonies could have resulted in the triumph of L colony (to be discussed later in Section 2.4 when diffusion will be included).

2.3.2. $L_0=D_0$

If $L_0=D_0$ at one time, then $L=D$ for all times

From Equation (11a), if $L_0=D_0$

$$\frac{\mathrm{d}\ln L}{\mathrm{d}\ln D}=\frac{D-\alpha}{L-\alpha}=1,\qquad (L\neq\alpha) \tag{18}$$

$$\frac{\mathrm{d}L}{\mathrm{d}t}=k_1L-k_2'L^2, \tag{19}$$

and

$$\frac{\alpha\,\mathrm{d}L}{L(\alpha-L)}=k_1\,\mathrm{d}t$$

---

* It is clear that in this model L and D can co-exist until concentration of L and D increases sufficiently so that the non-linear product LD becomes significant (see Equations (10) and (11)) and from then on the destructive mechanism for one population begins rapidly.

which, after integration, becomes

$$L\frac{L_0-\alpha}{L-\alpha}=L_0\, e^{k_1 t},$$

and solving for L,

$$L=\frac{\alpha}{1-\left(1-\frac{\alpha}{L_0}\right)e^{-k_1 t}}, \qquad \alpha=k_1/k_2', \tag{20}$$

in agreement with the solution to a similar equation for population growth by Lotka (1924). We see that if $L_0>\alpha$, $L>\alpha$ at all times, and as $t\to\infty$, $L\to\alpha$ (see Figure 2g). If $L_0<\alpha$, $L<\alpha$ at all times, and as $t\to\infty$, $L\to\alpha$ (see Figure 2h).

In examining Figure 2 we see that this condition of $L=D$ *at all times is very improbable*, because this solution is very unstable to any fluctuation in L and in D, i.e., to any deviation from $L=D$.

2.3.3. *General Relation Between L and D*

From Equations (10) and (11),

$$\left(1-\frac{\alpha}{L}\right)\mathrm{d}L=\left(1-\frac{\alpha}{D}\right)\mathrm{d}D, \qquad (L\neq\alpha),$$

and integration gives

$$L-\alpha\ln L=D-\alpha\ln D+A$$

$$e^L L^{-\alpha}=e^D D^{-\alpha} e^A,$$

and rearranging yields the general relation between L and D at any time (except for $L=\alpha$) with $K=1/\alpha=k_2'/k_1$:

$$\frac{L}{D}=\frac{L_0}{D_0}\, e^{K[L-D-(L_0-D_0)]}. \tag{21}$$

2.3.4. *Linear Solution for Short Times*

Let $\tau=k_1 t$, then from Equations (14)–(17) we obtain for $\tau\ll 1$,

$$\left.\begin{aligned} L&\simeq L_0[1+(1-KD_0)\,\tau],\\ D&\simeq D_0[1+(1-KL_0)\,\tau],\\ \frac{L}{D}&\simeq\frac{L_0}{D_0}[1+K(L_0-D_0)\,\tau]. \end{aligned}\right\} \tag{22}$$

We see that for short times, the behavior is linear with time, with an initial increase or decrease in L and D depending on the value of KL_0 and KD_0. Only afterward does the nonlinear amplification and destruction take place. We also see that L/D for $L_0 > D_0$ and D/L for $D_0 > L_0$ always increase in time.

2.3.5. *Characteristic Times*

First Stage $(k_1 t < 1)$. Linear solution (see Equation (22)). Decrease or increase in one or both.

Second Stage $(t \geqslant t_1)$. One of the types (L or D) begins to disappear. We can write for e-fold increase in L over D (or D over L),

$$\frac{L}{D} = \frac{L_0}{D_0} e^1.$$

Then from Equation (12), if $L_0 \neq D_0$,

$$K(L_0 - D_0)(e^{k_1 t_1} - 1) = 1$$

and

$$t_1 = \frac{1}{k_1} \ln\left(1 + \frac{k_1/k_2'}{|L_0 - D_0|}\right) \tag{23}$$

(different from that defined by Frank (1953)).

Third Stage $(t > t_1)$. One component increases only, e.g., if $L_0 > D_0$, D practically vanishes, and from Equation (11b)

$$L \simeq (L_0 - D_0) e^{k_1 t}.$$

If we define time t_2 as the time it takes for L to roughly double, i.e.,

$$L \simeq (L_0 - D_0) e^{k_1 t_2} = L_0 + D_0 (\text{or } 2L_0),$$

then

$$t_2 = \frac{1}{k_1} \ln \frac{L_0 + D_0}{|L_0 - D_0|}. \tag{24}$$

This is the same quantity as that defined by Frank (1953), but we have not found this quite as useful as t_1. For small $L_0 - D_0$, L and D may undergo changes of many orders of magnitude before one of them begins to die out (e.g., see Figure 4).

2.3.6. *Role of Continuous Fluctuations and Effective* $L_0 - D_0$

One may ask if the previous results were correct at all, since we only assumed one fluctuation at $t = 0$, when $L_0 \neq D_0$. Let us take Δt as the time during which one fluctuation in the number of L or D is expected to occur. Let at $t = 0$ the initial disproportion be $L_0 - D_0 = q_0$, then at $t = \Delta t$, from Equation (11b) $L - D = q_0 e^a$, where $a = k_1 \Delta t$, $a > 0$. Let us further assume that at $t = \Delta t$, another fluctuation in the number of L or of D occurred (e.g., from diffusion, mixing, etc.) so that at $t = \Delta t$, $L - D = q_0 e^a + q_1$, where q_1 is the fluctuation in $L - D$. Then at $t = 2\Delta t$ (assuming fluctuation occurring in equal spaced times Δt; can be easily generated to different Δt's).

$$L - D = (q_0 e^a + q_1) e^a + q_2 = q_0 e^{2a} + q_1 e^a + q_2.$$

After $n\Delta t$ times,

$$t=n\Delta t,\ L-D=q_0\,e^{na}+q_1\,e^{(n-1)a}+q_2\,e^{(n-2)a}+\cdots+q_{n-1}\,e^{a}+q_n.$$

Replacing $n\Delta t$ with t

$$L-D=e^{k_1t}\left[q_0+q_1\,e^{-a}+q_2\,e^{-2a}+\cdots+q_{n-1}\,e^{-(n-1)a}+q_n\,e^{-na}\right]. \tag{25}$$

We see that this is the same equation as Equation (11b), if we write

$$(L_0-D_0)_{\text{effective}}=L_0-D_0+q_1\,e^{-a}+q_2\,e^{-2a}+\cdots+q_n\,e^{-na}=\text{constant}. \tag{26}$$

Since a is positive, $q_1, q_2, \ldots, q_n$ are random positive and negative numbers from a class of distribution (e.g., normal) weighted with decreasing exponential factor. We see that the role of later fluctuations is smaller and smaller, although it is always a chance that, for example $|q_2\,e^{-2a}|>|q_0|$ and that even the sign of $(L_0-D_0)_{\text{effective}}$ will reverse (e.g.,

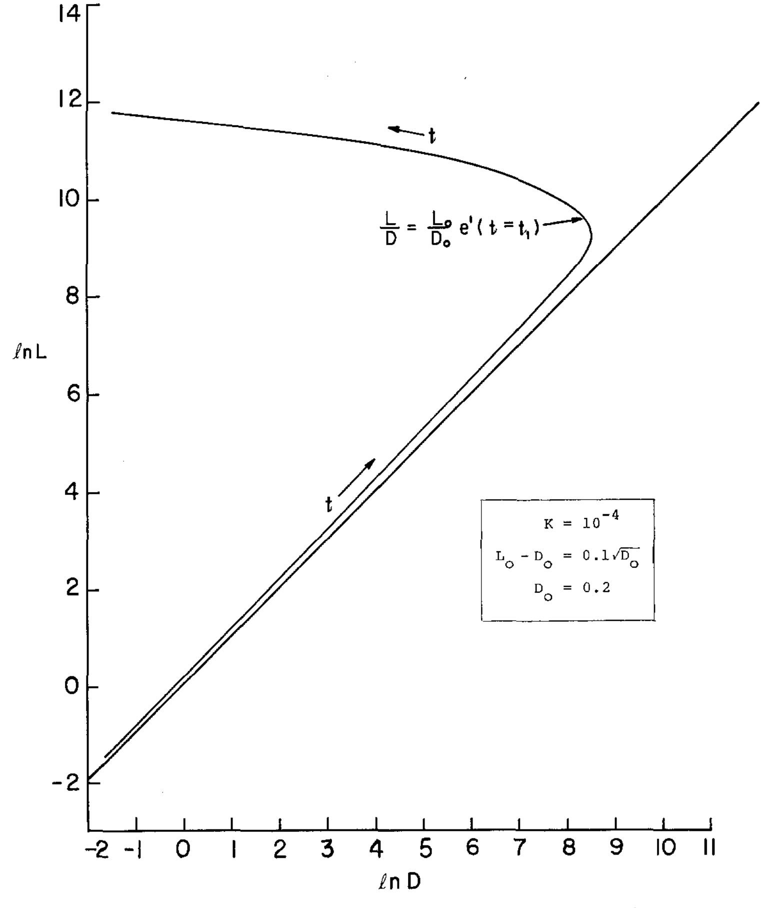

Fig. 4. $\ln L$ vs $\ln D$ for an initial condition of very slight excess of L. Note that it takes almost 11 magnitudes change in growth of L before 'the memory' of the excess of L takes place and then D rapidly disappears. In this calculation only one fluctuation was used. Arrows indicate the time sequence.

$q_2 < 0$). If $a = k_1 \Delta t \gg 1$, then

$$(L_0 - D_0)_{\text{effective}} \simeq L_0 - D_0 .$$

This is the case considered in previous sections and corresponds to the condition that $\Delta t \gg 1/k_1$, i.e., the time between occurrence of fluctuations need to be longer than the time of reproduction $1/k_1$. Thus under this condition all results of previous initial disproportion, are correct. In general, we see that we can interpret an 'initial disproportion' $(L_0 - D_0)_{\text{effective}}$ to signify weighted sum of many fluctuations in the beginning of growth only (e.g., neglecting $q_2\ e^{-2a}$ etc.).

If on the other hand, $a = k_1 \Delta t \ll 1$,

$$(L_0 - D_0)_{\text{effective}} = q_0 + q_1 + q_2 + \cdots = \sum \langle q_i \rangle \simeq 0 ,$$

since there are equal number of positive and negative fluctuations. This means that if fluctuations occur in times so frequently that the system cannot adjust itself (e.g., cannot grow or die by the kinetic mechanisms), then obviously $L = L_0 \pm |\delta l_i|$, $D = D_0 \pm \pm |\delta D_i|$, and on the average $\bar{L} = \bar{D}$ at all times.

The effect in L/D and in L and in D is highly nonlinear and can be studied numerically by placing random fluctuations at random times, while solving Equations (10) and (11). The results, however, are expected to be the same as with only initial disproportion $(L_0 \neq D_0)$ if $k_1 \Delta t \gg 1$.

2.3.7. *Conclusions*

We have seen that for the process of reproduction of L and D polymers (or cells) with the same rate constant k_1, i.e. for

$$L \xrightarrow{k_1} L + L, \qquad D \xrightarrow{k_1} D + D$$

the L and D enanthiomorphs will grow exponentially in time. The presence of an initial disproportion, due to fluctuation, (e.g., $L_0 > D_0$) will also cause an exponential growth of the difference $L - D$. In case when k_1 for both processes are not equal, due to some external conditions, both initial disproportion and the difference in rate constant will contribute to an increased difference of $L - D$. In this simple system there is no instability and no sudden death of one component (e.g., D).

By adding the additional process of lethal interaction (F. C. Frank, 1953) whenever L encounters D (death rate with rate constant k_2')

$$L + D \xrightarrow{k_2'} X \qquad (X \text{ is a 'dead' object})$$

the equations become nonlinear, i.e. the rate of production $k_1 L$ and $k_1 D$ is counteracted by the rate of destruction $k_2 LD$ (see Equations (10) and (11)). The system is now only stable if initially $L = D$, exactly. If at any time L will exceed D by a small number, due to a fluctuation, L will increase exponentially, but D after some time will die faster than exponentially (as $e^{-A e^{k_1 t}}$, see Equation (17)). If at any time D will exceed L by a small number, due to a fluctuation, D will grow exponentially and L will be destroyed extremely fast. Frank's comparison of a marble rolling downhill, starting on the crown of the road is very appropriate. The marble is likely to continue toward the side where it happens to be at the moment, even if it is only slightly off the center. The probability that L will grow and that $L=D$ at all times is extremely small and the situation is very unstable. Since in Equations (10) and (11) the product LD appears, the equations are nonlinear and the fluctuation is said to be amplified by a nonlinear kinetic process.

It was shown that if rate of reproduction is greater than the rate of destruction and if fluctuations occur very rarely, L and D may grow from an initial fluctuation through many orders of magnitude (Figure 4), and then suddenly (when k_2LD become appreciable in comparison with k_1L and k_1D, see Equations (10) and (11)) the long memory will cause the destruction of one (e.g., D).

It was shown that if the time between fluctuations is greater than the time of reproduction, initial disproportion (fluctuation) predominates over consecutive fluctuations, leading to growth of one and death of the other isomer.

If the fluctuations are very small and occur very frequently, (faster than rate of reproduction) L and D will become nearly equal. But due to the fact that after a long time, greater and greater fluctuations are probable (Gumble distribution), one of the forms (L or D) will take over and the other will be destroyed.

2.4. EXTENSION OF FRANK MODEL TO THE CASE WITH ASYMMETRIC RATE CONSTANTS

We are going to generalize Frank's model by considering the following reactions[†]

$$L\xrightarrow{k_1}L+L, \qquad D\xrightarrow{k_1^*}D+D, \qquad L+D\xrightarrow{k_2'}X,$$

where $k_1 \neq k_1^*$, due to some unspecified external asymmetry. We are interested in comparing such an effect with the effect of fluctuations, i.e., $L_0 \neq D_0$. Thus

$$\frac{dL}{dt}=k_1L-k_2'LD \tag{27}$$

$$\frac{dD}{dt}=k_1^*D-k_2'LD. \tag{28}$$

The case $k_1=k_1^*$ was discussed earlier. There does not appear to exist a solution in a closed analytic form. Series expansion exists by Abate and Hofelich (1968, 1969), and a straightforward Taylor series expansion (Milne's Method) to be derived later, is also applicable for a numerical solution. We will also derive here close form approximate solutions, which are convenient for general conclusion comparing effects of asymmetry in k (k_1 vs k_1^*) with the effect of fluctuations (i.e., $L_0 \neq D_0$).

2.4.1. *General Relation Between L and D*

Eliminating time, we obtain (if $L \neq k_1^*/k_2'$)

$$\frac{d\ln L}{d\ln D}=\frac{k_1-k_2'D}{k_1^*-k_2'L}$$

[†] See text after Equation (2), p. 322, and the Conclusion of this section.

and we can sketch $\ln L$ vs $\ln D$, similar to the previous case, when $k_1 = k_1^*$. We see that the character of the curves and solutions is the same, except that the singular (stationary) point is moved from $D = k_1/k_2'$ and $L = k_1/k_2'$ to the same D but $L = k_1^*/k_2'$. Integrating and rearranging, we obtain an exact expression:

$$\frac{L}{D} = \frac{L_0}{D_0}\left(\frac{L}{L_0}\right)^{\varepsilon} e^{K[L-D-(L_0-D_0)]}, \tag{29}$$

where we define ε as

$$\varepsilon = \frac{k_1 - k_1^*}{k_1} \equiv 1 - \beta \tag{30}$$

$$K = k_2'/k_1. \tag{31}$$

2.4.2. *Short Time Approximations*

(a) *First Method.* Subtracting Equation (28) from Equation (27) gives

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(L-D) &= k_1(L-D) + (k_1 - k_1^*)\, D \\ &\simeq k_1(L-D) + \varepsilon k_1 \bar{D}, \end{aligned} \tag{32}$$

where we have used

$$D(t) = \bar{D} + \delta(t),$$

and where we have neglected the $\varepsilon\delta$ term. Integrating, we obtain

$$L - D \simeq (L_0 - D_0 + \varepsilon\bar{D})\, e^{k_1 t} - \varepsilon\bar{D}. \tag{33}$$

Combining with Equation (29) and rearranging exponents yields

$$\frac{L}{D} \simeq \frac{L_0}{D_0}\left(\frac{L}{L_0}\right)^{\varepsilon} e^{K(L_0 - D_0 + \varepsilon\bar{D})\,(e^{k_1 t} - 1)}. \tag{34}$$

For *short times*, we can set $\bar{D} \simeq D_0$. When $k_1^* = k_1$, $\varepsilon = 0$ and we obtain exactly Equation (12). If $\varepsilon < 1$, we can neglect $\varepsilon\bar{D}$ [also *after long time* L will be dominant and $D(t) \simeq \bar{D}$ will be very small], so that

$$\frac{L}{D} \simeq \frac{L_0}{D_0}\left(\frac{L}{L_0}\right)^{\varepsilon} e^{K(L_0 - D_0)\,(e^{k_1 t} - 1)}, \tag{35}$$

and using approximately Equation (16) for L, with $L_0 > D_0$ and $\tau = k_1 t$,

$$\frac{L}{D} \simeq \frac{L_0}{D_0}\, e^{\varepsilon\tau + K(L_0 - D_0)\,(\varepsilon^{\tau} - 1)}. \tag{36}$$

The conclusion is that the asymmetry in the rate constant (ε) contributes to the exponential increase of L/D, whereas the statistical fluctuation in the initial condition (i.e., $L_0 - D_0$) contributes exponentially in the exponent. Thus, the fluctuations have much faster amplification than asymmetry in the rate constant. It is not relevant how fast L increases, but how fast D is 'dying out', i.e., L/D ratio is more relevant than just L. For short times,

$$\frac{L}{D} \simeq \frac{L_0}{D_0} e^{[k_1 - k_1^* + k_2'(L_0 - D_0)]t}, \tag{37}$$

and the effect of asymmetry has the same role as the effect of fluctuation. Thus, *initial fluctuations are more significant if*

$$L_0 - D_0 > \frac{k_1 - k_1^*}{k_2'}. \tag{38}$$

(b) *Second Method.* Using Montroll's technique (1972) of extended nonlinearization approximation,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \ln(L/D) &= k_1 - k_1^* + k_2'(L - D) = \\ &= (k_1 - k_1^*) + ck_2'\left(\frac{L}{c} - \frac{D}{c}\right) \simeq \\ &\simeq \varepsilon k_1 + ck_2' \ln(L/D).\end{aligned}$$

$$\ln(L/D) \simeq B\, e^{ck_2't} - \frac{\varepsilon k_1}{ck_2'}$$

$$\ln(L/D) = \left(\ln\frac{L_0}{D_0} + \frac{\varepsilon k_1}{ck_2'}\right) e^{ck_2't} - \frac{\varepsilon k_1}{ck_2'}$$

and

$$\frac{L}{D} \simeq \frac{L_0}{D_0} \exp\left\{\left[\ln(L_0/D_0) + \frac{\varepsilon k_1}{ck_2'}\right](e^{ck_2't} - 1)\right\}.$$

From the example in the previous case where $k_1 = k_1^*$, we believe this to be a good approximation for short times, i.e.,

$$\frac{L}{D} \simeq \frac{L_0}{D_0} \exp\left[\left(c \ln\frac{L_0}{D_0} + \frac{\varepsilon k_1}{k_2'}\right) k_2't\right].$$

At initial times to the fluctuation $L_0 - D_0$ (or $D_0 \ln(L_0/D_0)$ is added effectively $(k_1 - k_1^*)\, D_0$ and if we use as in the previous case,

$$\ln(L_0/D_0) \simeq \frac{L_0 - D_0}{D_0} \quad \text{and} \quad c = D_0,$$

then

$$\frac{L}{D} \simeq \frac{L_0}{D_0} e^{[k_1 - k_1^* + k_2'(L_0 - D_0)]t}, \tag{38a}$$

which is the same as Equation (37).

2.4.3. *Taylor Series Expansion*

A suitable expression for numerical evaluation is obtained from Taylor series expansion of $\ln L$ and $\ln D$ (Milne's method):

$$\ln L(\tau+\Delta\tau)=\ln L+(1-KD)\,\Delta\tau-\tfrac{1}{2}KD(\beta-KL)\,(\Delta\tau)^2+$$
$$+[-KD(\beta-KL)^2+K^2DL(1-KD)]\,\frac{(\Delta\tau)^3}{6}+\cdots \tag{38b}$$

$$\ln D(\tau+\Delta\tau)=\ln D+(\beta-KL)\,\Delta\tau-\tfrac{1}{2}KL(1-KD)\,(\Delta\tau)^2+$$
$$+[-KL(1-KD)^2+K^2DL(1-KL)]\,\frac{(\Delta\tau)^3}{6}+\cdots, \tag{38c}$$

where $\beta=k_1^*/k_1=1-\varepsilon$, $K=k_2'/k_1$, $\tau=k_1t$ and all L and D on the right hand side of the above relations are evaluated at normalized time τ. Subtracting,

$$\ln\frac{L(\tau+\Delta\tau)}{D(\tau+\Delta\tau)}=\ln(L/D)+[\varepsilon+K(L-D)]\,\Delta\tau+\tfrac{1}{2}K(L-\beta D)^2(\Delta\tau)^2+$$
$$+\tfrac{1}{6}K[L(1-KD)^2-D(\beta-KL)^2+K^2LD(L-D)]\,(\Delta\tau)^3+\cdots. \tag{38d}$$

If $\tau=0$,

$$\frac{L(\Delta\tau)}{D(\Delta\tau)}\simeq\frac{L_0}{D_0}\,e^{[\varepsilon+K(L_0-D_0)]\,\Delta\tau+[K(L_0-\beta D_0)^2\,(\Delta\tau)^2]/2} \tag{38e}$$

and we see that Equation (36) agrees exactly to order Δt and the $(\Delta t)^2$ term from the expanded Equations (36) or (38) only agrees if $\beta=1$. We also see that if $L_0=D_0$ at $\tau=0$, then $L\neq D$ at other times!

2.4.4. *An Exact Integral Equation*

From Equations (27) and (28), with $\tau=k_1t$,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(L-D)=L-D+\varepsilon D \tag{39}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\ln(L/D)=K(L-D)+\varepsilon. \tag{39a}$$

Integrating Equation (39),

$$L-D=[L_0-D_0+\varepsilon A(\tau)]\,e^{\tau}, \tag{39b}$$

where

$$A(\tau)=\int_0^{\tau}D(\tau')\,e^{-\tau'}\,\mathrm{d}\tau'.$$

Substituting Equation (39b) into Equation (39a) and integrating, we obtain an exact expression convenient for analysis:

$$\ln(L/D)=K(L_0-D_0)\,(e^{\tau}-1)+\varepsilon KB(\tau)+\varepsilon\tau+\ln\frac{L_0}{D_0}$$

or

$$\frac{L}{D}=\frac{L_0}{D_0}\,e^{\varepsilon\tau+K(L_0-D_0)\,(e^{\tau}-1)+\varepsilon KB(\tau)}, \tag{39c}$$

where

$$B(\tau)=\int_0^{\tau}e^{\tau''}\int_0^{\tau''}D(\tau')\,e^{-\tau'}\,\mathrm{d}\tau'\,\mathrm{d}\tau''.$$

We see that the previously derived approximations for short times (Equations (35)–(37)) can be directly reproduced by neglecting the A and B integrals. Furthermore, we see that the integral contributes a positive term, since A and B are positive, increasing $L-D$ and L/D values. As $\varepsilon\to 0$, we see that we can recover exactly Equations (12) and (13).

2.4.5. *Approximation for Long Times*

If $L_0 > D_0$ and/or $\varepsilon > 0$, then after long time, L will increase and D will rapidly decrease, so that the integral

$$A(\tau) = \int_0^\tau D(\tau')\, e^{-\tau'}\, \mathrm{d}\tau' \to \bar{A} = \text{constant}. \tag{39d}$$

Because of the rapidly decreasing integrand, we may assume to the first approximation,

$$\bar{A} \simeq D_0 \tag{39e}$$

and

$$B(\tau) \simeq \int_0^\tau e^{\tau''} D_0\, \mathrm{d}\tau'' = D_0 (e^\tau - 1) \tag{39f}$$

and

$$L - D \simeq (L_0 - D_0 + \varepsilon D_0)\, e^\tau \tag{39g}$$

$$\frac{L}{D} \simeq \frac{L_0}{D_0}\, e^{\varepsilon\tau + K[(L_0 - D_0) + \varepsilon D_0]\,(e^\tau - 1)}. \tag{39h}$$

2.4.6. *Comparison of Role of Fluctuations with Possible Asymmetry in Rate Constants*

In this section we will present an argument to show the probable dominance of fluctuations over asymmetry in rate constants.

In Equation (37) we have shown that short times fluctuations are more important than the asymmetry in the rate constants when (e.g., $k_1 > k_1^*$ and $L_0 > D_0$)

$$\frac{k_2'}{k_1}(L_0 - D_0) > \varepsilon = \frac{k_1 - k_1^*}{k_1}. \tag{39i}$$

Now from Equation (9), we have for the rms initial conditions

$$L_0 - D_0 = \frac{N_L^0 - N_D^0}{V} \simeq g \frac{\sqrt{2N_L^0}}{V}, \tag{39j}$$

where N_L^0 and N_D^0 are total initial numbers of L and D, respectively in the volume V and g is a factor, $g \geqslant 1$ due to Gumbel distribution of extreme fluctuations (see p. 324).

Defining the rate of population growth as $R_1^{(L)}$ and $R_1^{(D)}$ and the rate of population destruction as R_2, we have

$$\frac{\mathrm{d}L}{\mathrm{d}t} = R_1^{(L)} - R_2; \qquad \frac{\mathrm{d}D}{\mathrm{d}t} = R_1^{(D)} - R_2 \tag{39k}$$

so that

$$R_1^{(L)} = k_1 L, \qquad R_1^{(D)} = k_1 D, \qquad R_2 = k_2' L D \tag{39l}$$

we obtain

$$\begin{aligned} \frac{k_2'}{k_1}(L_0 - D_0) &= \frac{k_2' L_0 D_0}{k_1 L_0 D_0}(L_0 - D_0) = \frac{R_2^0}{R_1^0} \frac{L_0 - D_0}{D_0} = \frac{R_2^0}{R_1^0} \frac{N_L^0 - N_D^0}{N_D^0} = \\ &= \frac{R_2^0}{R_1^0} g \frac{\sqrt{2N_0}}{N_0} = g \frac{R_2^0}{R_1^0} \sqrt{\frac{2}{N_0}}, \end{aligned} \tag{39m}$$

where R_1^0 and R_2^0 are initial rates, $\bar{N}_L^0 \simeq \bar{N}_D^0 = N_0$.

Thus the initial disproportion (fluctuation) is more significant than the difference in the rate constants ($k_1 \neq k_1^*$) if

$$\frac{R_2^0}{R_1^0} g \sqrt{\frac{2}{N_0}} > \varepsilon. \tag{39n}$$

In Figure 5 we have plotted approximately Equation (39n). For example, had we measured a value of $\varepsilon = 7 \times 10^{-10}$, then as long as (for $R_1/R_2 = 20$, i.e. for twenty times faster rate of reproduction then destruction) total number of L's and D's is less than 10^{16}, the fluctuations are more important, and for $R_1/R_2 = 10^4$ as long as $N < 10^{11}$. In order to estimate ε we can make the following analysis. Let us consider only simple linear system with $L \xrightarrow{k_1} L + L$, $D \xrightarrow{k_1^*} D + D$ and $L_0 = D_0$. Then

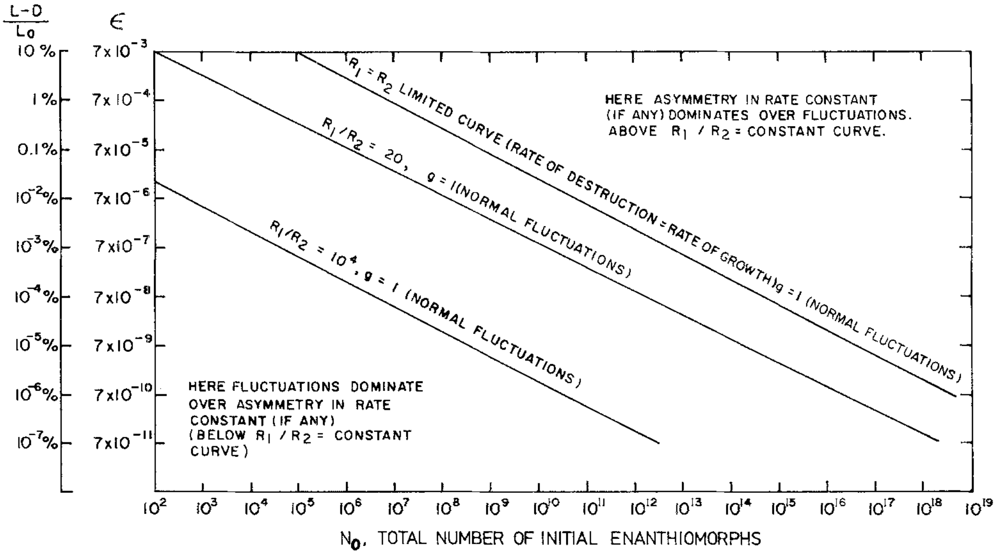

Fig. 5. Role of fluctuations and possible asymmetry in rate constants as a function of R_1/R_2, N_0 and ε.

$$\frac{dL}{dt} = k_1 L, \qquad \frac{dD}{dt} = k_1^* D \tag{39o}$$

$$L = L_0\, e^{k_1 t}, \qquad D = L_0\, e^{k_1^* t} \tag{39p}$$

$$\frac{L-D}{L_0} = e^{k_1 t} - e^{k_1^* t} = e^{k_1 t}\left(1 - e^{-\varepsilon k_1 t}\right) \tag{39q}$$

or

$$\varepsilon = -\frac{1}{k_1 t} \ln\left[1 - \frac{L-D}{L_0} e^{-k_1 t}\right] \simeq \frac{e^{-k_1 t}}{k_1 t} \frac{L-D}{L_0}. \tag{39r}$$

If we consider any experiment of racemization (e.g., via crystallization) we can consider

$k_1 t = 2$ (L and D increase by a factor of 7.4), then

$$\varepsilon \simeq 0.07 \frac{L-D}{L_0}. \tag{39s}$$

In seeking through literature we could not find any conclusive data on the effect of external sources on asymmetric growth of any L and D isomers. Assuming the optimistic laboratory results $(L-D)/L_0 \lesssim 0.1\%$, we obtain

$$\frac{L-D}{L_0} \lesssim 10^{-3} \quad \text{and} \quad \varepsilon \lesssim 7 \times 10^{-5}. \tag{39t}$$

For fluctuations to dominate, if $R_1/R_2 \simeq 20$ we get $N_0 < 10^{6+}$ and for $R_1 = R_2$, we get $N_0 < 10^{9+}$. Since we expect N_0 to be small and ε to be extremely small, if not zero, we may assume that fluctuations probably played a more significant role than any possible asymmetry in rate constants.

2.4.7. *Conclusions*

In Equation (37) we showed that for short times, the effect of asymmetry in rate constants ($k_1 \neq k_1^*$) is only important if

$$\varepsilon = \frac{k_1 - k_1^*}{k_1} \gtrsim K(L_0 - D_0).$$

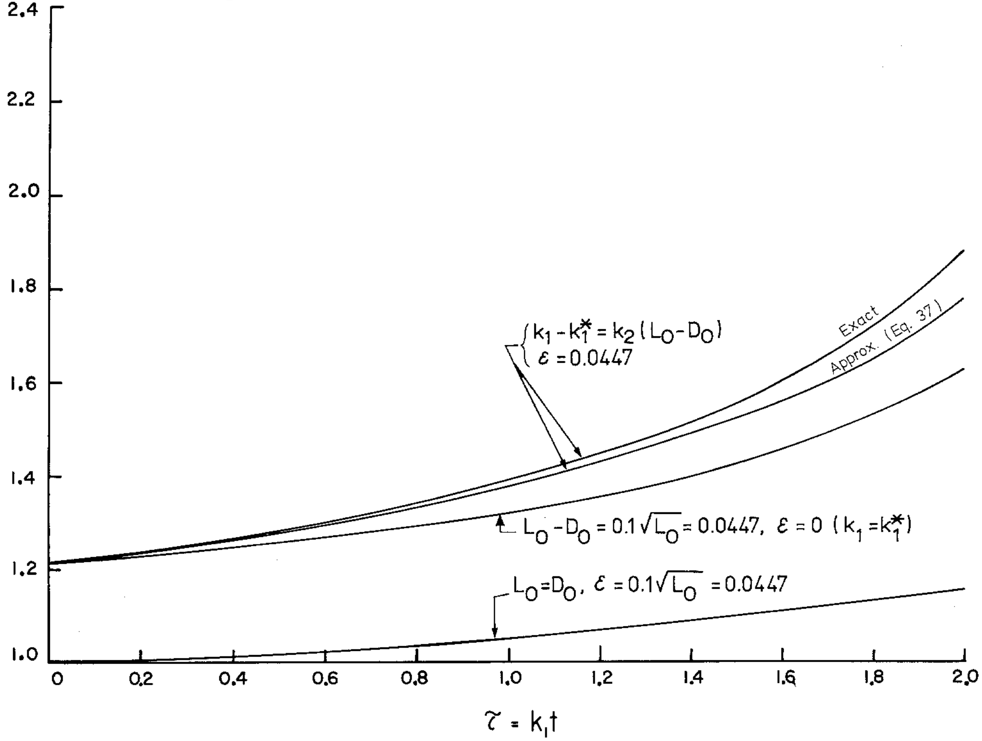

Fig. 6. Comparison of effect of nonequal rate constants ($k_1 \neq k_1^*$) vs fluctuations ($L_0 \neq D_0$). L/D as a function of normalized time τ.

In Figure 6 we see the validity of Equation (36) (or Equation (37)) for short times for the cases when $\varepsilon = k(L_0 - D_0)$, $\varepsilon = 0$, $L_0 \neq D_0$, and $\varepsilon \neq 0$, $L_0 = D_0$. Statistical fluctuations are more significant then the asymmetry in the rate constant, when

$$\varepsilon < \frac{R_2^0}{R_1^0}\sqrt{\frac{2}{N_0}}.$$

Thus we conclude that, in our model, *even if there were some small effects of asymmetry in the rate constants due to some external conditions, the statistical fluctuations would have been most likely to dominate.* Again, we should point out what we stated in the introduction, that until now, no definite mechanism has been proposed for any

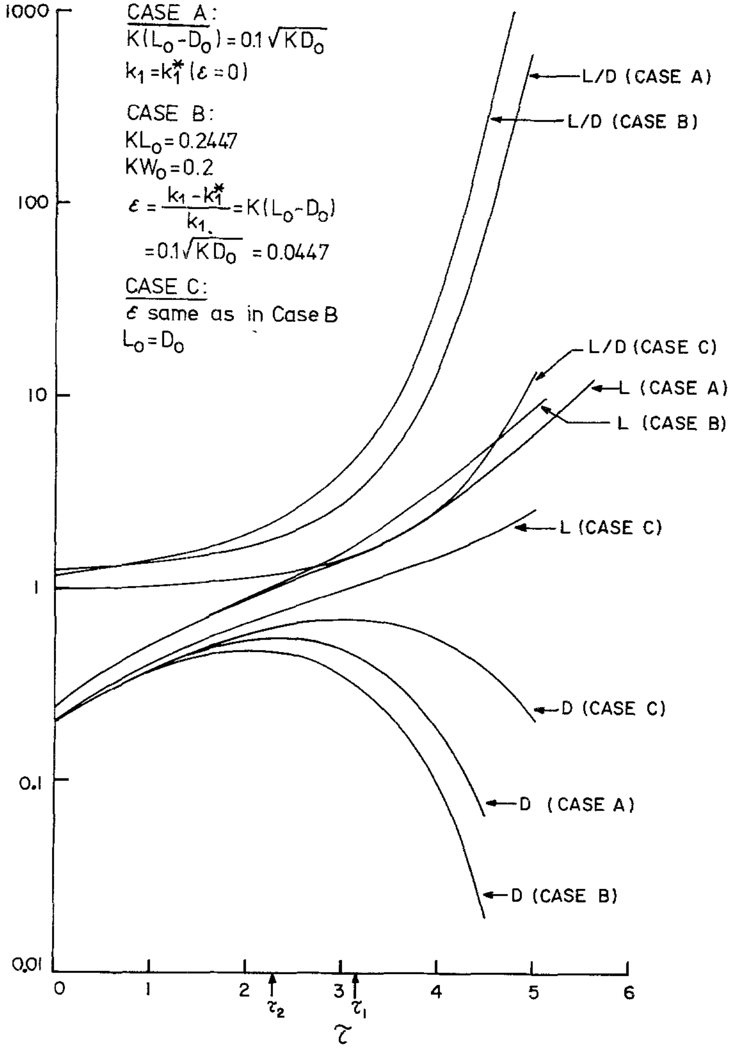

Fig. 7. Similar to Figure 6 for longer times for 3 different cases.

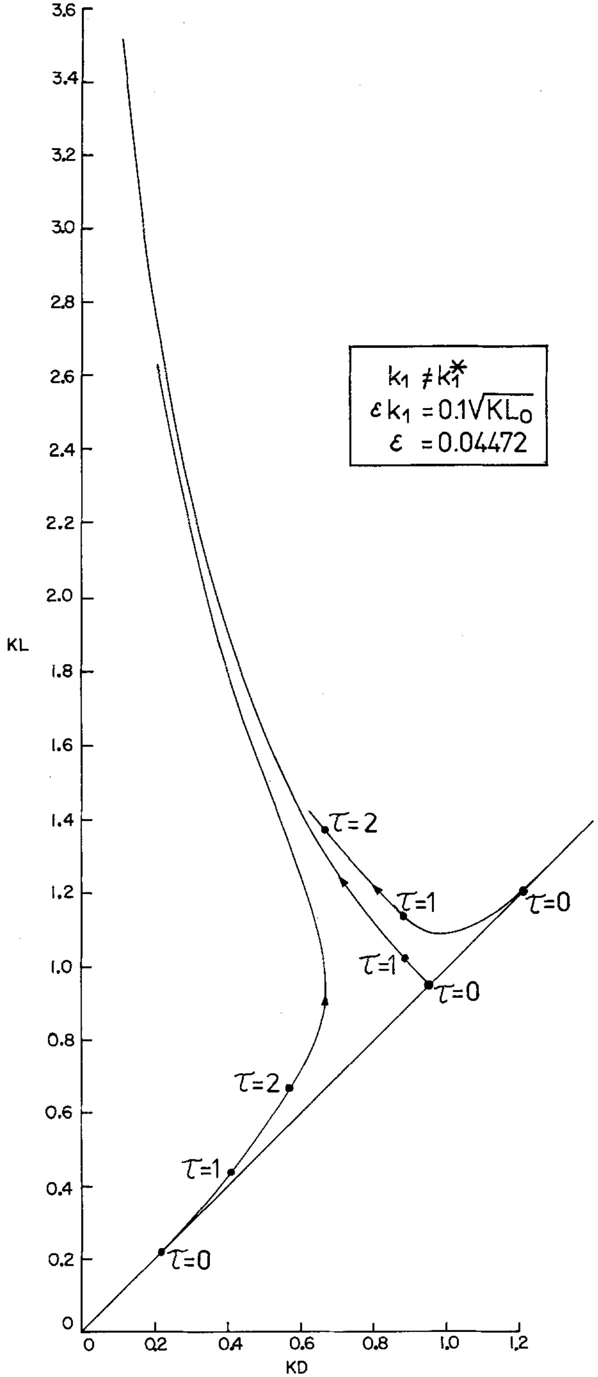

Fig. 8. *KL* vs *KD* for the asymmetric rate constants ($k_1 \neq k_1^*$) and in the absence of fluctuation ($L_0 = D_0$) as a function of normalized time τ for three different initial conditions.

asymmetry in rate constants due to natural, external processes, which could have led to the present dominance of L amino acids in living organisms. The fluctuations are most plausible source of such selection.

It is interesting to note (see Figure 7) that the time in which one enanthiomorph begins to die at a significantly fast rate is roughly the same whether it is due to (a) pure statistical fluctuations, $L_0 - D_0$ (case A); or (b) equal contribution from asymmetry in rate constants and statistical fluctuations (case B); or (c) asymmetry in rate constant, but in the (unlikely) absence of any fluctuations, i.e., $L_0 = D_0$ (case C).

The absolute magnitude of growth of the winning enanthiomorph is actually of no significance, only the death of the antagonist counts in this model; and L/D is such an indicator. L/D increases rapidly when D begins to die out (see Figure 7).

Figure 8 illustrates the plot of L vs D for three cases when $\varepsilon \neq 0$, but when $L_0 = D_0$. We see that when one starts with $L_0 = D_0$ and $\varepsilon \neq 0$, the equations are unstable and tend to increase* L if $\varepsilon > 0$ (or increase D when $\varepsilon < 0$). If $L_0 = D_0$ and the asymmetry in the rate constant ($\varepsilon \neq 0$) is present, the nonlinear term ($k_2 LD$) will amplify the difference in rate constant causing the death of one enanthiomorph.

3. Generalization of Frank's Model by Including Diffusion

3.1. INTRODUCTION

We are considering L and D organisms (or growing polymers) in a liquid, gaseous or porous medium. In the presence of diffusion from one concentration of a mixture of L and D organisms to another concentration of L and D cells, and in the presence of Frank's growth and annihilation reactions, Equations (10) and (11) are generalized to

$$\frac{\partial L}{\partial t} = k_1 L - k_2' LD + \nabla \cdot (\gamma \nabla L) \tag{40}$$

$$\frac{\partial D}{\partial t} = k_1 D - k_2' LD + \nabla \cdot (\gamma \nabla D), \tag{41}$$

where γ is the diffusion coefficient or effective diffusion coefficient (in porous medium), and we have assumed no flow (stationary medium).

Subtracting Equation (41) from Equation (40), we obtain

$$\frac{\partial}{\partial t}(L - D) = k_1 (L - D) + \nabla \cdot \gamma \nabla (L - D) \tag{42}$$

which can be reduced to a generalized diffusion equation. *If $L = D$ in all space at one time, $L = D$ in all space at all other times.* We are going to restrict our solutions to the one dimensional case with a constant diffusion coefficient. The theory of nonlinear diffusion can be found in Montroll and West (1973).

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* Except at the stationary point, where $L_0 = k_1^*/k_2$ and $D_0 = k_1/k_2$ (see Section 2.4.1 and Equation (75e)).

3.2. EXACT LINEAR SOLUTION FOR $L-D$

With $\gamma=$constant, in one dimension Equations (40) and (41) become*

$$\frac{\partial L}{\partial t}=k_1 L-k_2' LD+\gamma\frac{\partial^2 L}{\partial x^2}, \tag{40a}$$

$$\frac{\partial D}{\partial t}=k_1 D-k_2' LD+\gamma\frac{\partial^2 D}{\partial x^2} \tag{41a}$$

and

$$\frac{\partial}{\partial t}(L-D)=k_1(L-D)+\gamma\frac{\partial^2}{\partial x^2}(L-D). \tag{42a}$$

Let (e.g., Richardson, 1961) $L-D=e^{k_1 t}\,\phi(x,t)$, then one obtains the normal diffusion equation

$$\frac{\partial \Phi}{\partial t}=\gamma\frac{\partial^2 \phi}{\partial x^2}. \tag{42b}$$

3.2.1. *Infinite-Region Exact Solution*

The infinite-region exact solution is (e.g., Richardson, 1961)

$$L(x,t)-D(x,t)=\frac{e^{k_1 t}}{\sqrt{4\pi\gamma t}}\int_{-\infty}^{\infty} e^{-(x'-x)^2/4\gamma t}\,[L(x',0)-D(x',0)]\,\mathrm{d}x', \tag{43}$$

where $L(x',0)$ and $D(x',0)$ are initial ($t=0$) distribution of L and D organisms. We can write this expression in a different form:

$$L(x,t)-D(x,t)=\frac{1}{\sqrt{\pi}}\,e^{k_1 t}\int_{-\infty}^{\infty} e^{-u^2}\,[L(u,0)-D(u,0)]\,\mathrm{d}u, \tag{43a}$$

where

$$u=\frac{x'-x}{\sqrt{4\gamma t}},\qquad W=x+u\sqrt{4\gamma t}.$$

If after long time L increases and D becomes very small, then Equation (43) gives the rate of increase of L only. We will illustrate various examples of growths for different initial distributions.

(a) If a local fluctuation occurred in one place only, i.e., assume

$$L(x,0)-D(x,0)=(L_0-D_0)\,e^{-bx^2}, \tag{44}$$

where $\varepsilon_0=L_0-D_0=$positive constant. Then it follows from Equation (43) that at any other time at any x

$$L(x,t)-D(x,t)=(L_0-D_0)\frac{e^{k_1 t}}{\sqrt{1+4b\gamma t}}\,e^{-bx^2/(1+4b\gamma t)}, \tag{45}$$

---

* An introduction to the role of chemical instabilities in some nonlinear kinetic equations with diffusion was discussed by Prigogine (1967).

i.e., the growth of the fluctuation at given x is increasing in time with reduced exponential and the width of the distribution spreads also in time, with the dispersion

$$\sigma^2=\frac{1}{2b}+2\gamma t.$$

(b) If a local fluctuation occurred in one place only, i.e., assume

$$L(x,0)-D(x,0)=(L_0-D_0)\,\delta(x), \tag{46}$$

where $\delta(x)$ is the Dirac Delta function,* then

$$L(x,t)-D(x,t)=(L_0-D_0)\frac{e^{k_1t}}{\sqrt{4\pi\gamma t}}\,e^{-x^2/4\gamma t}, \tag{47}$$

which is very similar to the previous example for $t>1/4\beta\gamma$ and with $\beta=\pi$. As $t\to\infty$,

$$L-D\to(L_0-D_0)\frac{e^{k_1t}}{\sqrt{4\pi\gamma t}}, \tag{48}$$

i.e., at every point, its growth in time is slower than exponential.

(c) If initially $L(x,0)-D(x,0)=L_0-D_0=\text{constant}$ for all x, then from Equation (43) (or with $\beta=0$),

$$L(x,t)-D(x,t)=(L_0-D_0)\,e^{k_1t},$$

in agreement with the previous solution (Equation (11b)) in the absence of diffusion.

(d) Let

$$L(x,0)-D(x,0)=a_1\sin(x/\lambda). \tag{49}$$

Then

$$L(x,t)-D(x,t)=a_1\frac{e^{k_1t}}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-u^2}\sin\left[\frac{1}{\lambda}\left(x+u\sqrt{4\gamma t}\right)\right]\mathrm{d}u$$

and (e.g., see Gradstein and Ryzyk, 1963)

$$L(x,t)-D(x,t)=a_1\,e^{(k_1-\lambda^{-2}\gamma)t}\sin(x/\lambda). \tag{50}$$

The growth of $L-D$ will continue if the wavelength λ is such that

$$\lambda>\sqrt{\frac{\gamma}{k_1}}. \tag{51}$$

We note that if we approximate Equation (42)

$$\frac{\partial I}{\partial t}=k_1I+\gamma\frac{\partial^2 I}{\partial x^2}\simeq k_1I-(\gamma/l^2)\,I, \tag{52}$$

where $I=L-D$, l is some characteristic length, we get

$$I\simeq I_0\,e^{(k_1-\gamma/l^2)t} \tag{53}$$

which has similar time dependence to Equation (50) with $l=\lambda$.

(e) If

$$L(x,0)-D(x,0)=b_1\cos(x/\lambda) \tag{54}$$

then

$$L(x,t)-D(x,t)=b_1\,e^{(k_1-\lambda^{-2}\gamma)t}\cos(x/\lambda). \tag{55}$$

(f) If we represent the original distribution of $L-D$ by a Fourier series, i.e., if

$$L(x,0)-D(x,0)=\sum_{n=0}^{\infty}\left[a_n\sin(nx/\lambda)+b_n\cos(nx/\lambda)\right], \tag{56}$$

---

* $\delta(x)=0$ for $x\neq0$ and so much infinite at $x=0$ that $\int_{-\infty}^{\infty}\delta(x)\,\mathrm{d}x=1$. One of the properties used here is $\int_{-\infty}^{\infty}f(x)\,\delta(x-a)\,\mathrm{d}x=f(a)$.

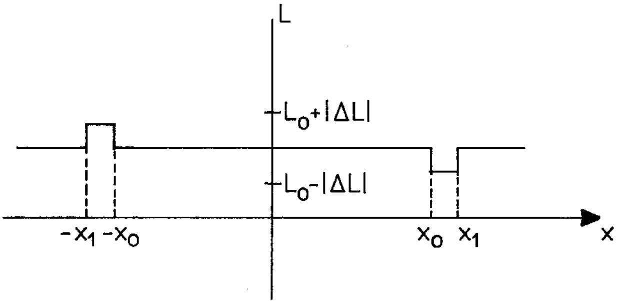
then

$$L(x,t)-D(x,t)=\sum_{n=0}^{\infty} e^{(k_1-n^2\gamma\lambda^{-2})t}\times\left[a_n\sin(nx/\lambda)+b_n\cos(nx/\lambda)\right], \tag{57}$$

and for all integers $n<\lambda\sqrt{k_1/\gamma}=g$, amplitudes will grow. i.e.. all lower harmonics $(n<g)$ will cause a spatial separation of L over D, and higher harmonics $(n>g)$ will be quickly damped out.

(g) If initially $L=D$ for all x, then exactly $L=D$ at all times.

(h) Let us assume



We assume

$$\begin{aligned}
&D(x,0)=D_0 && \text{for all } x\\
&L(x,0)=L_0=D_0 && \text{for all } x \text{ except:}\\
&L(x,0)=L_0+|\Delta L| && \text{for } -x_1\leqslant x\leqslant -x_0,\\
&L(x,0)=L_0-|\Delta L| && \text{for } x_0\leqslant x\leqslant x_1.
\end{aligned} \tag{58}$$

and

$$\int_{-\infty}^{\infty} L(x)\,\mathrm{d}x=\int_{-\infty}^{\infty} D(x,0)\,\mathrm{d}x.$$

Thus

$$L(x,t)-D(x,t)=|\Delta L|\frac{e^{k_1t}}{\sqrt{4\pi\gamma t}}\left(\int_{-x_1}^{-x_0} Q\,\mathrm{d}x'-\int_{x_0}^{x_1} Q\,\mathrm{d}x'\right), \tag{59}$$

where for abbreviation

$$Q=e^{-(x'-x)^2/4\gamma t}.$$

$$L(x,t)-D(x,t)=\tfrac{1}{2}|\Delta L|\,e^{k_1t}\left[\Phi(u_1)-\Phi(u_0)-\Phi(u_3)+\Phi(u_2)\right], \tag{60}$$

where

$$u=\frac{x'-x}{\sqrt{4\gamma t}},\qquad u_0=\frac{-x_1-x}{\sqrt{4\gamma t}},\qquad u_1=\frac{-x_0-x}{\sqrt{4\gamma t}},\qquad u_2=\frac{x_0-x}{\sqrt{4\gamma t}},\qquad u_3=\frac{x_1-x}{\sqrt{4\gamma t}},$$

Φ is the error function

$$\Phi(\alpha)=\frac{2}{\sqrt{\pi}}\int_0^{\alpha} e^{-u^2}\,\mathrm{d}u.$$

For $t\to\infty$, $u\to 0$ and $\Phi(u)\simeq\frac{2}{\sqrt{\pi}}e^{-u^2}u\simeq\frac{2}{\sqrt{\pi}}(1-u^2)\,u$, and thus

$$L(x,t)-D(x,t)\xrightarrow{t\to\infty}\frac{|\Delta L|}{\sqrt{\pi}}e^{k_1t}\left(u_0^3-u_1^3+u_3^3-u_2^3\right)=-\frac{6\Delta L}{\sqrt{\pi}}\frac{e^{k_1t}}{(4\gamma t)^{3/2}}\,x\left(x_1^2-x_0^2\right). \tag{61}$$

This means that

$L-D$ will increase with time for $x<0$
$D-L$ will increase with time for $x>0$.

$L-D$ will increase with time for $x<0$, with subsequent destruction via nonlinear reaction of D and then with L growing alone. $D-L$ will increase with time for $x>0$, with subsequent destruction via nonlinear reaction of L and then with D growing alone. We obtained *interesting separation in space of L and D.* The reason that L increases on the negative axis is that at $t=0$ we took $L_0>D_0$ on the negative axis.

(i) Let an initial excess of L at $x=-a$ and exactly the same excess of D at $x=a$ be represented by

$$L(x,0)-D(x,0)=\varepsilon_0\delta(x+a)-\varepsilon_0\delta(x-a). \tag{62}$$

Then (see Figure 9)

$$L(x,t)-D(x,t)=\varepsilon_0\frac{e^{k_1t}}{\sqrt{4\pi\gamma t}}\left[e^{-(a+x)^2/4\gamma t}-e^{-(a-x)^2/4\gamma t}\right] \tag{63}$$

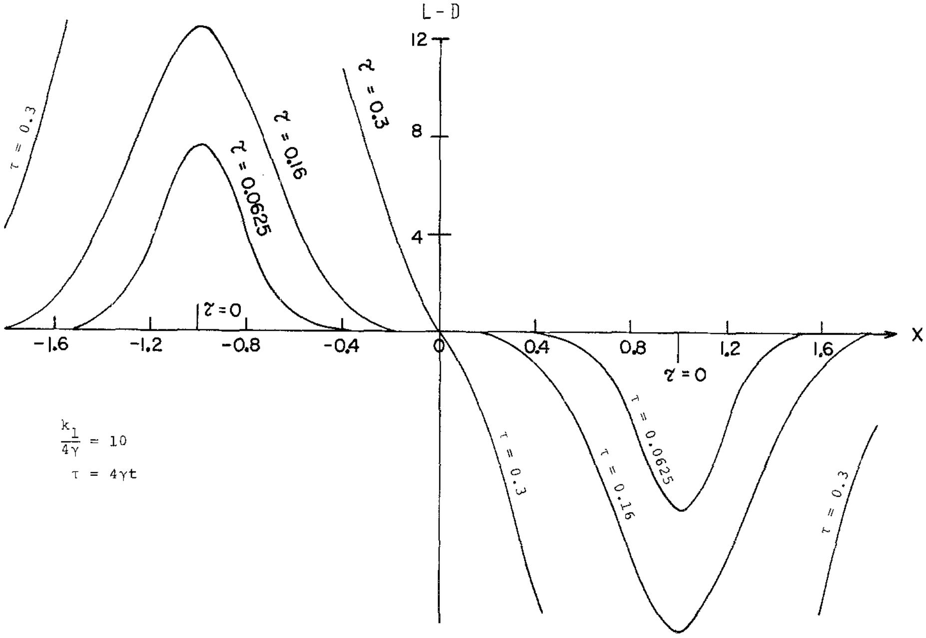


Fig. 9. The growth in time of *L-D* in space (as a function of *x*) for an initial excess of L at $x=-1$ and exactly the same excess of D at $x=+1$. $k_1/4\gamma=10$, $\tau=4\gamma t$.

For $x=\pm\infty$, $L-D=0$, at $x=0$ $L-D=0$ and for any x for $t>x/4\gamma$
for $x<0$

$$L(x,t)-D(x,t)\xrightarrow{t\to\infty}\frac{4a\varepsilon_0}{\sqrt{\pi}}\frac{e^{k_1t}}{(4\gamma t)^{3/2}}|x| \tag{64}$$

for $x>0$

$$D(x,t)-L(x,t)\xrightarrow{t\to\infty}\frac{4a\varepsilon_0}{\sqrt{\pi}}\frac{e^{k_1t}}{(4\gamma t)^{3/2}}x. \tag{65}$$

After nonlinear interaction L will survive for $x<0$ and D for $x>0$. Thus, the result of initial separation causes spatial separation of L and D populations.

(j) In a similar process, but with

$$L(x,0)-D(x,0)=\varepsilon_1\delta(x+a)-\varepsilon_2\delta(x-a), \tag{66}$$

and if $\varepsilon_1>\varepsilon_2$, i.e., there is an excess of $L-D$ at $x=-a$, then as $t\to\infty$

$$L(x,t)-D(x,t)\xrightarrow{t\to\infty}|\varepsilon_1-\varepsilon_2|\,\frac{e^{k_1t}}{\sqrt{4\pi\gamma t}}, \tag{67}$$

i.e., there will be growth increasing $L-D$ (and L subsequently through nonlinear interaction) for all x.

(k) If

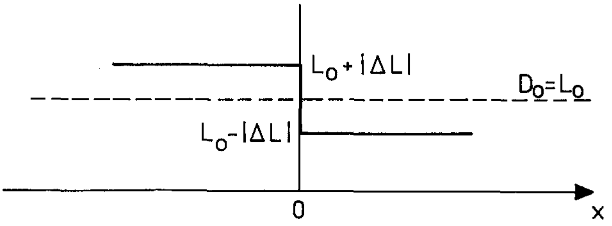
$$L(x,0)-D(x,0)=\varepsilon_1\delta(x-b)+\varepsilon_2\delta(x+a)-\varepsilon_2\delta(x-a), \tag{68}$$

then

$$L(x,t)-D(x,t)\xrightarrow{t\to\infty}\varepsilon_1\frac{e^{k_1t}}{\sqrt{4\pi\gamma t}}, \tag{69}$$

i.e., the excess of L over D at $x=-a$ will cancel the equal decrease at $x=a$, and only the initial perturbation at $x=b$ will increase $L-D$ and subsequently L for all x.

(l)



Let $t=0$;

$D=D_0=L_0$

$$\left.\begin{aligned}L(x,0)&=L_0+|\Delta L|\quad\text{for}\quad x<0\\ L(x,0)&=L_0-|\Delta L|\quad\text{for}\quad x>0,\end{aligned}\right. \tag{70}$$

then

$$L(x,t)-D(x,t)=\frac{e^{k_1t}}{\sqrt{4\pi\gamma t}}|\Delta L|\left[\int_{-\infty}^{0}e^{-(x'-x)^2/4\gamma t}\,\mathrm{d}x'-\int_{0}^{\infty}e^{-(x'-x)^2/4\gamma t}\,\mathrm{d}x'\right]=$$

$$=\frac{e^{k_1t}}{\sqrt{\pi}}|\Delta L|\left(\int_{-\infty}^{-x/\sqrt{4\gamma t}}e^{-u^2}\,\mathrm{d}u-\int_{-x/\sqrt{4\gamma t}}^{\infty}e^{-u^2}\,\mathrm{d}u\right),$$

where

$$u=\frac{x'-x}{\sqrt{4\gamma t}}$$

$$L(x,t)-D(x,t)=-2\,|\Delta L|\frac{e^{k_1t}}{\sqrt{\pi}}\int_{0}^{x/\sqrt{4\gamma t}}e^{-u^2}\,\mathrm{d}u=-|\Delta L|\,e^{k_1t}\Phi\left(\frac{x}{\sqrt{4\gamma t}}\right) \tag{71}$$

As $t\to\infty$

$$L(x,t)-D(x,t)=-|\Delta L|\,e^{k_1t}\frac{2}{\sqrt{\pi}}\,e^{-x^2/4\gamma t}\frac{x}{\sqrt{4\gamma t}}=2\,|\Delta L|\frac{e^{k_1t}}{\sqrt{4\pi\gamma t}}\,e^{-x^2/4\gamma t}(-x). \tag{72}$$

$$\left.\begin{aligned}
&\text{For } |x|<\sqrt{4\gamma t} \text{ and for } x<0:\\
&L(x,t)\simeq L(x,t)-D(x,t)\xrightarrow{t\to\infty} 2\,|\Delta L|\,\frac{e^{k_1 t}}{\sqrt{4\pi\gamma t}}\,|x|\\
&\text{and for } x>0:\\
&D(x,t)\simeq D(x,t)-L(x,t)\xrightarrow{t\to\infty} 2\,|\Delta L|\,\frac{e^{k_1 t}}{\sqrt{4\pi\gamma t}}\,x.
\end{aligned}\right\} \tag{73}$$

The numerical example is plotted in Figure 10. We see that for $x<0$ (where $L_0>D_0$) $L-D$ increases with time, and for $x>0$ (where $D_0>L_0$) $D-L$ increases with time. As time increases the near straight line behavior for small x has increasing slope, approaching in the limit at $x=0$ the same step function shape as at $\tau=0$.

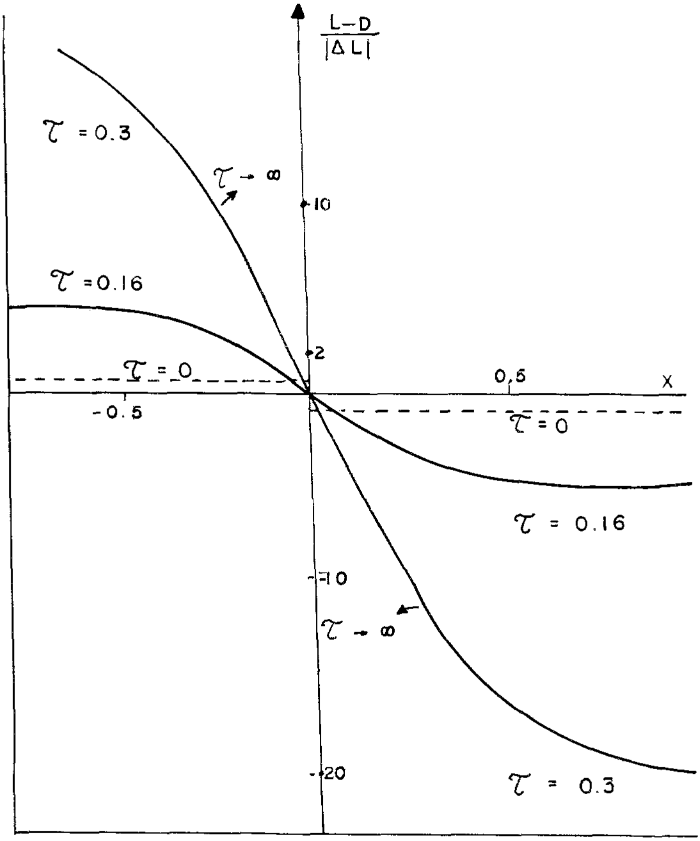


Fig. 10. The growth in time of L-D in space for an initial excess of L for $x<0$ and initial equal decrease of D for $x>0$.

3.2.2. *Finite Slab Solution*

If $I(x,t)=L(x,t)-D(x,t)$ vanishes at the boundary of a slab at $x=0$ and $x=a$, i.e.,

$$I(0,t)=I(a,t)=0,$$

and I is specified initially at $t=0$ everywhere within the slab, i.e.,

$$I(x, 0)=L(x, 0)-D(x, 0) \quad \text{for} \quad 0<x<a,$$

then the exact solution is given by

$$I(x, t)=\frac{2}{a} e^{k_1 t} \sum_{n=0}^{\infty} A_n \exp\left(-\frac{n^2\pi^2\gamma}{a^2} t\right) \sin\left(\frac{n\pi}{a} x\right),$$

where

$$A_n=\int_0^a I(u, 0) \sin\left(\frac{n\pi}{a} u\right) \mathrm{d}u. \tag{74}$$

We see that I will grow in time if

$$k_1>\frac{\pi^2\gamma}{a^2}$$

and that only lower harmonics $n<(a/\pi)\sqrt{k_1/\gamma}$ will contribute to it. It is interesting to note that this is a similar solution to the infinite region solution with sinusoidal perturbation and wavelength λ (see Equation (57) and discussion afterwards), where $\lambda=a/\pi$ in the finite slab solution.

3.2.3. *Stability Analysis in the Neighborhood of Time Independent Homogeneous Solution*

If $\partial L/\partial t=0$, $\partial D/\partial t=0$ and for both $L=L_0=\text{constant}$ and $D=D_0=\text{constant}$

$$k_1 L_0-k_2' L_0 D_0=0$$

$$k_1 D_0-k_2' L_0 D_0=0$$

with the stationary solution $L_0=D_0=k_1/k_2'$ (see limiting value solution in Equation (23)).

Following the standard perturbation technique in the analysis of nonlinear equations (e.g., see Prigogine, 1967), let us assume a small perturbation in space with wavelength λ and frequency ω, around the L_0, D_0 values, i.e., let

$$\left.\begin{aligned} L&=L_0+\varepsilon_L\, e^{\omega t+ix/\lambda}, \quad \text{with} \quad |\varepsilon_L|\ll L_0 \\ D&=D_0+\varepsilon_D\, e^{\omega t+ix/\lambda}, \quad \text{with} \quad |\varepsilon_D|\ll D_0. \end{aligned}\right\} \tag{75a}$$

Substituting these in Equations (40a) and (41a) we obtain, after neglecting the term containing $\varepsilon_L\varepsilon_D$

$$\left.\begin{aligned} &\varepsilon_L\left(\omega+\frac{\gamma}{\lambda^2}\right)+\varepsilon_D(k_2' L_0)=0 \\ &\varepsilon_L(k_2' L_0)+\varepsilon_D\left(\omega+\frac{\gamma}{\lambda^2}\right)=0. \end{aligned}\right\} \tag{75b}$$

These equations are satisfied if the determinant vanishes, i.e.,

$$\begin{vmatrix} \omega+\dfrac{\gamma}{\lambda^2} & k_2L_0 \\ k_2L_0 & \omega+\dfrac{\gamma}{\lambda^2} \end{vmatrix} = 0$$

and we obtain the so-called dispersion equation for ω in our case

$$\left(\omega+\frac{\gamma}{\lambda^2}\right)^2 = k_2^2L_0^2 = k_1^2$$

which has two roots

$$\left.\begin{aligned} \omega_1 &= -k_1 - \frac{\gamma}{\lambda^2} \\ \omega_2 &= +k_1 - \frac{\gamma}{\lambda^2}. \end{aligned}\right\} \quad (75c)$$

ω_1 is always negative, ω_2 can be negative, zero or positive, depending on the value of λ. The instability will only be present if L and D will grow away from L_0, D_0, i.e., only if $\omega > 0$ (see Equation (75a)); this corresponds only to

$$\omega = \omega_2 = k_1 - \frac{\gamma}{\lambda^2} > 0,$$

i.e.,

$$\lambda > \sqrt{\frac{\gamma}{k_1}}, \quad \text{or} \quad k_1 > \frac{\gamma}{\lambda^2}. \quad (75d)$$

Thus, only for wavelengths greater than $\sqrt{\gamma/k_1}$ will there be instability and growth of L and D away from the homogeneous, stationary solution $L_0 = D_0 = k_1/k_2$. It is interesting that the same result was obtained (without small perturbations) and exactly for the growth of $L-D$ (see Equation (51)) for an initial (also sinusoidal) variation in $L-D$.

If $k_1 \neq k_1^*$ (see Equations (27) and (28)), then the analysis can be made in exactly the same way, with the conclusion that the instability from stationary solution will only be present if

$$k_1\sqrt{k_1^*/k_1} > \gamma/\lambda^2. \quad (75e)$$

3.3. APPROXIMATE SOLUTIONS FOR L AND D

3.3.1. *Short Time Solution*

From Equations (40) and (41) we obtain, upon dividing Equation (40) by L, Equation (41) by D and subtracting the second from the first,

$$\frac{\partial}{\partial t}\ln(L/D) = k_2 I + \gamma\left(\frac{1}{L}\frac{\partial^2 L}{\partial x^2} - \frac{1}{D}\frac{\partial^2 D}{\partial x^2}\right), \quad (76)$$

where

$$I(x,t)=L(x,t)-D(x,t)$$

is given in Equations (43)–(46). Now, if at the beginning, when $L\simeq D$ for all x, $L=L_0\simeq D=D_0$, then

$$\frac{\partial}{\partial t}\ln(L/D)\simeq k_2 I+\frac{\gamma}{L_0}\frac{\partial^2 I}{\partial x^2}\equiv f(x,t),$$

where $f(x,t)$ is a known function. Then for *short times*

$$\ln(L/D)\simeq\int\limits_0^t f(x,t')\,\mathrm{d}t'+c(x) \tag{76a}$$

$$c(x)=\ln\frac{L(x,0)}{D(x,0)}$$

so that

$$\frac{L(x,t)}{D(x,t)}\simeq\frac{L(x,0)}{D(x,0)}\exp\left[\int\limits_0^t f(x,t')\,\mathrm{d}t'\right] \tag{76b}$$

and D and L can be obtained from the identity

$$D=\frac{I}{\dfrac{L}{D}-1},\qquad L=D\cdot\frac{L}{D}. \tag{76c}$$

For example, if $I(x,0)=(L_0-D_0)\,e^{-bx^2}$, then from Equation (44) at *short times*,

$$\frac{L(x,t)}{D(x,t)}\simeq\frac{L(x,0)}{D(x,0)}\exp\left\{(L_0-D_0)\,e^{-bx^2}\left[K-\frac{2b\gamma}{k_1L_0}+\frac{4b^2\gamma}{L_0k_1}\,x^2\right][e^{k_1t}-1]\right\} \tag{77}$$

which reduces exactly [sic!] to Equation (12) in the absence of diffusion ($\gamma=0$) and with $b=0$ (no Gaussian fluctuation).

3.3.2. *Asymptotic Approximation for Long Times*

Let us assume that starting with an excess of L

$$L(x,t)\simeq I(x,t)+\eta(t),\qquad \eta<I \tag{78}$$

$$D(x,t)\simeq\eta(t), \tag{79}$$

where $I(x,t)=L(x,t)-D(x,t)$ was obtained in Equation (43) and is an increasing function with time. The correction η we assume to be only a function of time, which is equivalent to an assumption that there is no time for diffusion of D, which is quickly disappearing due to destructive reaction. From Equation (42a)

$$\frac{\partial I}{\partial t}=k_1 I+\gamma\frac{\partial^2 I}{\partial x^2} \tag{80}$$

and using Equation (41a) we obtain

$$\frac{\mathrm{d}\eta}{\mathrm{d}\tau}+(I-\alpha)\,\eta+\eta^2=0, \tag{81}$$

with $\tau=k_2t$, $\alpha=k_1/k_2'$. This is the Bernoulli nonlinear differential equation. Let $z=1/\eta$. Then we obtain the linear equation

$$\frac{\mathrm{d}z}{\mathrm{d}\tau}+(\alpha-I)\,z=1, \tag{82}$$

which has a known exact solution given by

$$z=e^{-F}\left(a+\int\limits_0^\tau e^F\,\mathrm{d}\tau\right),\qquad a=1/D_0, \tag{83}$$

where

$$F=\int_0^\tau (\alpha-I)\,\mathrm{d}\tau'=\alpha t-\int_0^\tau I(x,\tau')\,\mathrm{d}\tau'$$

$$\eta(\tau)=\frac{e^F}{\dfrac{1}{D_0}+\displaystyle\int_0^\tau e^F\,\mathrm{d}\tau'}=D_0\frac{\exp\left[\alpha\tau-\displaystyle\int_0^\tau I(x,\tau')\,\mathrm{d}\tau'\right]}{1+D_0\displaystyle\int_0^\tau \exp\left[\alpha\tau''-\displaystyle\int_0^{\tau''} I(x,\tau')\,\mathrm{d}\tau'\right]\mathrm{d}\tau''}. \tag{84}$$

In agreement with Bernoulli's equation (Equation (81)). Neglecting η^2, as $t\to\infty$ and $D\to 0$, then $L\to I$.
If we are only interested in time development, we can use (see Equation (53)) the following approximations:

$$\frac{\partial^2 L}{\partial x^2}\simeq-\frac{L}{l_1^2};\qquad \frac{\partial^2 D}{\partial x^2}\simeq-\frac{D}{l_2^2}, \tag{85}$$

where l_1 and l_2 are some unspecified characteristic diffusion lengths. Then,

$$\frac{\partial L}{\partial\tau}-gL+L^2\simeq 0 \tag{86}$$

and

$$\frac{\partial D}{\partial\tau}-hD+D^2\simeq 0, \tag{87}$$

where $\tau=k_2't$, $\alpha=k_1/k_2'$ and where

$$\left.\begin{aligned} g&=\alpha+I-\frac{\gamma}{k_2' l_1^2}\\ \text{and}\qquad & \\ h&=\alpha-I-\frac{\gamma}{k_2' l_2^2}.\end{aligned}\right\} \tag{88}$$

These again are Bernoulli's differential equations, and from the previous case, we can write immediately the solutions

$$L(x,\tau)\simeq\frac{L_0\,e^{F_L}}{1+L_0\displaystyle\int_0^\tau e^{F_L}\,\mathrm{d}\tau'} \tag{89}$$

$$D(x,\tau)\simeq\frac{D_0\,e^{F_D}}{1+D_0\displaystyle\int_0^\tau e^{F_D}\,\mathrm{d}\tau'}, \tag{90}$$

where

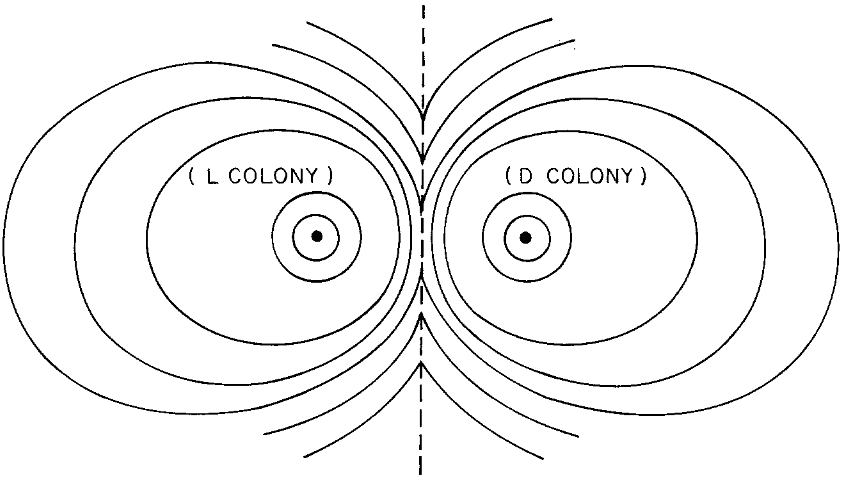
$$F_L=\left(\alpha-\frac{\gamma}{k_2' l_1^2}\right)\tau+\int_0^\tau I(x,\tau')\,\mathrm{d}\tau' \tag{91}$$

$$F_D=\left(\alpha-\frac{\gamma}{k_2' l_2^2}\right)\tau-\int_0^\tau I(x,\tau')\,\mathrm{d}\tau'. \tag{92}$$

3.3.3. *Visualization of Two Dimensional Diffusion with Reproduction and Destruction*

The case of growth, destruction and diffusion can be best visualized by the following thought experiment. Feed by a pipet at an exponentially increasing rate drops of an acid at one point of a porous inert surface. At some other point on the inert surface we feed tiny drops of base (equal strength) with the same exponentially increasing rate. Let us assume acid to be of one color, base of another, and the interacting two drops of base and acid a third color (and neutral).

Now if we start with an exact amount of acid and base growing with the same rate, these will first form colored rings around feeding points, then touch the interacting boundary creating color No. 3. Thus we expect:



After a long time we expect the boundary becoming sharper and sharper, with the acid spreading to the one side, and the base to the other.

If we had a slightest excess of acid to the left, after a long time there will be acidic environment everywhere emanating in rings from the acid feeding point. Again wave patterns may be created.

The above illustration can help to visualize the growth of L and D organims (or polymers) and their diffusion, including 'wars' between many colonies (many acid and base feeding points).

3.3.4. *Conclusions*

Generalized Frank's model with diffusion, i.e. with reproduction and annihilation reactions was solved for few cases. Some of the results obtained are listed below.

If at one time $L=D$ everywhere, then at all times $L=D$ everywhere (see Equation (42)).

If everywhere $L=D$, except if in some region fluctuation occurred, so that initially there was an increase of Ls in form of a Gaussian distribution over the distance, then $L-D$ will grow in time with a rate slower than exponential and with the width of the distribution spreading linearly in time (see Equation (45), case 3.2.1 (a)).

If everywhere the fluctuation in $L-D$ are sinusoidal with a wavelength λ, then $|L-D|$ will grow with exponential envelope, with $L \geqslant D$ and $L \leqslant D$ in between, and as long as $\lambda > \sqrt{\gamma k_1}$ (see Equation (51), case 3.2.1 (d)), where γ is diffusion coefficient and k_1 the rate of growth. Similar results were obtained for a finite slab (case 3.2.2) and from a stability analysis (case 3.2.3).

If at a one place $L > D$, and at other point $D > L$ and at all other locations $L = D$, then $L-D$ will increase with time in such a way, that $L-D$ will be increasing on one side and $D-L$ on the other side, with L and D separating in space (see case 3.2.1 (h, i, j)).

If there is an excess of L in one place and exactly same excess of D in other place, then after a long time the local fluctuations will disappear and L and D will grow in time while $L = D$ everywhere (see case 3.2.1 (j, k)).

If L was in small excess everywhere to the left, D in same excess to the right, then L and D will grow, with the L winning to the left, D to the right and with sharp boundary in between (case 3.2.1 (1)).

In Equation (77) we have shown that for short times the results are similar to solutions without diffusion and that diffusion contributes to effective skewness of the spread of fluctuation.

In Section 3.3.3 we have extrapolated the results from one dimensional analysis to a two dimensional sketch and illustrated the results with a simple analogy between spatially interacting acid and base.

4. Generalization to Various *n*-Order Nonlinear Symmetric Rate Processes with Reproduction, Destruction and Diffusion

We can also assume that in a more complicated way, two protocells (or polymers) could have led to multiplication (e.g., in their catalytic presence)

$$L+L \xrightarrow{k_2} L+L+L$$

$$D+D \xrightarrow{k_2} D+D+D$$

We may even generalize that n cells could have lead to the formation of an additional cell, i.e.,

$$\underbrace{L+L+\cdots L}_{n} \xrightarrow{k_n} \underbrace{L+L+\cdots L}_{n+1}$$

and

$$\underbrace{D+D+\cdots D}_{n} \xrightarrow{k_n} \underbrace{D+D+\cdots D}_{n+1}.$$

We may also include annihilation by the presence of one different isomer ('end effect'), i.e.,

$$L+\underline{D}\xrightarrow{k_2'}X \qquad \text{(considered earlier in Equation (9))}$$

$$\begin{cases} L+L+\underline{D}\xrightarrow{k_3'}X \\ D+D+\underline{L}\xrightarrow{k_3'}X \end{cases}$$

$$\vdots$$

$$\begin{cases} \underbrace{L+L+L\cdots L}_{n-1}+\underline{D}\xrightarrow{k_n'}X \\ \underbrace{D+D+D\cdots D}_{n-1}+\underline{L}\xrightarrow{k_n'}X, \end{cases}$$

where X symbolically defines any 'dead' product. And for simplicity, we consider (although it is easy to extend) that no more than one different type of assimilation leads to the death. We may also include that under the influence of L and D cells, either an L or D cell is formed, i.e.,

$$L+D\xrightarrow{r_2}L+L+D$$

$$L+D\xrightarrow{r_2}D+D+L, \quad \text{etc.}$$

Finally we will include the class of destruction due to natural death (k_1' will be included in k_1)

$$L\xrightarrow{k_1'}X$$

$$D\xrightarrow{k_1'}X$$

and due to crowding with members of own species (as used in Frank's (1953) modified model).

$$L+L\xrightarrow{k''}X$$

$$D+D\xrightarrow{k''}X.$$

For all the processes outlined the overall rates are given by

$$\begin{aligned} \frac{\partial L}{\partial t} &= k_1L+k_2L^2+\cdots k_{n_1}L^{n_1}-k_2'LD-2k_3'L^2D- \\ &\quad -k_3'D^2L-\cdots-(n-1)\,k_n'L^{n-1}D-k_n'D^{n-1}L+r_2LD-2k''L^2+\nabla\cdot(\gamma\nabla L) \end{aligned} \tag{93}$$

$$\begin{aligned} \frac{\partial D}{\partial t} &= k_1D+k_2D^2+\cdots k_{n_1}D^{n_1}-k_2'DL-k_3'L^2D- \\ &\quad -2k_3'D^2L-\cdots-k_n'L^{n-1}D-(n-1)\,k_n'D^{n-1}L+r_2LD- \\ &\quad -2k''D^2+\nabla\cdot(\gamma\nabla D), \end{aligned} \tag{94}$$

where we added the general three-dimensional diffusion term, with $\gamma=\gamma(x, y, z)$. By subtracting Equation (94) from Equation (93) we obtain the n-order nonlinear diffusion equation

$$\frac{\partial}{\partial t}(L-D)=(L-D)\,(G_{+}-G_{-})+\nabla\cdot\gamma\nabla(L-D), \tag{95}$$

where $G_{+}(L, D)$ and $G_{-}(L, D)$ are positive quantities, namely

$$\begin{aligned} G_{+}=&k_1+k_2(L+D)+\cdots k_{n_1}(L^{n_1-1}+L^{n_1-2}D+\cdots+D^{n_1-2})+\\ &+k'_3LD+k'_nLD(L^{n-3}+\cdots+D^{n-3}| \end{aligned} \tag{96}$$

$$G_{-}=2k'_3LD+\cdots(n-1)\,k'_nLD(D^{n-3}+\cdots+L^{n-3})+2k''(L+D) \tag{97}$$

Thus, *if at any time in all space* $L=D$, *then at all time at all space* $L=D$. Furthermore, we see that if at any time $L>D$ at any place, G_{+} will contribute to the increase of L over D, and G_{-} will try to keep the balance the other way. In any event, we see that the stable solutions $L=D$ are very improbable and that the L or D will grow in time depending on the sign of the term $G_{+}-G_{-}$. For quadratic nonlinearities in original equations $G_{+}-G_{-}=k_1+(k_2-k'')\,(L+D)$.

In the absence of diffusion, if $L\neq D$ we see that there may be numerous possible positive values (roots) of L and D for which $\partial L/\partial t=\partial D/\partial t=0$, i.e., there may be numerous possible steady states, corresponding to $G_{+}=G_{-}$. In the absence of diffusion, if we divide the first equation by L, the second by D and subtract, we obtain

$$\frac{\mathrm{d}\ln L/D}{\mathrm{d}t}=(L-D)\,(\Gamma_{+}-\Gamma_{-}), \tag{98}$$

where

$$\begin{aligned} \Gamma_{+}=&k_2+\cdots k_{n_1}(L^{n_1-2}+\cdots+D^{n_1-2})+k'_2+k'_3(L+D)+\cdots+\\ &+k'_n(L^{n-2}+\cdots D^{n-2}) \end{aligned} \tag{99}$$

and

$$\Gamma_{-}=+\cdots(n-1)\,k'_nLD(L^{n-4}+\cdots+D^{n-4})+r_2+2k'', \tag{100}$$

and if $L\neq D$, L/D will increase if $\Gamma_{+}>\Gamma_{-}$. For quadratic nonlinearities in original equations $\Gamma_{+}-\Gamma_{-}=(k_2-r_2)+(k_2-2k'')$.

5. Mathematical Analysis of Stereoscopic Autocatalysis

Based on the experimental results of Havinga (1954), who claimed spontaneous formation of optically active substances, Calvin (1969) felt that the notion of stereoscopic autocatalysis could have played an important role for the origin of chirality in organic molecules. According to this process, equal mixtures of left-handed (A_L) and right-handed material (A_D) due to small fluctuation would, by chance, be transformed to another material B_L or B_D. We will describe this model mathematically and show how fluctuations are important, and demonstrate when the model can account for significant separation of one enantiomorph. Furthermore, if one assumes that one type of B material (e.g. right-handed) could have been formed in one place, then one must admit the other material (e.g., left-handed) at another place. In the absence of mutual antagonism, this model by itself cannot explain the origin on the entire Earth of one type of material. Bonner (1972) also points out that Havinga's experiments have never been repeated, nor extended to other experimental systems. It is still of interest to describe the model mathematically, to see under what conditions could have been important. We show that under certain special conditions it is indeed possible at one location to have growth of only B_L (or B_D) and also that under more common conditions both B_L and B_D will grow simultaneously.

Consider substance A in two chiral forms A_L and A_D (Calvin, 1969), with

$$A_L \xrightarrow{k_1} A_D, \qquad A_D \xrightarrow{k_1} A_L$$

and almost equal (racemic) mixture of A_L and A_D and very fast rate constant k_1 for conversion from left- to right-handed and right- to left-handed forms of A. Further, let substance A be converted to right- and left-handed substance B by two processes:

Slow process:

$$A_L \xrightarrow{k_2} B_L, \qquad A_D \xrightarrow{k_2} B_D,$$

and

Fast Catalytic Process:

$$A_L + B_L \xrightarrow{k_c} B_L + B_L$$

$$A_D + B_D \xrightarrow{k_c} B_D + B_D,$$

where we assumed that the catalytic process can be represented by this reaction. In this process, B_L is catalytic for the conversion of A_L to B_L, and B_D is catalytic for the conversion of A_D to B_D. The assumption here is that this process is many times faster than the previous one and that rate constants are the same under identical external conditions due to mirror symmetry (see discussion after Equation (2)).

The rate equations for this process are

$$\frac{dB_L}{dt} = k_2 A_L + k_c A_L B_L, \tag{101}$$

$$\frac{dB_D}{dt} = k_2 A_D + k_c A_D B_D, \tag{102}$$

$$\frac{dA_L}{dt} = k_1 (A_D - A_L) - k_2 A_L - k_c A_L B_L, \tag{103}$$

$$\frac{dA_D}{dt} = k_1 (A_L - A_D) - k_2 A_D - k_c A_D B_D. \tag{104}$$

Note that these equations are nonlinear, because of the appearance of $A_L B_L$ and $A_D B_D$ products. Because of the absence of mutual antagonism, dB_L/dt and dB_D/dt are always positive and B_L and/or B_D can only grow in time. Note also that (by adding equations and integrating)

$$A_D+A_L+B_D+B_L=A_L^0+A_D^0+B_D^0+B_L^0, \tag{105}$$

where the superscript indicates initial concentrations.

If the catalytic rate is much greater than the non-catalytic rate of conversion of A to B, i.e., if

$$k_2A_L \ll k_cA_LB_L \quad \text{and} \quad k_2A_D \ll k_cA_DB_D,$$

then

$$\frac{\mathrm{d}B_L}{\mathrm{d}t}=k_cA_LB_L, \tag{106}$$

$$\frac{\mathrm{d}B_D}{\mathrm{d}t}=k_cA_DB_D, \tag{107}$$

$$\frac{\mathrm{d}A_L}{\mathrm{d}t}=k_1(A_D-A_L)-k_cA_LB_L, \tag{108}$$

$$\frac{\mathrm{d}A_D}{\mathrm{d}t}=k_1(A_L-A_D)-k_cA_DB_D. \tag{109}$$

We will consider two different subcases:

(1) Assuming that the *rate of racemization* of $A_L \overset{k_1}{\rightleftarrows} A_D$ *is smaller than the rate of catalytic formation of B_L and B_D*, we get

$$\frac{\mathrm{d}A_L}{\mathrm{d}t}=-k_cA_LB_L, \tag{110}$$

$$\frac{\mathrm{d}A_D}{\mathrm{d}t}=-k_cA_DB_D. \tag{111}$$

Comparing Equation (110) with (106) and Equation (111) with (107), we have

$$B_L=A_L^0-A_L+B_L^0, \tag{112}$$

$$B_D=A_D^0-A_D+B_D^0, \tag{113}$$

where at $t=0$ $B_L=B_L^0$, $B_D=B_D^0$, $A_D=A_D^0$ and $A_L=A_L^0$. Note that if at $t=0$, $B_L^0=0$, $B_L=0$ at all times. Similarly, if $B_D^0=0$, $B_D=0$ at all times.

Substituting Equation (112) into (106) and Equation (113) into (107), we obtain

$$\frac{\mathrm{d}B_L}{\mathrm{d}t}=k_c(A_L^0+B_L^0-B_L)\,B_L, \tag{114}$$

$$\frac{\mathrm{d}B_D}{\mathrm{d}t}=k_c(A_D^0+B_D^0-B_D)\,B_D. \tag{115}$$

Integrating directly from 0 to t we get, after some manipulation,

$$B_L = B_L^0 \cdot \frac{A_L^0 + B_L^0}{B_L^0 + A_L^0 e^{-\tau_1}}, \qquad \tau_1 = k_c (A_L^0 + B_L^0)\, t, \tag{116}$$

$$B_D = B_D^0 \cdot \frac{A_D^0 + B_D^0}{B_D^0 + A_D^0 e^{-\tau_2}}, \qquad \tau_2 = k_c (A_D^0 + B_L^0)\, t \tag{117}$$

and

$$A_L = A_L^0 + B_L^0 - B_L; \quad \text{and} \quad A_D = A_D^0 + B_D^0 - B_D. \tag{118}$$

For $t \to \infty B_L \to A_L^0 + B_L^0$ and $B_D \to A_D^0 + B_D^0$.

If $B_L^0 \neq 0$ and $B_D^0 = 0$ there will be growth of only B_L, leaving A_D untouched and as $t \to \infty B_L \to B_L^0 + A_L^0$, $B_D \to 0$. Subsequent fluctuation causing $B_D \neq 0$ will make B_D grow from A_D. There are no instabilities, only normal growth of B_L and B_D. Incidentally, if $B_L^0 = B_D^0 = 0$, then $B_L = B_D = 0$ at all times.

(2) In this subcase, we assume that $A_L = A_D$ for all times, which is equivalent to *the rate of racemization of* $A_L \overset{k_1}{\rightleftarrows} A_D$ *is infinite.* Adding Equations (103) and (104), with $A = A_L + A_D$, $A_L^0 = A_D^0 = \frac{1}{2} A_0$, $A_0 = A_L^0 + A_D^0$, we obtain the following equations:

$$\frac{\mathrm{d}B_L}{\mathrm{d}t} = \tfrac{1}{2}(k_2 + k_c B_L)\, A \tag{119}$$

$$\frac{\mathrm{d}B_D}{\mathrm{d}t} = \tfrac{1}{2}(k_2 + k_c B_D)\, A \tag{120}$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\left[k_2 + \tfrac{1}{2} k_c (B_L + B_D)\right] A \tag{121}$$

Adding and subtracting Equation (119) and Equation (120) we obtain

$$\frac{\mathrm{d}(B_L + B_D)}{\mathrm{d}t} = \left[k_2 + \tfrac{1}{2} k_c (B_L + B_D)\right] A \tag{122}$$

$$\frac{\mathrm{d}(B_L - B_D)}{\mathrm{d}t} = \tfrac{1}{2} k_c (B_L - B_D)\, A. \tag{123}$$

Adding Equations (119), (120) and (121) we see that

$$B_L + B_D + A = B_L^0 + B_D^0 + A_0 = C_1 \tag{124}$$

and Equations (122) and (123) can be rewritten as

$$\frac{\mathrm{d}(B_L + B_D)}{\mathrm{d}t} = \left[k_2 + \tfrac{1}{2} k_c (B_L + B_D)\right] \left[C_1 - (B_L + B_D)\right] \tag{125}$$

$$\frac{\mathrm{d}(B_L - B_D)}{\mathrm{d}t} = \tfrac{1}{2} k_c (B_L - B_D) \left[C_1 - (B_L + B_D)\right]. \tag{126}$$

Let

$$u = B_L + B_D, \qquad u_0 = B_L^0 + B_D^0,$$

then integrating Equation (A.25) from 0 to t

$$t(k_2+\tfrac{1}{2}k_c C_1)=\ln\left[\frac{a+u}{C_1-u}\cdot\frac{C_1-u_0}{a+u_0}\right], \tag{127}$$

where $a=2k_2/k_c$, and inverting Equation (127) yields

$$u=B_L+B_D=\frac{C_1 p_0\, e^{\tau}-a}{1+p_0\, e^{\tau}}, \tag{128}$$

where

$$p_0=\frac{a+u_0}{c_1-u_0}=\frac{a+u_0}{A_0}$$

and

$$\tau=(k_2+\tfrac{1}{2}k_c C_1)\, t.$$

Eliminating time from Equations (122) and (123) and with $u=B_L+B_D$, $v=B_L-B_D$, then if $B_L^0\neq B_D^0$ we have

$$\frac{\mathrm{d}u}{\mathrm{d}v}=\frac{k_2+\frac{1}{2}k_c u}{\frac{1}{2}k_c v}=\frac{a+u}{v}. \tag{129}$$

Integrating, we obtain

$$a+u=C_2 v$$

$$a+B_L+B_D=C_2(B_L-B_D), \tag{130}$$

with

$$C_2=\frac{a+B_L^0+B_D^0}{B_L^0-B_D^0} \tag{131}$$

Using Equation (128) in Equation (103) yields

$$B_L-B_D=b\frac{p_0\, e^{\tau}}{1+p_0\, e^{\tau}},$$

where

$$b=(B_L^0-B_D^0)\left(1+\frac{A_0}{a+B_L^0+B_D^0}\right)$$

From Equation (128) to Equation (131)

$$B_L=\frac{(C_1+b)\, p_0\, e^{\tau}-a}{2(1+p_0\, e^{\tau})} \tag{132}$$

$$B_D=\frac{(C_1-b)\, p_0\, e^{\tau}-a}{2(1+p_0\, e^{\tau})}. \tag{133}$$

[If $B_L^0=B_D^0=0$, then from Equation (126) $B_L=B_D$ and from Equation (128),

$$B_L=B_D=\frac{C_1 p_0\, e^{\tau}-a}{2(1+p_0\, e^{\tau})}=\frac{a(e^{\tau}-1)}{2\left(1+\frac{a}{A_0}\, e^{\tau}\right)}, \tag{134}$$

since $C_1 = A_0$, $p_0 = a/A_0$]. There are no instabilities, no mechanism of extreme separation of B_L from B_D if $B_L^0 \neq 0$ and $B_D^0 \neq 0$. For $t \to \infty$:

$$B_L \to \frac{C_1 + b}{2} = \tfrac{1}{2}(B_L^0 + B_D^0 + A_0) + \tfrac{1}{2}(B_L^0 - B_D^0)\left(1 + \frac{A_0}{a + B_L^0 + B_D^0}\right) \tag{135}$$

$$B_D \to \frac{C_1 - b}{2} = \tfrac{1}{2}(B_L^0 + B_D^0 + A_0) - \tfrac{1}{2}(B_L^0 - B_D^0)\left(1 + \frac{A_0}{a + B_L^0 + B_D^0}\right) \tag{136}$$

$$\begin{aligned} &B_L + B_D \to B_L^0 + B_D^0 + A_0 \\ &B_L - B_D \to b(B_L^0 - B_D^0)\left(1 + \frac{A_0}{a + B_L^0 + B_D^0}\right) \\ &A \to 0. \end{aligned} \tag{137}$$

If $B_L^0 = \varepsilon \ll 1$, $B_D^0 = 0$, then as $t \to \infty$

$$B_L \to \tfrac{1}{2}A_0 + \tfrac{1}{2}\varepsilon \frac{A_0}{a + \varepsilon} \tag{138}$$

$$B_D \to \tfrac{1}{2}A_0 - \tfrac{1}{2}\varepsilon \frac{A_0}{a + \varepsilon}. \tag{139}$$

If $B_L^9 = B_D^0 = 0$, then as $t \to \infty$

$$B_L \to \tfrac{1}{2}A_0, \qquad B_D \to \tfrac{1}{2}A_0.$$

In both cases above there is no separation of B_L and B_D.

One can achieve separation of B_L (or B_D) *if a single large initial disproportion* was such that

$$B_L^0 \gg a = 2k_2/k_c \quad \text{and} \quad B_D^0 \ll B_L^0.$$

Then indeed, as $t \to \infty$ (i.e. for $\tau \gg 1$)

$$\left.\begin{aligned} &B_L \to B_L^0 + \tfrac{1}{2}A_0 + \tfrac{1}{2}\frac{A_0 B_L^0}{a + B_L^0} \simeq A_0 + B_L^0 - \frac{A_0}{2}\cdot\frac{a}{B_L^0} \simeq 2A_L^0 + B_L^0 \\ &B_D \to \tfrac{1}{2}A_0 - \tfrac{1}{2}\frac{A_0 B_L^0}{a + B_L^0} \simeq A_D^0 \frac{a}{B_L^0} \ll 1. \end{aligned}\right\} \tag{141}$$

Thus, consistently with Calvin (1969) the requirements for stereoscopic autocatalysis are (see, inequalities before Equation A.6):

(1a) $k_c B_L \gg k_2$ or (1b) $k_c B_D \gg k_2$, and
(2) very fast rate of racemization.

The added requirement here that the initial B_L^0 satisfies

$$B_L^0 \gg 2k_2/k_c \quad \text{and} \quad B_L^0 \gg B_D^0,$$

for B_L to grow, is consistent (within factor of two) with the requirement (1a) and for B_D to grow

$$B_D^0 \gg 2k_2/k_c \quad \text{and} \quad B_D^0 \gg B_L^0,$$

is consistent with the requirement (1b).

The importance of only initial value of B_L^0 (or B_D^0) for a growth of B_L only is that the conversion of A to B_L be essentially completed before another fluctuation occurs.

Series Expansion for General Solution for Short Times

Let $\mathbf{t}=k_c t$, $K=k_2/k_c$, $\alpha=k_1/k_c$ and let d/dt and d^2/dt^2 be denoted by primes, then Equation (101)–(104) become (no approximations)

$$B_L'=(K+B_L)\,A_L$$

$$B_D'=(K+B_D)\,A_D$$

$$A_L'=\alpha(A_D-A_L)-(K+B_L)\,A_L$$

$$A_D'=\alpha(A_L-A_D)-(K+B_D)\,A_D.$$

Differentiating again yields

$$B_L''=(K+B_L)\,A_L'+B_L'A_L$$

$$B_D''=(K+B_D)\,A_D'+B_D'A_D, \quad \text{etc.}$$

Taylor series expansion gives for B_L and B_D at $\mathbf{t}=k_c\Delta t$:

$$B_L(\Delta\mathbf{t})=B_L^0+B_L'(0)\cdot\Delta\mathbf{t}+\tfrac{1}{2}B_L''(0)\cdot(\Delta\mathbf{t})^2+\cdots$$

$$B_D(\Delta\mathbf{t})=B_D^0+B_D'(0)\cdot\Delta\mathbf{t}+\tfrac{1}{2}B_D''(0)\cdot(\Delta\mathbf{t})^2+\cdots$$

and

$$\begin{aligned}B_L(\Delta\mathbf{t})=B_L^0+[(K+B_L^0)\,A_L^0]\,\Delta\mathbf{t}+\tfrac{1}{2}\{(K+B_L^0)\,[\alpha(A_D^0-A_L^0)-\\ -(K+B_L^0)\,A_L^0]+(A_L^0)^2\,[(K+B_L^0]\}\,(\Delta\mathbf{t})^2+\ldots\end{aligned} \tag{142}$$

$$\begin{aligned}B_D(\Delta\mathbf{t})=B_D^0+[(K+B_D^0)\,A_D^0]\,\Delta\mathbf{t}+\tfrac{1}{2}\{(K+B_D^0)\,[\alpha(A_L^0-A_D^0)-\\ -(K+B_D^0)\,A_D^0]+A_D^0[(K+B_D^0)\,A_D^0]\}\,(\Delta\mathbf{t})^2+\ldots.\end{aligned} \tag{143}$$

Note that the rate constant of racemization of A, i.e. k_1 (in α) contributes only to $(\Delta t)^2$ and higher order terms.

$$\begin{aligned}B_L(\Delta\mathbf{t})-B_0(\Delta\mathbf{t})=(B_L^0-B_D^0)+[K(A_L^0-A_D^0)+(A_L^0B_L^0-A_D^0B_D^0)]\,\Delta\mathbf{t}+\\ +\tfrac{1}{2}\{(A_D^0-A_L^0)\,[\alpha(B_L^0-B_D^0)+k^2-K(A_L^0+A_D^0)]+\\ +A_D^0(B_D^0)^2-A_L^0(B_L^0)^2+B_L^0(A_L^0)^2-B_D^0(A_D^0)^2\}\,(\Delta\mathbf{t})^2+\cdots\end{aligned} \tag{144}$$

Thus B_L-B_D grows due to initial disproportion both in A and in B, i.e., due to $B_L^0-B_D^0\neq 0$, and due to $A^0{}_L-A_D^0\neq 0$.

If $B_L^0 = B_D^0 = 0$, then

$$B_L(\Delta t) - B_D(\Delta t) = K(A_L^0 - A_D^0)\, \Delta t + \tfrac{1}{2}(A_L^0 - A_D^0)\, K[A_L^0 + A_D^0 - K]\, (\Delta t)^2 + \cdots \tag{145}$$

We see that the difference $B_L - B_D$ will grow due to initial disproportion in A, i.e., due to $A_L^0 - A_D^0 \neq 0$, and if $A_L^0 > A_D^0$ then $B_L > B_D$.

If $A_L^0 = A_D^0 = \bar{A}_0$, $B_D^0 = 0$, but $B_L^0 \neq 0$, then

$$B_L = B_L^0 + (K + B_L^0)\, \bar{A}_0 \Delta t + \tfrac{1}{2}(K + B_L^0)\, \bar{A}_0 [A_0 - K - B_L^0]\, (\Delta t)^2 + \cdots \tag{146}$$

and

$$\begin{aligned} B_D = K\{ & A_0 \Delta t + \tfrac{1}{2}\bar{A}_0(\bar{A}_0 - K)\, (\Delta t)^2 + \\ & + \tfrac{1}{6}\bar{A}_0(\bar{A}_0^2 + K^2 - \alpha B_L^0 - 4K\bar{A}_0)\, (\Delta t)^3 + \cdots\}. \end{aligned} \tag{147}$$

Thus, also B_L will grow fast, B_D will grow almost independently of B_L^0. If $K = k_2/k_c \ll 1$, we obtain separation (i.e. B_L grows, $B_D \simeq 0$). This condition is consistent with conditions (1a) and (2) (see after Equation (141)).

Summary of Asymptotic Results

Condition: Catalytic rate is much greater than the non-catalytic rate of conversion of A to B_L and A_D to B_D

$$k_2/k_c \ll B_L \quad \text{and} \quad k_2/k_c \ll B_D.$$

$$B_L \xrightarrow{t\to\infty} \begin{cases} \overset{*}{\to}\ A_L^0 + B_L^0 \\ \qquad |\overset{*\dagger}{\to}\ A_L^0 + B_L^0 \qquad \text{(only } A_L \text{ converted to } B_L) \\ \overset{**}{\to}\ A_L^0 + \dfrac{B_L^0 + B_D^0}{2} + \dfrac{(B_L^0 - B_D^0)}{2}\left(1 + \dfrac{2A_L^0}{a + B_L^0 + B_D^0}\right) \overset{***}{\to} \\ \qquad \hookrightarrow A_L^0 + A_D^0 + B_L^0 \qquad \text{(all } A_L \text{ and } A_D \text{ converted to } B_L!) \end{cases}$$

$$B_D \xrightarrow{t\to\infty} \begin{cases} \overset{*}{\to}\ A_D^0 + B_D^0 \\ \qquad \overset{*\dagger}{\to}\ 0 \qquad (A_D \text{ remains unchanged}) \\ \overset{**}{\to}\ A_D^0 + \dfrac{B_D^0 + B_L^0}{2} - \dfrac{(B_L^0 - B_D^0)}{2}\left(1 + \dfrac{2A_D^0}{a + B_L^0 + B_D^0}\right) \overset{***}{\to} \\ \qquad \hookrightarrow \sim A_D^0 \dfrac{a}{B_L^0} \ll A_D^0 \qquad (\text{almost no } B_D). \end{cases}$$

Conclusions

It has been shown that under normally expected conditions, with non-zero values of

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* Rate of racemization $A_L \overset{k_1}{\rightleftarrows} A_D$ is smaller than the rate of catalytic formation of B_L and B_D. See Equations (116) and (117).

** Infinite rate of racemization. See Equations (135) and (136) $a = 2k_2/k_c$.

*** If $B_L^0 \gg a = 2k_2/k_c$ and $B_D^0 \ll B_L^0$. See Equations (141).

*† If $B_L^0 \neq 0$, $B_D^0 = 0$.

$A_L^0 - A_D^0$, B_L^0 and B_D^0, both B_L and B_D will grow and there will be no formation of pure B_L (or B_D).

Under rather special local conditions, when by chance, $B_L^0 \gg 2k_2/k_c$, but $B_D^0 \ll B_L^0$, B_L will be formed only from all A_L and all A_D (or if $B_D^0 \gg 2k_2/k_c$, but $B_L^0 \ll B_D^0$, B_D will be formed only from all A_L and all A_D). This condition will be valid only if (1) catalytic rate is much greater than non-catalytic rate of conversion of A_L to B_L and A_D to B_D and (2) rate of racemization of A_L to A_D and vice versa is extremely fast, so that at all times $A_L \simeq A_D$.

If the rate of racemization is smaller than the rate of catalytic formation of B_L and B_D, then one can achieve separation also due to growth of A_L to B_L but without change in A_D (or A_D to B_D without change in A_L). Subsequent fluctuation in the remaining form of A will convert it to the corresponding form of B.

A large local fluctuation is required for the separation in the above process (e.g. $B_L^0 \gg 2k_2/k_c$). If the value of $2k_2/k_c$ is such that normal fluctuations or initial disproportion (e.g., B_L^0) are expected to exceed it, then at different location B_L or B_D will be separated. If the value of $2k_2/k_c$ is such that the inequality is satisfied only for an unusually large fluctuation (or initial disproportion), then once separation was achieved at one location, it will spread by diffusion and mixing.

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