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HIGH-ORDER SLIDING MODES IN CONTROL SYSTEMS

S. V. Emel'yanov, S. K. Korovin,
and A. Levant

The article considers the properties of a new type of solutions that arise in discontinuous dynamic systems. A specific feature of these solutions is the tangency of the phase velocity vectors to the manifold of the right-hand side discontinuities, instead of the transversal intersection of the manifold typically observed for ordinary sliding modes. The solutions identified in this case are high-order sliding modes, and the order of the mode is determined by the smoothness of tangency of the sliding manifold. Second-order sliding modes are considered in detail. Examples of systems with such modes are given; application of the theory to stabilization of uncertain dynamic systems is described. It is shown that the sensitivity of high-order sliding modes to small variations in the right-hand side of the discontinuous system is an order of magnitude higher than for ordinary sliding modes.

INTRODUCTION

One of the most efficient approaches to solving control problems under uncertainty involves forced decomposition of the control system. Formally, this decomposition is achieved by tracking as accurately as possible some special constraint on the system variables. This approach, in particular, is applied when the controlled plant is essentially nonstationary, and its parameters vary unpredictably but in a known range. The constraint lowers the dimension of the control system, and a properly chosen constraint also compensates for the effect of plant uncertainty on the operation of the system. The performance of the control system depends directly on the accuracy with which the constraint is tracked.

In this paper, we formulate and analyze separately the problem of defining and tracking the constraint. This problem is typically solved using large, and in the limit infinite, gains [1], sliding modes [2, 3], and binary algorithms [4-11]. The algorithms considered in this paper are binary algorithms.

Sliding modes in control systems are characterized by controls that are discontinuous in time. Exact tracking of the constraint is achieved in the limit by infinitely frequent switching. Binary control algorithms use bounded, time-continuous controls, although the earliest control algorithms of this type ensured only approximate, and in some cases asymptotically exact, constraint tracking.

The constraint is satisfied when the point is contained in a certain manifold in the phase space of the closed-loop control system. In the simplest case, when the constraint manifold is a hyperplane, the ordinary sliding mode has a geometrical description.

Feedback discontinuity on the constraint manifold corresponds to a discontinuity of the vector field of phase velocities. The phase curves approach the constraint manifold from different sides with velocity vectors that intersect the manifold. The phase-space image point reaches the constraint manifold in a finite time, and does not leave it after that. The motion of the point on the constraint manifold is described by Filippov's procedure [12]. This motion can be interpreted as limiting motion, which arises when switching delay tends to zero (Fig. 1). Such sliding modes [17, 18, 22] are called 1st order sliding modes (the order of a sliding mode is defined in Sec. 2).

A 2nd order sliding mode also arises in discontinuous dynamic systems. However, in contrast to the ordinary sliding mode, it arises at the points of the constraint manifold where any phase velocity vector obtained by passage to the limit approaching the constraint manifold over the continuity region of the phase velocity field is tangent to the manifold. A 2nd

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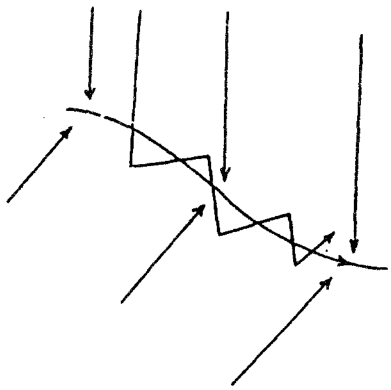

Fig. 1

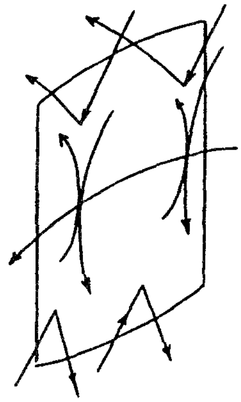

Fig. 2

order sliding mode arises, by definition, when the set of these points of the manifold consists of phase paths of the discontinuous vector field (Fig. 2). Figure 2 is an example of an unstable 2nd order sliding mode.

Such sliding modes arise in a natural manner in binary dynamic systems [13]. It is intuitively clear that, with small switching delays, a 2nd order sliding mode can ensure a higher constraint tracking accuracy than an ordinary sliding mode. This practically important consideration has stimulated further research in this area. Below we consider the organization of stable 2nd order sliding modes for control under uncertainty.

The proposed approach attains the constraint in a finite time and ensures exact tracking of the constraint by a time-bounded Lipschitzian control. If there are restrictions on switching frequency, the constraint tracking is 2nd order accurate in interswitching time. Under the same conditions, constraint tracking by the ordinary 1st order sliding mode is only 1st order accurate.

1. EXAMPLES OF SLIDING MODES

1.1. In this section, we consider a prototype example that qualitatively demonstrates 1st and 2nd order sliding modes and their potential applications.

Consider the dynamic system

$$\ddot{y} = g(y, \dot{y}) + u \,, \tag{1.1}$$

where $y \in \mathbf{R}$, $u \in \mathbf{R}$ are controls, the function g is continuously differentiable with bounded partial derivatives, $|g| \leq 1$.

The objective is to stabilize the system asymptotically at the point $y = \dot{y} = 0$. The control should be bounded in any bounded region in the y, $\dot{y}$ plane. The phase coordinates y, $\dot{y}$ are assumed observable.

To solve the problem, it is clearly sufficient to track the constraint $y + \dot{y} = 0$. A standard approach of the theory of variable structure systems leads to the algorithm

$$u = 2\mu\sqrt{2 + \dot{y}^2} \,, \tag{1.2}$$

$$\mu = -\operatorname{sign}(y + \dot{y}) \,. \tag{1.3}$$

The control law (1.2), (1.3) ensures prevalence of control in the time derivative of the constraint function $\sigma = y + \dot{y}$ evaluated on the system:

$$\dot{\sigma} = g + \dot{y} + u = g + \dot{y} - 2\sqrt{2 + \dot{y}^2}\operatorname{sign}\sigma \,.$$

The constraint function σ vanishes in a finite time, and subsequently it tracks the zero constraint in a sliding mode (Fig. 3). However, exact constraint tracking and exact stabilization are actually possible only in the limit, with infinitely frequent switching.

1.2. In the theory of binary dynamic systems, this problem is solved, in particular, by the so-called $\mathcal{A}_\mu$-algorithm [4-11]:

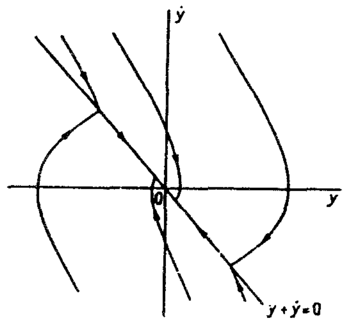

Fig. 3

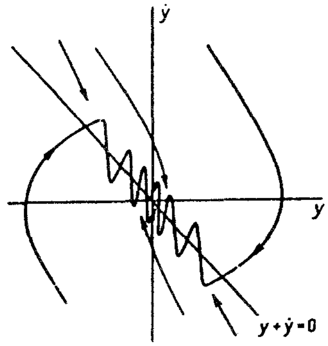

Fig. 4

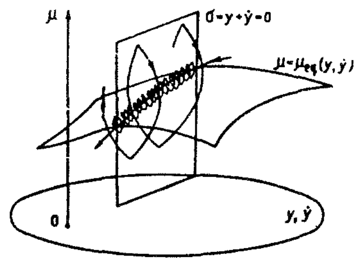

Fig. 5

$$\dot{\mu} = \begin{cases} -\mu & \text{for } |\mu| > 1\,, \\ -\alpha\,\mathrm{sign}\sigma & \text{for } |\mu| \le 1\,, \end{cases} \tag{1.4}$$

where $\alpha > 0$. As $\alpha \to \infty$, the properties of the $\mathcal{A}_\mu$-algorithm approach those of algorithm (1.3).

The projection of the phase curves of system (1.2), (1.3), (1.4) on the y, $\dot{y}$ plane is shown in Fig. 4. Eventually we have the inequality

$$\frac{|\sigma|}{\sqrt{1+\dot{y}^2}} \le \frac{c}{\alpha}\,,$$

where $c > 0$ is a constant. Stabilization is achieved approximately.

Consider the phase portrait of system (1.1), (1.2), (1.4) in the coordinates y, $\dot{y}$, μ. Note that by (1.4) the sign of μ is determined by the sign of σ, and the sign of $\dot{\sigma}$ is determined by the sign of the difference $\mu - \mu_{eq}$, where

$$\mu = -\frac{g(y,\dot{y}) + \dot{y}}{2\sqrt{2+\dot{y}^2}}\,.$$

The function $\mu_{\mathrm{eq}}(y, \dot{y})$ is obtained from the condition

$$\dot{\sigma}(y,\dot{y},\mu) = g + \dot{y} + \mu\sqrt{2+\dot{y}^2} = 0\,.$$

notation "μ_{eq}" stands for "equivalent μ" (this term is borrowed from [2, 3]).

It is easy to see that for large α the phase path of system (1.1), (1.2), (1.4) rotates around the curve

$$\sigma = 0\,, \qquad \dot{\sigma} = 0\,,$$

which is equivalent to $\sigma = 0$, $\mu = \mu_{\mathrm{eq}}(y, \dot{y})$ (Fig. 5). By an appropriate choice of initial conditions we can obviously ensure that the phase curve is "pressed" as closely as desired to the curve $\sigma = \dot{\sigma} = 0$. Thus, the curve $\sigma = \dot{\sigma} = 0$ may be regarded as a limiting path of the discontinuous dynamic system (1.1), (1.2), (1.4), and it is also a solution path of the system [13, 14]. This argument will be mathematically justified in Sec. 4.

The curve $\sigma = \dot{\sigma} = 0$ is the set of all points of the plane $\sigma = 0$ where the vector field of system (1.1), (1.2), (1.4), continued from any continuity region, is tangent to the plane $\sigma = 0$ (Fig. 5). Integrality of the set $\sigma = \dot{\sigma} = 0$ shows that for large α the system develops a 2nd order sliding mode on the constraint $\sigma = 0$.

The motion in this mode is obviously described by the equality $\dot{y} = -y$ and becomes possible (as in the case of a 1st order sliding mode) only with infinitely frequent switching. The control in this case is a continuous function of time.

The existence of this mode, however, is only of theoretical interest if it is unstable. Indeed, the phase paths must "coil" around the sliding mode path in a finite or infinite time. We can show that in system (1.1), (1.2), (1.4) only "asymptotic coiling" is possible (i.e., in infinite time). Other algorithms are needed in order to achieve a finite transition time to a 2nd order sliding mode.

2. IDEAL AND REAL SLIDING

In this section, we consider a number of formal definitions and concepts that will be used below.

The controlled plant is described by the differential equation

$$\dot{x} = f(t, x, u) ,$$

where $x \in X$, X is a smooth finite-dimensional real manifold, t is time, $u \in \mathbf{R}^l$ is control, and f is a smooth vector function. A smooth constraint function is given:

$$\sigma : (t, x) \rightarrow \sigma(t, x) \in \mathbf{R}^s$$

The problem is to ensure exact or approximate tracking of the constraint $\sigma(t, x) = 0$. This problem is called the sliding problem in what follows.

2.1. A control algorithm is a relationship that generates the controls in response to observations. Any exact motion on the constraint manifold $\sigma = 0$ is called ideal sliding, and any motion in a small neighborhood of the constraint manifold is called real sliding. An algorithm that tracks the constraint exactly is called an ideal sliding algorithm; an algorithm that tracks the constraint approximately is called a real sliding algorithm. The adjective "ideal" is often omitted in application to sliding.

We give a definition that makes it possible to estimate the accuracy of real sliding. Suppose that the real sliding algorithm $\mathcal{A}(\varepsilon)$ depends on a small parameter $\varepsilon \in \mathbf{R}^\lambda$, and for some function γ: $\mathbf{R}^\lambda \rightarrow \mathbf{R}^{\lambda_1}$ we have $\gamma \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Definition 2.1. The algorithm $\mathcal{A}(\varepsilon)$ is called a real sliding algorithm of order r ($r \geq 0$) in γ if for some $c > 0$ for sufficiently small ε with any initial conditions t_0, x_0 there exists an ε-independent convergence-time bound $\Delta t(t_0, x_0)$ such that after the time Δt the algorithm $\mathcal{A}(\varepsilon)$ permanently maintains the inequality

$$\|\sigma(t, x)\| \leq c\|\gamma(\varepsilon)\|^r .$$

If the initial conditions t_0, x_0 are from some fixed region, and the process is viewed on a time interval of length T, where $\Delta t < T$, we speak of a local real sliding algorithm of order r. If for sufficiently small ε and initial conditions t_0, x_0 from some set D we can choose a common constant Δt independent of t_0, x_0 and ε, then we say that the algorithm converges uniformly on the set of initial conditions D.

In this paper, we study algorithms that provide switching from one smooth dynamic system to another depending on observation results. We naturally assume that smoothness of the control is lost at the switching points. The next definition makes it possible to estimate the accuracy of such algorithms subject to restrictions on switching frequency.

Definition 2.2. Assume that a real sliding algorithm on the constraint $\sigma = 0$ depends on a small parameter $\varepsilon \in \mathbf{R}^\lambda$ and generates a control that is piecewise-smooth in time with smoothness intervals not less than $\tau(\varepsilon) > 0$, $\tau(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. If the algorithm produces sliding of order r in τ, it is called an r-th order sliding algorithm on the constraint $\sigma = 0$ (without mentioning τ).

We similarly define the order of real sliding for a family of real sliding modes that depend on a small parameter, i.e., for a family of paths $(t, t(x))$ indexed by the parameter ε.

Under standard assumptions, real sliding algorithms using a large gain k are of 1st order in k (and not higher).

2.2. Assume that the algorithm $\mathcal{A}(\varepsilon)$ produces r-th order sliding on the constraint $\sigma = 0$, and the time intervals where the control is smooth are not less than $\tau(\varepsilon)$ (see Definition 2.2). The algorithm $\mathcal{A}(\varepsilon)$ produces and maintains the inequality

$$\|\sigma\| \leq c\tau(\varepsilon)^r , \tag{2.1}$$

where c is from Definition 2.1 (we take $\gamma = \tau$). The mode that tracks the inequality (2.1) is called steady-state.

Proposition 2.1. Let $\ell = [r]$ be the whole part of r. Then uniform boundedness of the ℓ-th time derivative evaluated along an arbitrary path

$$\sigma^{(\ell)} = \left(\frac{d}{dt}\right)^{\ell} \sigma(t, x(t))$$

implies the existence of constants $c_1, c_2, \ldots, c_{\ell-1}$ such that in the steady-state mode (2.1) we have on the smoothness intervals

$$\|\dot\sigma\| \le c_1\tau^{\ell-1},\ \|\ddot\sigma\| \le c_2\tau^{\ell-2}, \ldots,\ \|\sigma^{(\ell-1)}\| \le c_{\ell-1}\tau .$$

Proposition 2.2. Assume that for some $\delta > 0$ and integer $P > 0$ for any path realized by the algorithm $\mathcal{A}(\varepsilon)$ in the steady-state mode (2.1) there ultimately occur intervals of smooth control on which some of the scalar components σ_j of the vector function $\sigma\colon X \to \mathbf{R}^s$ has a p-th time derivative separated from zero: $|\sigma_j^{(p)}| \ge \delta$. Then the real sliding is of order $r \le p$.

To satisfy the conditions of Proposition 2.2, it is sufficient to have bounded $\sigma^{(p+1)}$ and to ensure the inequality $\|\sigma^{(p)}\| \ge \delta_1 > 0$. Both propositions follow from the results of [15]. By Proposition 2.2, the sliding algorithms used in the theory of variable structure systems have, under standard assumptions, real sliding of order not higher than 1 with discrete switching. We give an independent proof of both propositions.

LEMMA 2.1. Let an r_1-smooth real function $\omega(t)$ be given on some time interval of length τ. Then there exists a constant $\Gamma > 0$ independent of the choice of τ and ω such that at some point t_1 of the interval we have the inequality

$$|\omega^{(r_1)}(t_1)| \le \Gamma \sup |\omega| \cdot \tau^{-r_1} .$$

To prove Lemma 2.1, it suffices to apply the Lagrange theorem $r_1(r_1 + 1)/2$ times. Proposition 2.2 follows from Lemma 2.1 (proof by contradiction). To prove Proposition 2.1, we have to apply Lemma 2.1 for all $r_1 \le \ell - 1$ and then integrate σ^{ℓ} successively $\ell - 1$ times.

2.3. It follows from Propositions 2.1 and 2.2 that sliding accuracy can be increased by maintaining equality to zero not only of the constraint function itself, but also of some of its successive time derivatives along the phase curves. Below this conclusion is applied to define high-order sliding modes.

Consider a differential equation with a discontinuous right-hand side

$$\dot y = v(y) \tag{2.2}$$

Here $y \in \mathbf{R}^n$, v is a measurable locally bounded function.

Definition 2.3. Following [12], a solution of (2.2) is an absolutely continuous function $y(t)$ that almost everywhere satisfies the differential inclusion

$$\dot y \in V(y) ,$$

where $V(y) = \bigcap_{\delta>0} \bigcap_{\mu N=0} \operatorname{conv} v(U_\delta(y)\backslash N)$.

Here conv stands for the closed convex hull, μ is the Lebesgue measure, $U_\delta(y)$ is a sphere of radius δ centered at y. Existence of a solution of Eq. (2.2) is proved in [12].

Given is a smooth manifold Ω included in $\mathbf{R}^n$. The set of its points is called the set of 1st order sliding on the manifold Ω. The set of points y of the manifold Ω where $V(y)$ is included in the tangent space of the manifold Ω at the point y is called the set of 2nd order sliding on the manifold Ω.

Definition 2.4. Equation (2.2) defines in the region $\mathcal{D}$ a 2nd (1st) order sliding mode on the manifold Ω if the intersection of the region $\mathcal{D}$ with the set of 2nd (1st) order sliding on the manifold Ω is nonempty and is an integral set, i.e., consists of phase curves of Eq. (2.2).

Sliding modes discussed in the literature [2, 3] are 1st order modes.

Let the manifold Ω be defined by equality to zero of the smooth function $\omega\colon \mathbf{R}^n \to \mathbf{R}^\nu$. Assume that for almost all points from $\mathcal{D}$ there is a neighborhood where $r - 1$ successive time derivatives of the function ω exist along any phase curve of Eq. (2.2), and these derivatives evaluated at a fixed point of the region $\mathcal{D}$ are independent of the choice of the phase curve through that point. This property holds for piecewise-smooth $v(y)$.

Definition 2.5. The set $\mathcal{L}_r$ of r-th order sliding on the constraint $\omega = 0$ (on the constraint function $\omega = 0$) is the set of points where ω and its $r - 1$ successive derivatives along the phase curves $\dot{\omega}, \ddot{\omega}, \ldots, \omega^{(r-1)}$ (which, by assumptions, are functions of y) may be defined equal to zero by continuity.

Definition 2.6. Equation (2.2) defines in $\mathcal{D}$ an r-th order sliding mode on the constraint $\omega = 0$ (on the constraint function ω) if the r-th order sliding set $\mathcal{L}_r \cap \mathcal{D}$ is nonempty and is an integral set of Eq. (2.2).

Note that the notions of 2nd order sliding mode on the manifold $\omega = 0$ and on the constraint function ω are different. They are identical for $\omega_y{}^j = \text{codim}\ \Omega = \nu$ only if $\dot{\omega}$ is representable as a function of y.

If the solutions of Eq. (2.2) are time-unbounded, then we naturally define stability of the sliding mode as stability of the corresponding integral set. Unlike the 1st order sliding mode, high-order modes can be asymptotically stable.

A sliding mode is called "finite-time attracting" if

a) for any initial conditions, every path of Eq. (2.2) reaches in finite time the sliding set of corresponding order and subsequently remains there;

b) in some neighborhood of the sliding set, the time to reach the sliding set is uniformly bounded.

If the initial conditions that ensure attainment of the sliding mode in finite time are taken only in the neighborhood of the sliding set, then the mode is called "locally finite-time attracting".

Definitions 2.3-2.6 are of local character, and can therefore be extended to the case when the phase space is a smooth real manifold. We can thus speak of stability of the sliding mode if a metric is defined on the manifold.

The definitions are extended to the nonautonomous case by introducing the artificial control $\dot{t} = 1$. Note that the solution of the nonautonomous equation is not understood in Filippov's sense [12], because in Filippov's paradigm the time coordinate is not equivalent to the other coordinates. In most cases, however, these definitions are equivalent [12].

2.4. Let us return to the original control problem the problem of establishing and tracking the constraints $\sigma = 0$ during control of the equation

$$\dot{x} = f(t, x, u) \ . \tag{2.3}$$

Let

$$u = U(t, x, \xi) \ , \tag{2.4}$$

where the feedback operator U is indexed by the parameter $\xi \in \mathbb{R}^{s_1}$. The parameter is called an operator variable [5-11]. The variable dynamics is defined by the equation

$$\dot{\xi} = \psi(t, x, \xi) \ . \tag{2.5}$$

The initial value, in general, is defined by the function $\xi(t_0) = \xi_0(t_0, x(t_0))$. By default we assume an arbitrary initial value $\xi(t_0)$ from the feasible set.

The control algorithm (2.4), (2.5) and the system (2.3), (2.4), (2.5) are called binary. The right-hand sides of (2.4), (2.5) may be discontinuous functions. In this case, the closed-loop system is understood in the sense of the definition from subsec. 2.3.

We usually deal with binary systems with a continuous feedback operator U. In such systems, the derivative of the constraint function $\sigma(t, x)$ along the phase curve $\dot{\sigma}$ is independent of the choice of the phase curve, and depends only on the point (t, x, ξ) where it is evaluated. Under some additional assumptions, this also applies to the higher derivatives along the phase curve $\ddot{\sigma}, \dddot{\sigma}, \ldots$

Definition 2.7. Algorithm (2.4), (2.5) is called an ideal r-th order sliding algorithm on the constraint $\sigma = 0$ (on the constraint function σ) if in the closed-loop system (2.3), (2.4), (2.5) there is a finite-time attracting r-th order sliding mode on the constraint $\sigma = 0$.

In this paper, we consider 2nd order sliding modes and propose ideal and real 2nd order sliding algorithms.

3. STATEMENT OF THE PROBLEM

3.1. Consider the simplest version of the sliding problem, when the constraint function and the control are real scalars. The results can be extended to the case of vector controls and vector constraint functions by the standard technique of the theory of variable structure systems [3, 4] and binary systems [4-11].

To increase the generality of the problem, we state its conditions in axiomatic form, and the sliding problem itself is formulated as a control problem for a "black box" with one input variable u and one output variable – the constraint function σ. It is required to ensure exact or approximate tracking of the constraint $\sigma = 0$ by controls generated in response to observations of the variable σ.

We make the following assumptions:

3.1) The controlled plant is described by the differential equation

$$\dot{x} = f(t, x, u) , \tag{3.1}$$

where $x \in X$, X is a finite-dimensional smooth real manifold, t is time, $u \in \mathbf{R}$ is control. We assume that the control u does not exceed in absolute value the constant $\varkappa > 1$. The dimension of the state space X and the specific form of the function f are unknown. For any continuous control $|u(t)| \le \varkappa$ the solutions of (3.1) are infinitely continuable in time. The constraint function

$$\sigma : (t, x) \longmapsto \sigma(t, x) \in \mathbf{R}$$

and the vector function f are smooth.

In general, any algorithm may be used to drive the image point into a small neighborhood of the constraint manifold $\sigma = 0$.

3.2) There exists a constant u_1, $u_1 \in (0, 1)$ such that for any control $u(t)$ that is a continuous function of time the inequality $|u| > u_1$ ensures that the relationship $\sigma u > 0$ is achieved in a finite time.

Define the differential operator L_u such that for any differentiable function $z(t, x, u)$

$$L_u\, z(t, x, u) = \frac{\partial z}{\partial t}(t, x, u) + \frac{\partial z}{\partial x}(t, x, u) f(t, x, u) .$$

We assume that

$$\dot{\sigma}(t, x, u) = L_u\, \sigma(t, x) .$$

Suppose that in some neighborhood of the constraint manifold defined by the inequality $|\sigma| < \sigma_0$, where $\sigma_0 = \text{const} > 0$, the following conditions are satisfied for some positive constants $u_0 < 1$, $K_m < K_M$, C_0:

3.3) For $|\sigma(t, x)| < \sigma_0$, $|u| > u_0$ implies $\dot{\sigma} u > 0$.

3.4) For all $|u| \le \varkappa$ we have the inequality

$$K_m \le \frac{\partial}{\partial u} \dot{\sigma}(t, x, u) \le K_M ;$$

3.5) For $|\sigma(t, x\ u)| < \sigma_0$, $|u| \le \varkappa$ we have the inequality

$$|L_u L_u \sigma| \le C_0$$

Condition 3.3 implies the existence of a 1st order sliding mode defined by the algorithm $u = -\text{sign}\, \sigma$. Condition 3.4 implies sign-constancy of $\dot{\sigma}_u{}' = \sigma_x{}' f_u{}'$. In the theory of variable structure systems, where only systems linear in control are considered, this condition guarantees uniqueness of motion in the 1st order sliding mode [3, 16]. Condition 3.5 is new. Here $L_u L_u \sigma$ is the 2nd time derivative of σ evaluated on the system for a constant control.

Note that conditions 3.2-3.5 in principle are verifiable in the process of a "black box" experiment. Parametric synthesis of the algorithms considered below requires only knowledge of the constants σ_0, K_m, K_M, C_0, and the parameters can be chosen in the course of the experiment if these constants are unknown.

In what follows, the neighborhood $|\sigma| < \sigma_0$ of the constraint manifold is called a "linear zone" (because of approximate linearity of $\dot{\sigma}$ in control).

We finally state our problem: construct algorithms of ideal and real 2nd order sliding on the constraint $\sigma = 0$.

3.2. Relaxation of Smoothness Conditions. Control systems are often designed using functional elements with nonsmooth characteristics. We accordingly give modified conditions that relax the smoothness requirements:

3.1′) the vector function f is locally Lipschitzian, and the constraint function σ has locally Lipschitzian partial derivatives; in all other respects condition 3.1 remains valid;

3.2′) this condition is identical with condition 3.2;

3.3′) condition 3.3 remains unchanged;

3.4′) for $|\sigma(t, x)| < \sigma_0$, $|u| \leq \varkappa$ for any t, x, u and sufficiently small Δu we have the inequality

$$K_m|\Delta u| \leq [\dot\sigma(t,x,u+\Delta u) - \sigma(t,x,u)]\,\mathrm{sign}\,\Delta u \leq K_M|\Delta u|\ ;$$

3.5′) there exist an atlas of local charts on the manifold X and positive functions $l_1(t, x, u)$, $l_2(t, x, u)$ that serve as local Lipschitz constants of the function $\dot\sigma(t, x, u)$ by the variables t and x respectively at the point (t, x, u) such that for $|\sigma| < \sigma_0$, $|u| \leq \varkappa$ we have the following inequality in local coordinates:

$$l_1(t,x,u) + l_2(t,x,u)\|f(t,x,u)\| \leq C_0\ .$$

Smoothness of f, σ and conditions 3.1′-3.5′ obviously lead to conditions 3.1-3.5.

3.3. Comparison with Traditional Sliding Problem. The theory of variable structure systems [2, 3] and the traditional theory of binary systems [4-11] deal with dynamic systems linear in control:

$$\dot x = a(t,x) + b(t,x)u\ , \tag{3.2}$$

Here $x \in \mathbf{R}^n$. We only consider the case when $u \in \mathbf{R}$, and a, b are smooth functions. Let $\sigma(t, x)$ be a smooth constraint function, σ: $\mathbf{R}^{n+1} \to \mathbf{R}$. Usually, the right-hand side of (3.2) and the constraint function σ increase linearly in the variable x. In this case, conditions 3.1-3.5 in general are satisfied only in a bounded neighborhood of the point $x = 0$.

Assume that for some smooth positive function $\Phi(x)$ with uniformly bounded Φ_x' and $\Phi_{xx}''\Phi$ (an example of such a function is $\Phi = (x^t\mathcal{D}x + h)^{1/2}$, where $\mathcal{D}$ is a positive semidefinite matrix, $h = \mathrm{const} > 0$) we have the following conditions:

a) for all t, x and some $\delta > 0$, $\delta > 0$, $\sigma_x'b \geq \delta$;

b) σ_x', σ_t'/Φ, σ_{tx}'', $\sigma_{xx}''\Phi$, a_x', a/Φ, a_t'/Φ, b, $b_x'\Phi$, b_t' are uniformly bounded in t, x.

Let

$$u = \mu k \cdot \Phi(x)\ , \tag{3.3}$$

where μ is an operator variable, $k = \mathrm{const} > 0$.

Proposition 3.1. Assume that conditions a and b are satisfied. Then the differential equation

$$\dot x = a(t,x) + \mu k \cdot b(t,x) \cdot \Phi(x)\ ,$$

where μ is regarded as a new control, $|\mu| \leq \varkappa$, $\varkappa > 1$, and the constraint function $\varphi(t, x) = \sigma(t, x)/\Phi(x)$ satisfy conditions 3.1-3.5 for sufficiently large k.

If a, b are locally Lipschitzian functions, and σ and Φ are functions with a Lipschitzian derivative, then we formulate an analogue of condition 3.4, in which all the nonexisting variables are replaced with local Lipschitz constants. The condition of boundedness of $\Phi_{xx}''\Phi$ is similarly modified, and analogue of Proposition 3.1 is formulated.

Proof is by direct evaluation of $\dot\varphi$, $(\partial/\partial\mu)\varphi$, and $L_\mu L_\mu\varphi$. Conditions 3.3-3.4 are usually satisfied in the theory of variable structure systems. Their physical meaning is the existence of a locally bounded equivalent control [3] with a locally bounded rate of change. The standard algorithm of the theory of variable structure systems is defined by the feedback

$$u = -k\,\mathrm{sign}\,\sigma\,\Phi(x)\ .$$

4. EXISTENCE OF HIGH ORDER SLIDING MODES OF S_μ-SYSTEM

4.1. Assume that the controlled plant is defined by the equation

$$\dot x = f(t,x,u)\ . \tag{4.1}$$

The vector function f and the constraint function σ satisfy conditions 3.1′, 3.3′, 3.5′. Then within the limits of the linear zone $|\sigma| < \sigma_0$ there exists a unique function $u_{eq}(t, x)$ that satisfies the equation $\dot\sigma(t, x, u_{eq}(t, x)) = 0$. The function u_{eq} is locally Lipschitzian and uniformly bounded in absolute value by the constant u_0 (by condition 3.3′, $u_0 \in (0, 1)$). It is easy to see that condition 3.5′ leads to the following proposition.

Proposition 4.1. Assume that conditions 3.1′, 3.3′, 3.5′ are satisfied and some control $u(t)$ tracks the constant value of the constraint function $\sigma(t, x) = \sigma_1$, $|\sigma_1| < \sigma_0$ on the solution $x(t)$ of Eq. (4.1). Then $u(t) = u_{eq}(t, x(t))$ and $u(t)$ is a Lipschitzian function with the Lipschitz constant C_0/K_m.

Note that if the inequality in condition 3.4′ is replaced with the inequality

$$\ell_1(t, xu) + \ell_2(t, x, u)\|f(t, x, v)\| \le C_0 ,$$

where v is not necessarily identical with $|v| \le \varkappa$, then the function $u_{eq}(t, x(t))$ evaluated on the solution $x(t)$ of Eq. (4.1) corresponding to an arbitrary continuous control $u(t)$ has the Lipschitz constant C_0/K_m. Hence we see the relationship of this problem with the problem posed in [14, 17-19].

Suppose that the control algorithm is defined by the equation

$$\dot{u} = \Psi(t, x, u) , \tag{4.2}$$

where Ψ is a bounded measurable function. The closed system (4.1), (4.2) is understood in the form described in Sec. 2. Existence of solutions of system (4.1), (4.2) has been proved in [12].

THEOREM 4.1. Assume that in any neighborhood of each point of the 2nd order sliding set for the constraint $\sigma = 0$ (i.e., the set $\sigma = \dot{\sigma} = 0$) the function Ψ takes on sets of nonzero measure both values that are not less than C_0/K_m and values that are not greater than $-C_0/K_m$. Then system (4.1), (4.2) has a 2nd order sliding mode on the constraint $\sigma = 0$. The operation of the controlled plant in this mode is described by the equation

$$\dot{x} = f(t, x, u_{eq}(t, x)) . \tag{4.3}$$

Proof. The 2nd order sliding set is defined by the equalities $\sigma(t, x) = 0$, $u = u_{eq}(t, x)$. This set is nonempty. It remains to show that this is an integral set for system (4.1), (4.2). Through an arbitrary point of the set $\sigma = 0$, $u = u_{eq}$ pass a curve $\gamma(t) = (t, x(t), u(t))$ that satisfies Eq. (4.3). Then $u(t) = u_{eq}(t, x(t))$. Such a curve is contained entirely in the set $\sigma = \dot{\sigma} = 0$.

From the definition of Sec. 2, the system (4.1), (4.2) is equivalent to a differential inclusion. By the condition of the theorem, the right-hand side of this inclusion at points of the set $\sigma = \dot{\sigma} = 0$ contains vectors whose t and x components are equal to 1 and $f(t, x, u_{eq}(t, x))$, while the u component is contained in the interval $[-C_0/K_m, C_0/K_m]$. By Proposition 4.1, the velocity vector of the curve γ at the differentiability points of the curve is contained in this subset of the right-hand side of the differential inclusion. Since the curve γ is obviously absolutely continuous, it is a solution of system (4.1), (4.2). Q.E.D.

Note that the method proposed by the theorem replacing the control u with the equivalent control u_{eq} for the description of the control system dynamics in the sliding mode is known as the "equivalent control method" in the literature [3]. It has been proved for ordinary sliding modes and equations (4.1) linear in control.

We now give sufficient conditions for the existence of a sliding mode of an arbitrary order r (where r is an integer, $r \ge 2$).

Suppose that conditions 3.1-3.3 are satisfied. Consider the system

$$\begin{aligned} \dot{x} &= f(t, x, u) , \\ \dot{u} &= \xi_1 , \\ \dot{\xi}_1 &= \xi_2 , \\ &\dots \\ \dot{\xi}_{r-2} &= \psi(t, x, u, \xi) , \end{aligned}$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_{r-2}) \in \mathbb{R}^{r-2}$. For local existence of an r-th order sliding mode in the neighborhood of the point $M(t_0, x_0, \xi^0)$ from the set $\sigma = \dot{\sigma} = \dots = \sigma^{(r-1)} = 0$, it is sufficient that in any neighborhood of the point M the function ψ takes on a set of nonzero measure both values not less than $u_{eq}^{(r-1)}(M) + \delta_1$ and values not greater than $u_{eq}^{(r-1)}(M) - \delta_1$. Here $\delta_1 = \text{const} > 0$, $u_{eq}^{(r-1)}(M)$ is the $(r - 1)$-th successive derivative on the system of the smooth function $u_{eq}(t, x)$ at the point M.

The proof of this proposition is similar to the proof of Theorem 4.1. The r-th order sliding set for σ is nonempty, because the parameter ξ_{j-1} can be expressed from the equation $\sigma^{(j)} = 0, j \ge 2$ in terms of $t, x, u, \xi_1, \dots, \xi_{j-2}$ (we assume that $\xi_0 = u$).

4.2. Consider the algorithm

$$\dot{u} = \begin{cases} -u & \text{for } |u| > 1\,, \\ -\alpha\,\mathrm{sign}\,\sigma & \text{for } |u| \le 1\,, \end{cases} \tag{4.4}$$

where $\alpha > 0$. It is called A_μ-algorithm, and system (4.1), (4.4) is called S_μ-system [5-7]. The example of the S_μ-system from Sec. 1 reduces to this case by the substitution described in subsec. 3.3.

By Theorem 4.1, system (4.2), (4.4) under conditions 3.1′, 3.3′, 3.4′, 3.5′ has a 2nd order sliding mode on σ. In general, however, this mode is unstable. Application of algorithm (4.4) is based on the fact that it is a real sliding algorithm on σ of 1st order in α^{-1}. Consider a more general algorithm

$$\dot{u} = \begin{cases} -u & \text{for } |u| > 1\,, \\ -\alpha(t,x,u)\,\mathrm{sign}\,\sigma(t,x) & \text{for } |u| \le 1\,,\ \alpha \ge \alpha_0\,, \end{cases} \tag{4.5}$$

where α is a measurable locally bounded function, $\alpha_0 = \text{const} > 0$.

Proposition 4.2. Assume that Eq. (4.1) and the constraint function σ satisfy assumptions 3.1′-3.4′ and for almost all t and all x, u such that $\sigma(t, x)\dot{\sigma}(t, x, u) > 0$, we have the inequality

$$\alpha(-t,x,u) > 4K_M/\xi\,, \tag{4.6}$$

where $0 < \xi \le \sigma_0$. Then for any initial conditions t_0, $x(t_0)$, $u(t_0)$ the solution of system (4.1), (4.5) reaches in finite time the region $\mathcal{G}_\xi = \{(t, x, u) \mid |\sigma(t, x)| < \xi\}$ and subsequently remains there.

Proof. For any initial conditions t_0, $x(t_0)$, $u(t_0)$, $\sigma(t, x(t))$ vanishes after a finite time with control taking a value $|u(t)| \le 1$. Indeed, otherwise the control u stabilizes on the value $u = -\mathrm{sign}\,\sigma$, and we obtain a contradiction with assumption 3.2′.

Let $\sigma(t_0, x(t_0)) = 0$, $|u(t_0)| \le 1$. We will show that the point $(t, x(t))$ never leaves the set $\mathcal{G}_\xi$ for $t \ge t_0$. Assume the contrary: suppose that $t_1 \ge t_0$ is the first instant when we have the equality $\sigma(t_1)x(t_1)$ (the case $\sigma = -\xi$ is analyzed similarly). Then obviously $\dot{\sigma}(t_1, x(t_1)) = 0$. Let t_* be the nearest previous instant when $\sigma(t_*, x(t_*)) = 0$. For $t_* < t < t_1$, $\sigma > 0$ and therefore during the time $t_1 - t_*$ the control u was decreasing or stabilized at the value $u = -1$.

Note that by condition 3.4′

$$|\dot{\sigma}(t,x,u)| \le K_M(u - u_{eq}(t,x))\,.$$

Since $|u| \le 1$, $|u_{eq}| < u_0 < 1$, we obtain

$$|\dot{\sigma}(t,x,u)| < 2K_M\,.$$

Let $T = \{t/t_* \le t \le t_1,\ \dot{\sigma}(t, x(t)u(t)) > 0\}$. Then

$$\xi = \sigma(t_1,x(t_1)) = \int_T \sigma\,dt + \int_{[t_*,t_1]\setminus T} \dot{\sigma}\,dt \le \int_T \dot{\sigma}\,dt \le 2K_M\lambda(T)\,. \tag{4.7}$$

Here $\lambda(t)$ is the Lebesgue measure of the set T. By (4.7), $\lambda(T) \ge \xi/(2K_M)$. But by (4.6) in time $t_1 - t_* \ge \lambda(T)$ the control u "runs" over the entire interval $[-1, 1]$ and stabilizes at $u = -1$. By conditions 3.3′, 3.4′, $\dot{\sigma}(t_1, x(t_1), -1) < 0$, a contradiction. Q.E.D.

Let us consider other simple sliding algorithms. Under conditions 3.1′-3.4′ the algorithm $u = -\mathrm{sign}\,\sigma$ is an (ideal) 1st order sliding algorithm on σ. In this case, however, the motion in the sliding mode in general is not described by the equivalent control method and, moreover, with some natural definitions of solutions of discontinuous differential equations (see, e.g., [21]) motion in the sliding mode is not necessarily single-valued. In this connection note that the description of the 2nd order sliding mode in Theorem 4.1 does not depend on the choice of definition [21].

Under conditions 3.1′-3.4′, the algorithm $u(t) = -\mathrm{sign}\,\sigma(t - \tau, x(t - \tau))$, where $\tau = \text{const} > 0$, is a real 1st order sliding algorithm for $\tau \to 0$. Under the same conditions, the high-gain algorithm

$$u = \begin{cases} -\mathrm{sign}\,\sigma & \text{for } k|\sigma| > 1\,, \\ -k\sigma & \text{for } k|\sigma| \le 1 \end{cases} \tag{4.8}$$

is a real sliding algorithm on σ of first order in k^{-1}. It is interesting that under conditions 3.1′-3.5′ the same algorithm produces 1st order sliding on $\dot{\sigma}$. The equivalent control method is thus applicable for large k.

5. WINDING ALGORITHM: IDEAL AND REAL SLIDING OF 2ND ORDER

5.1. Consider a controlled plant described by the differential equation

$$\dot{x} = f(t, x, u) . \tag{5.1}$$

Equation (5.1) and the constraint function σ satisfy assumptions 3.1′-3.5′. Let

$$\dot{u} = \begin{cases} -u & \text{for } |u| > 1 , \\ -\alpha_M \operatorname{sign} \sigma & \text{for } \sigma\dot{\sigma} > 0 , |u| \leq 1 , \\ -\alpha_m \operatorname{sign} \sigma & \text{for } \sigma\dot{\sigma} \leq 0 , |u| \leq 1 . \end{cases} \tag{5.2}$$

Here the constants $\alpha_M > \alpha_m > 0$. We stipulate that

$$\alpha_m > C_0 \,/\, K_m , \tag{5.3}$$

$$(K_M \alpha_m + C_0) \,/\, (K_m \alpha_M - C_0) < 1 , \tag{5.4}$$

$$\alpha_M > \min \{ \frac{4K_M}{\sigma_0} , \frac{K_M^2 (1 + u_0)^2}{2K_m \sigma_0} + \frac{C_0}{K_m} \} . \tag{5.5}$$

THEOREM 5.1. Assume that conditions 3.1′-3.5′ (or 3.1-3.5) and inequalities (5.3)-(5.5) are satisfied. Then (5.2) defines a 2nd order sliding algorithm on the constraint $\sigma = 0$ for the dynamic system (5.1).

By Theorem 4.1, the system is described in the sliding mode by an equation that follows from the equivalent control method. If assumption 3.2′ (or 3.2) is not satisfied, the image point should be driven to the neighborhood $|\sigma| \leq \delta$, $\delta < \sigma_0$ of the constraint manifold by a different algorithm, subsequently switching to algorithm (5.2). In this case, replacement of (5.5) with the inequality

$$\alpha_M > \frac{K_M^2 \,(1 + u_0)^2}{2K_m \,(\sigma_0 - \delta)} + \frac{C_0}{K_M} \tag{5.6}$$

guarantees convergence of the algorithm, and the phase curve does not go outside the linear zone $|\sigma| < \sigma_0$. Inequalities (5.3)-(5.6) are always satisfied for sufficiently large α_m and α_M/α_m.

Algorithm (5.2) is called a "winding algorithm" [13, 20]. This name is attributable to the fact that the phase path of system (5.1), (5.2) turns around the 2nd order sliding set (Fig. 6) and winds around it in a finite time.

5.2. Let us give the characteristics of the transient process. Let

$$q_m = \sqrt{\frac{K_m \alpha_m - C_0}{K_M \alpha_M + C_0}} , \qquad q_M = \sqrt{\frac{K_M \alpha_m + C_0}{K_m \alpha_M - C_0}} . \tag{5.7}$$

By (5.4), $q_m < q_M < 1$. For an arbitrary path $(x(t), u(t))$ of system (5.1), (5.2), let $t_0' = \sup\{t \,|\, u(t)| > 1, t \geq t_0\}$. Clearly, $t_0' - t_0 < \varkappa - 1$, and in practice we naturally specify $|u(t_0)| < 1$, $t_0' = t_0$. Let $t_1 = \inf\{t \,|\, \sigma(t, x(t)) = 0, t \geq t_0'\}$, $t_2 = \sup\{t \,|\, \sigma(t, x(t)) \neq 0\}$. Using Theorem 5.1 we state the following proposition.

Proposition 5.1. Under the conditions of Theorem 5.1, for all solutions of (5.1), (5.2) we have the following bound on the transient time:

$$t_2 - t_1 \leq (\frac{1}{K_m \alpha_M - C_0} + \frac{q_M}{K_m \alpha_m - C_0}) \frac{K_M (1 + u_0)}{1 - q_M} T_0 ,$$

where $T_0 = 0$ for $q_M < [K_m(1 - u_0)]/K_m(1 + u_0)]$. Otherwise,

$$T_0 = \frac{\sigma_0}{K_m (1 - u_0)} \frac{1}{1 - q_M^2} .$$

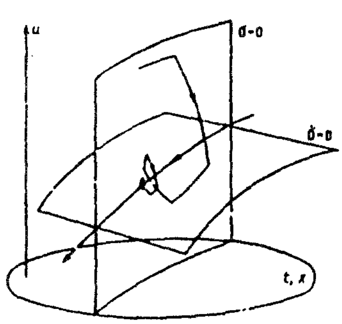

Fig. 6

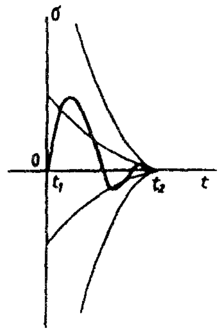

Fig. 7

For simplicity, let

$$q_M < \frac{K_m(1-u_0)}{K_M(1+u_0)}\,. \tag{5.8}$$

Proposition 5.2. Under the conditions of Theorem 5.1, if inequality (5.8) is satisfied and $t_1 \le t \le t_2$, then we have the inequalities

$$\frac{(1-q_M)^2(K_m\alpha_m - C_0)^2}{2(K_M\alpha_M + C_0)(\dfrac{K_m\alpha_m - C_0}{K_m\alpha_M - C_0} + q_M)^2} \le \sup_{\tau\in[t_1,t_2]} \frac{|\sigma(\tau, x(\tau))|}{(\tau - t_2)^2} \le \frac{(K_M\alpha_m + C_0)^2(1-q_m)^2}{2q_m^2(K_m\alpha_M - C_0)}\,, \tag{5.9}$$

$$\frac{(1-q_M)(K_m\alpha_m - C_0)}{\dfrac{K_m\alpha_m - C_0}{K_m\alpha_M - C_0} + q_M} \le \sup_{\tau\in[t_1,t_2]} \frac{|\dot\sigma(\tau, x(\tau), u(\tau))|}{(t_2 - \tau)} \le (K_M\alpha_m + C_0)\frac{(1-q_m)}{q_m}\,. \tag{5.10}$$

If (5.8) does not hold, then the bounds (5.9), (5.10) are valid on some time interval $t \in [t_1', t_2]$, where $t_1' > t_1$.

Given these bounds, the graphs of $\sigma(t)$ and $\dot\sigma(t)$ in the neighborhood of the point $t = t_2$ can be shown as in Figs. 7 and 8.

5.3. The winding algorithm (5.2) assumes differentiation of the observed variable $\sigma(t, x(t))$ in real time. In practice, this differentiation is often undesirable. Below we propose an algorithm free from this weakness.

In what follows we often omit the arguments $x(t)$ and $u(t)$ of the function evaluated at the point $(t, x(t), u(t))$ of the phase path, and write simply $\sigma(t)$.

Assume that the constraint function $\sigma(t, x(t))$ is observed at discrete time instants $t_0, t_1, t_2, \ldots$ with the increment $\tau_i = t_{i+1} - t_i \ge \tau_m = \text{const} > 0$. Also assume that at the current instant $t \in [t_i, t_{i+1})$. Denote

$$\delta_i\xi = \begin{cases} 0 & \text{for } i = 0\,, \\ \xi(t_i) - \xi(t_{i-1}) & \text{for } i \ge 1\,, \end{cases}$$

where ξ is an arbitrary function of t, x, u.

A discrete-increment winding algorithm is an algorithm of the form

$$\dot u = \begin{cases} -u(t_i) & \text{for } |u(t_i)| > 1\,, \\ -\alpha_M \operatorname{sign}\sigma(t_i) & \text{for } \sigma(t_i)\delta_i\sigma > 0,\ |u(t_i)| \le 1\,, \\ -\alpha_m \operatorname{sign}\sigma(t_i) & \text{for } \sigma(t_i)\delta_i\sigma \le 0,\ |u(t_i)| \le 1\,, \end{cases} \tag{5.11}$$

where $\alpha_M > \alpha_m > 0$, $t \in [t_i, t_{i+1}]$, $\tau_i \le (\varkappa - 1)/\alpha_M$.

Let $\tau_i = \tau = \text{const} > 0$, τ is a small parameter.

THEOREM 5.2. Under the conditions of Theorem 5.1, the algorithm (5.11) for $\tau_i = \tau = \text{const}$ is a real 2nd order sliding algorithm on the constraint $\sigma = 0$. The algorithm converges uniformly on the set of initial conditions from some neighborhood $|\sigma| < \varepsilon$, $|\dot\sigma| < \varepsilon_1$ of the 2nd order sliding set $\sigma = \dot\sigma = 0$.

6. PROOF OF THEOREM 5.1 ON WINDING ALGORITHM. IDEAL SLIDING

6.1. Lemmas on Majorizing and Minorizing Curves. We formalize a number of arguments, that will be often repeated in what follows.

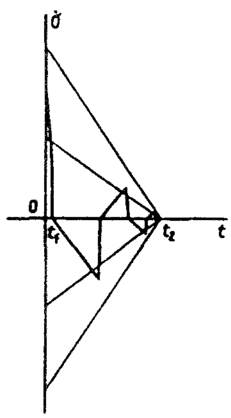

Fig. 8

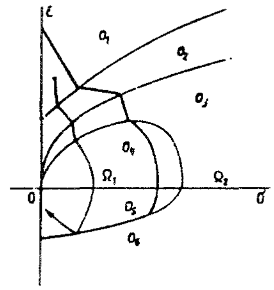

Fig. 9

Suppose that in the half-plane $\sigma \geq 0$ of the σ, ε plane we have an upper semicontinuous [12] field $f(\sigma, \varepsilon)$ of nonempty closed bounded convex sets, and let a piecewise-smooth curve without self-intersections $\Gamma: \tau \to (\sigma(\tau), \varepsilon(\tau))$, $\tau \in [0, 1]$ be given, where $\sigma(0) = 0$, $\sigma(\tau) > 0$ for $\tau \in (0, 1)$, $\sigma(1) = 0$, $\varepsilon(1) \neq \varepsilon(0)$. The curve Γ partitions the half-plane $\sigma \geq 0$ into two closed regions Ω_1 and Ω_2, whose intersection is Γ. Let Ω_1 be a bounded region, and Ω_2 an unbounded region.

Definition 6.1. The curve is called majorizing in the half-plane $\sigma > 0$ for the differential inclusion

$$(\dot{\sigma}, \dot{\varepsilon}) \in F(\sigma, \varepsilon) ,$$

if every solution of this inclusion originating from an arbitrary point of the region Ω_1 at time $t = 0$ is either entirely contained in Ω_1 for all $t > 0$ or can be continued forward in time until its intersection with the boundary of Ω_1 along the straight line $\sigma = 0$. The curve Γ is called minorizing if the same invariance conditions apply to the region Ω_2.

These concepts are basically used in the case shown in Fig. 9. The half-plane $\{(\sigma, \varepsilon) | \sigma \geq 0\}$ is partitioned into nonintersecting open regions $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_\ell$ with finitely many smooth curves γ_j, $j = 1, 2, \ldots, m$, which include the ray $\varepsilon = 0$, $\sigma \geq 0$ and the straight line $\sigma = 0$. To each region $\mathcal{O}_i$ we associate two numbers: R_{mi} and R_{Mi}, $R_{mi} \leq R_{Mi}$. Also given is the number $\bar{K} > 0$. We consider the differential inclusion

$$(\dot{\sigma}, \dot{\varepsilon}) \in F(\sigma, \varepsilon) , \tag{6.1}$$

defined in the interior points of the regions $\mathcal{O}_i$ by the set

$$F(\sigma, \varepsilon) = \{(\bar{K}\varepsilon, -R) \,|\, R \in [R_{mi}, R_{Mi}]\} . \tag{6.2}$$

In common boundary points of the regions $\mathcal{O}_i$ the field of sets F is defined by upper semicontinuity.

In each region $\mathcal{O}_i$ we consider two vector fields

$$V_i(\sigma, \varepsilon) = (\bar{K}\varepsilon, -R_{*i}) , \qquad v_i(\sigma, \varepsilon) = (\bar{K}\varepsilon, -R_{**i})$$

where

$$R_{*i} = \begin{cases} R_{mi} & \text{for } \varepsilon > 0, \\ R_{Mi} & \text{for } \varepsilon \leq 0, \end{cases} \qquad R_{**i} = \begin{cases} R_{Mi} & \text{for } \varepsilon > 0, \\ R_{mi} & \text{for } \varepsilon \leq 0. \end{cases}$$

The vector fields v_i, V_i generate discontinuous piecewise-smooth vector fields v, V in the entire half-plane $\sigma \geq 0$.

Assume that the following conditions A, B, and C are satisfied.

A. The intersections of the boundary curves γ_j are pairwise transversal if the intersection is not at the origin $\sigma = \varepsilon = 0$.

B. Each field V_i may touch the boundary of the region $\mathcal{O}_i$ only at isolated points, which are not the intersection points of the boundary curves. All the contact points of the phase curves of the field V_i with the boundary of the region $\mathcal{O}_i$ are of 1st order.

C. The numbers R_{mi}, R_{Mi} satisfy the inequalities

$$R_{Mi} > 0 , \qquad i = 1, 2, \ldots, \ell ;$$
$$R_{mi} > 0 \qquad \text{for } \mathcal{O}_i \subset \{ (\sigma, \varepsilon) | \varepsilon \geq 0\}.$$

LEMMA 6.1. Assume that conditions A, B, C are satisfied. Then from each point $\varepsilon > 0$, $\sigma = 0$ originates a majorizing curve in the half-plane $\sigma \geq 0$ of the differential inclusion (6.1), (6.2), and this curve is a phase curve of the differential equation

$$(\dot{\sigma}, \dot{\varepsilon}) = V(\sigma, \varepsilon) .$$

Assume that condition B refers to the field v (and not V), and condition C is augmented with the condition

$$R_{mi} > 0 \qquad \text{for } \bar{O}_i \cap \{ (\sigma, \varepsilon) | \varepsilon \geq -\varepsilon_1, \sigma > 0 \} \neq \emptyset ,$$

where $\varepsilon_1 > 0$ is a constant. Then we have the following lemma.

LEMMA 6.2. Assume that condition A and the modified conditions B and C are satisfied. Then from each point $\varepsilon > 0$, $\sigma = 0$ originates a minorizing curve in the half-plane $\sigma \geq 0$ of the inclusion (6.1), (6.2), and this curve is a phase curve of the differential equation

$$(\dot{\sigma}, \dot{\varepsilon}) = v(\sigma, \varepsilon) .$$

Proof is the same for both lemmas. It geometrically follows from the fact that at each point of the majorizing (minorizing) curve the tangent vectors of all the phase curves of the inclusion (6.1), (6.2) point into the corresponding region Ω_1 (Ω_2) from the definition of the majorizing (minorizing) curve.

6.2. Let us prove Theorem 5.1. Consider the closed-loop dynamic system

$$\dot{x} = f(t, x, u) , \tag{6.3}$$

$$\dot{u} = \begin{cases} -u & \text{for } |u| > 1 , \\ -\alpha_M \operatorname{sign} \sigma & \text{for } \sigma\dot{\sigma} > 0,\ |u| \leq 1 , \\ -\alpha_m \operatorname{sign} \sigma & \text{for } \sigma\dot{\sigma} \leq 0,\ |u| \leq 1 , \end{cases} \tag{6.4}$$

where Eq. (6.3), the constraint function σ, and the constants α_M, α_m satisfy the conditions of Theorem 5.1. By (6.4), after a time interval not exceeding $\varkappa - 1$ we permanently have $|u| \leq 1$.

If σ does not vanish for a sufficiently long time, then because of sign constancy of σ we obtain the equality $u = -\operatorname{sign} \sigma$ after a time not exceeding $\varkappa - 1 + 2/\alpha_m$. After that, this equality is satisfied in 1st order sliding mode. By assumption 3.2, σ changes its sign after a finite time. Thus, for any initial conditions, $\sigma(t)$ crosses the zero after a finite time for $u \in [-1, 1]$.

Now consider the dynamics of system (6.1), (6.2) within the linear zone $|\sigma| < \sigma_0$ for $u \in [-1, 1]$. If conditions 3.1-3.5 hold, we have the equality

$$\ddot{\sigma} = L_u L_u \sigma + \frac{\partial \dot{\sigma}}{\partial u} \dot{u} , \tag{6.5}$$

where $|L_u L_u \sigma| \leq C_0$, $\partial\dot{\sigma}/\partial u \in [K_m, K_M]$. If conditions 3.1′-3.5′ hold with the relaxed smoothness requirements, $\dot{\sigma}(t, x, u)$ is a locally Lipschitzian function and thus by 3.3′-3.5′ the absolutely continuous function $\dot{\sigma}(t, x(t), u(t))$ satisfies at its points of differentiability the differential inclusion

$$\ddot{\sigma} \in [-C_0, C_0] + [K_m, K_M] \dot{u} . \tag{6.6}$$

Here and in what follows, a numerical operation between numerical sets is defined as the set of all possible results of the corresponding operations over all possible elements taken in the order of the corresponding sets.

Equation (6.5) and differential inclusion (6.6) are called respectively the real sliding equation and the real sliding differential inclusion.

Consider the dynamics of system (6.3), (6.4) in the linear zone $|\sigma| < \sigma_0$ for $|u| \leq 1$. In this case, the paths of the differential inclusion (6.6) are the paths of the following inclusion defined by upper semicontinuity:

$$\ddot{\sigma} \in \begin{cases} [-C_0, C_0] - [K_m, K_M] \alpha_M \operatorname{sign} \sigma & \text{for } \sigma\dot{\sigma} > 0 , \\ [-C_0, C_0] - [K_m, K_M] \alpha_m \operatorname{sign} \sigma & \text{for } -K_m(1 - u) \leq \dot{\sigma} \operatorname{sign} \sigma \leq 0 , \\ [-C_0, C_0] + [K_m, K_M] [-\alpha_m, 1] \operatorname{sign} \sigma & \text{for } \dot{\sigma} \operatorname{sign} \sigma \leq -K_m(1 - u_0) . \end{cases} \tag{6.7}$$

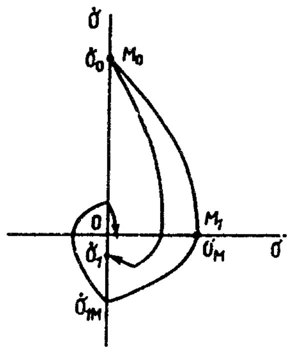

Fig. 10

In (6.7) we use the condition that the equality $u = 1$ for $\sigma \neq 0$ (and with initial conditions $\sigma = 0$, $|u| < 1$) may be reached only when $\dot{\sigma}\,\text{sign}\,\sigma \leq -K_m(1 - u_0)$. This follows from the inequality $|u_{eq}| < u_0 < 1$ and the inclusion

$$\dot{\sigma}(t, x, u) \in [K_m, K_M](u - u_{eq}(t, x)) \tag{6.8}$$

established by conditions 3.3′-3.4′. By Lemma 6.1 the majorizing curves of inclusion (6.7) in the half-planes $\sigma \geq 0$, $\sigma \leq 0$ have the form

$$\begin{aligned} &|\sigma| + \frac{1}{2}\dot{\sigma}^2 \,/\, (K_m\alpha_M - C_0) = \text{const} \quad \text{for } \sigma\dot{\sigma} > 0\,, \\ &|\sigma| + \frac{1}{2}\dot{\sigma}^2 \,/\, (K_M\alpha_m + C_0) = \text{const} \quad \text{for } \sigma\dot{\sigma} \leq 0\,. \end{aligned} \tag{6.9}$$

Suppose that the constraint function σ vanishes for the first time when the point (t, x, u) is represented in the σ, $\dot{\sigma}$ plane by the point $M_0(0, \dot{\sigma}_0)$ (Fig. 10), where for definiteness $\dot{\sigma} > 0$. Then the maximum possible deviation of σ from zero during the motion in the half-plane $\sigma > 0$ corresponds to the point $M_1(\sigma_M, 0)$. Issuing a majorizing curve (6.9) from the point M_0, we obtain a sufficient condition for the inequality $\sigma_M < \sigma_0$:

$$\frac{1}{2}\dot{\sigma}_0^2 \,/\, (K_m\alpha_M - C_0) < \sigma_0 \tag{6.10}$$

By inequality (6.8), the inequalities $|u| \leq 1$, $|u_{eq}| < u_0$, and one of the conditions of the theorem (inequality (5.5)), we have either (6.10) or

$$\alpha_M > 4K_M \,/\, \sigma_0\,.$$

The last inequality, by Proposition 4.2, is also sufficient for invariance of the linear zone. We may thus assume that all subsequent motion is in the linear zone.

Suppose that the next intersection of the majorizing curve (6.9) with the axis $\sigma = 0$ is at the point $(0, \dot{\sigma}_{1M})$, $\dot{\sigma}_{1M} < 0$. Then, setting $\sigma_{0M} = \dot{\sigma}_0$, we have

$$\left|\frac{\dot{\sigma}_{1M}}{\dot{\sigma}_{0M}}\right| = \sqrt{\frac{K_M\alpha_m + C_0}{K_m\alpha_M - C_0}} = q_M\,.$$

By the condition of the theorem, $q_M < 1$. Hence, the actual successive intersection points $\dot{\sigma}_0, \dot{\sigma}_1, \dot{\sigma}_2, \ldots$ where the projections of the solution of (6.3), (6.4) on the plane σ, $\dot{\sigma}$ cross the axis $\sigma = 0$ satisfy the inequality

$$\left|\frac{\dot{\sigma}_{i+1}}{\dot{\sigma}_i}\right| \leq q_M < 1\,.$$

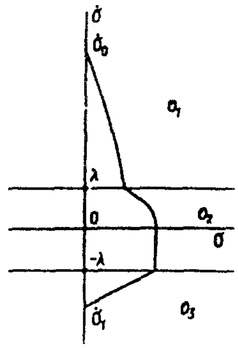

Fig. 11

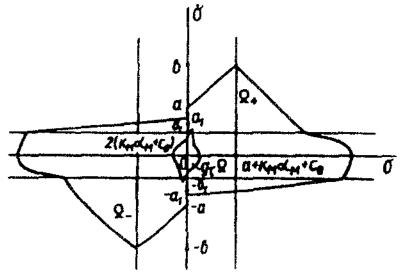

Fig. 12

Thus, the path ($\sigma(t)$, $\dot\sigma(t)$) is "clamped" inside the majorizing curve (6.9) that winds into the point $\sigma = \dot\sigma = 0$.

We will show that the algorithm converges in finite time. After a finite number of turns of the curve ($\sigma(t)$, $\dot\sigma(t)$) around the origin, we get permanently $|\dot\sigma| \le K_m(1 - u_0)$. This means that $|u| < 1$. Under these conditions, $\dot\sigma$ varies monotonically on sign-constancy sections of σ, and

$$K_m\alpha_m - C_0 \le |\ddot\sigma| \le K_M\alpha_M + C_0 \,.$$

The convergence time of the algorithm for $|\dot\sigma_0| \le K_m(1 - u_0)$ is thus bounded by

$$\frac{2}{K_m\alpha_m - C_0}(|\dot\sigma_0| + |\dot\sigma_1| + \ldots) \le \frac{2|\dot\sigma_0|}{(1 - q_M)(K_m\alpha_m - C_0)} \,.$$

The existence of a uniform convergence-time bound for initial conditions in some neighborhood of the sliding set $\sigma = \dot\sigma = 0$ is obvious. The convergence time bound in subsec. 5.2 has been obtained by the same technique, allowing the different values of $\ddot\sigma$ for $\sigma\dot\sigma > 0$ and $\sigma\dot\sigma < 0$. The constant T_0 is the time to reach the inequality $|\dot\sigma| \le K_m(1 - u_0)$. To complete the proof, it remains to show that the system path cannot leave the 2nd order sliding mode.

Assume the contrary. Take two arbitrarily close points P and P_1 of the path ($\sigma(t)$, $\dot\sigma(t)$). Let

$$P = (\sigma(t), \dot\sigma(t)) = (0,0) \,, \qquad P_1 = (\sigma(t+\tau), \dot\sigma(t+\tau)) \ne P \,,\ \tau > 0 \,.$$

Reverse the time in differential inclusion (6.6), (6.4). The family of curves (6.9) is now a family of minorizing curves in relation to the new reverse-time differential inclusion. Issuing the curve (6.9) from the point P_1, we verify that the point P cannot be at the origin $\sigma = \dot\sigma = 0$. Q.E.D.

6.3. Proof of Bounds (5.9), (5.10). Like the proof of the upper bound of convergence time in the theorem, we can prove the lower bound of convergence time. In this case, we also form the sum of a geometrical progression, but now with a common factor q_m. Since the first term of the progression is proportional to $\dot\sigma_0$, denoting the convergence time by T we obtain an inequality of the form

$$C_1 \le |\dot\sigma_0| \,/\, T \le C_2 \,,$$

where C_1, C_2 are constants. The bound (5.9) gives the specific form of these constants.

Using the lemma on majorizing curves, we obtain that the maximum value $|\sigma_M|$ of the variable $|\sigma|$ attained between successive crossings of the axis $\sigma = 0$ by the phase curve of the differential inclusion (6.6), (6.4) at the points $\dot\sigma_0$, $\dot\sigma_1$ satisfies the inequality

$$\frac{1}{2(K_M\alpha_M + C_0)}\dot\sigma_0^2 \le |\sigma_M| \le \frac{1}{2(K_m\alpha_M - C_0)}\dot\sigma_0^2 \,.$$

This and bound (5.9) give the bound (5.10).

7. PROOF OF THEOREM 5.2 ON WINDING ALGORITHM. REAL SLIDING

The proof of Theorem 5.2 consists of several lemmas.

7.1. Assume that the half-plane $\sigma \geq 0$ of the σ, ε plane is partitioned into a finite number of closed regions O_1, O_ℓ by a finite number of smooth curves γ_j. As in subsec. 6.1, the differential inclusion

$$(\dot{\sigma}, \dot{\varepsilon}) \in F(\sigma, \varepsilon) ,$$

$$F(\sigma, \varepsilon) = \{ (\bar{K}\varepsilon, -R) \mid \mathbf{R} \in [R_{mi}, R_{Mi}] \quad \text{for } (\sigma, \varepsilon) \in \operatorname{int} O_i \}$$

is defined in the half-plane $\sigma \geq 0$, and at the boundary points the field of sets F is defined by upper semicontinuity. Here $\bar{K} > 0$, $R_{mi} \leq R_{Mi}$ are some constants, $i = 1, \ldots, \ell$.

The linear operator

$$g_\nu : (\sigma, \varepsilon) \longmapsto (\nu^2 \sigma, \nu \varepsilon) ,$$

where $\nu > 0$, takes the partition $O_1, \ldots, O_\ell$ into the partition $g_\nu O_1, \ldots, g_\nu O_\ell$. The partition $g_\nu O_i$ and the same constants K, R_{mi}, R_{Mi}, $i = 1, \ldots, \ell$ define a new differential inclusion F_1.

LEMMA 7.1. The operator g_ν establishes orbital equivalence of the differential inclusions F and F_1. If the conditions of Lemmas 6.1 and 6.2 hold for F, then these conditions also hold for F_1, and the operator g_ν takes majorizing curves into majorizing curves, and minorizing curves into minorizing curves.

Proof. The operator g_ν takes the field of sets F into the field $g_{\nu*}F$, where

$$g_{\nu*}F(\sigma, \varepsilon) = g_\nu F(g_\nu^{-1}(\sigma, \varepsilon)) .$$

The lemma follows from the easily verified identity $g_{\nu*}F = \nu F_1$.

7.2. Suppose that the half-plane $\sigma \geq 0$ in the σ, $\dot{\sigma}$ plane is partitioned into three regions (Fig. 11),

$$\begin{aligned} O_1 &= \{(\sigma, \dot{\sigma}) \mid \dot{\sigma} \geq \lambda, \ \sigma \geq 0\} , \\ O_2 &= \{(\sigma, \dot{\sigma}) \mid \dot{\sigma} \leq \lambda; \ \sigma \geq 0\} , \\ O_3 &= \{(\sigma, \dot{\sigma}) \mid \dot{\sigma} \leq -\lambda, \ \sigma \geq 0\} , \end{aligned} \tag{7.1}$$

where $\lambda = \text{const} > 0$, and the following differential inclusion is given:

$$\ddot{\sigma} \in [R_{mi}, R_{Mi}] \quad \text{for } (\sigma, \dot{\sigma}) \in O_i, \ i = 1, 2, 3;$$

$$\begin{aligned} R_{m1} &= K_m \alpha_M - C_0 , & R_{M1} &= K_M \alpha_M + C_0 , \\ R_{m2} &= K_m \alpha_m - C_0 , & R_{M2} &= K_M \alpha_M + C_0 ; \\ R_{m3} &= K_m \alpha_m - C_0 , & R_{M3} &= K_m \alpha_M + C_0 . \end{aligned} \tag{7.2}$$

Here K_m, K_M, C_0 are defined in condition 3.3′, α_m, α_M are the parameters of the winding algorithm (subsec. 5.3).

LEMMA 7.2. Under the conditions of Theorem 5.2 there exist $\alpha > 0$ and $q_0 \in (0, 1)$ such that the majorizing curve of the differential inclusion (7.2) in the half-plane $\sigma \geq 0$ issuing according to Lemma 6.1 from the point $\sigma = 0$, $\dot{\sigma} = \dot{\sigma}_0 > a$ returns to the axis $\sigma = 0$ at a point $\dot{\sigma}_1 < 0$ such that

$$|\dot{\sigma}_1 / \dot{\sigma}_0| < q_0 .$$

To prove the lemma, we have to fix an arbitrary point a' on the axis $\sigma = 0$, issue from this point a majorizing curve Γ_ν that corresponds to the partition $g_\nu O_i$, and then start reducing ν.

The partition $g_\nu O_i$ differs from the partition O_i only by the parameter $\lambda' = \nu\lambda$. For sufficiently small $\nu_0 > 0$, the majorizing curve Γ_ν crosses the axis $\sigma = 0$ at the point $|a''/a'| = g_0 < 1$. Then we set $a = \nu_0^{-1} a'$.

7.3. Set $\lambda = 2(K_M \alpha_M + C_0)$ in Lemma 7.2 and choose the corresponding values $q_0 \in (0, 1)$ and $a > 0$. From the point $\sigma = 0$, $\dot{\sigma} = a$ in the σ, $\dot{\sigma}$ plane issue the phase curve of the differential equation

$$\ddot{\sigma} = K_M \alpha_M + C_0$$

and continue it to the intersection with the straight line $\sigma = a + K_M \alpha_M + C_0$ at the point with the coordinate $\dot{\sigma} = b$ (Fig. 12). From the point $(\sigma, \dot{\sigma}) = (a + K_M \alpha_M + C_0, b)$ issue a majorizing curve of inclusion (7.2) with the same constants R_{mi}, R_{Mi} as in Lemma 7.2, $\lambda = 2(K_M \alpha_M + C_0)$, and continue it to the next intersection with the straight line $\sigma = a + K_M \alpha_M +$

C_0 at the point with the coordinate $\dot{\sigma} = -b_1$ and onward to the intersection with the axis $\sigma = 0$ at the point with the coordinate $\dot{\sigma} = -a_1$.

Clearly, $b > a$, $a_1 > b_1$. It is easy to show that for large a, both $b - a$ and $a_1 - b_1$ are bounded. Hence, by Lemma 7.2, for large a we have

$$a_1 / a < q_0 < 1 . \tag{7.3}$$

Fix this value of a. Denote by $\Omega_+ (a)$ the bounded closed set of points in the σ, $\dot{\sigma}$ plane that are trapped between the constructed curve and the axis $\sigma = 0$ (Fig. 12). Let $\Omega_-(a)$ be the set obtained from $\Omega_+ (a)$ by central symmetry about the point $\sigma = \dot{\sigma} = 0$. Let

$$\Omega(a) = \Omega_+(a) \cup \Omega_-(a) .$$

Assume that τ is sufficiently small, so that

$$b\tau < K_m(1 - u_0 - \tau) \qquad \tau < 1 - u_0 . \tag{7.4}$$

Denote

$$G_\tau(a) = \{ (t, x, u) | (\sigma , \dot{\sigma}) \in g_\tau \Omega(a) \} .$$

Note that the projections of the set $g_\tau\Omega(a)$ on the axes σ and $\dot{\sigma}$ are of order τ^2 and τ, respectively.

LEMMA 7.3. Assume that conditions (7.3), (7.4) are satisfied. Then under the conditions of Theorem 5.2, every system path for which $\sigma(t') = 0$, $|\dot{\sigma}(t')| \leq a\tau$ at some instant t' does not leave the set $G_\tau(a)$ for all $t > t'$.

Proof. The incorrect switching zone is the set of points P in the σ, $\dot{\sigma}$ plane where sign $\sigma\dot{\sigma}$ and sign $\sigma \cdot \delta_i\ \sigma$, or sign σ and sign $\sigma(t_i)$ may be unequal for some path (σ, $\dot{\sigma}(t)$) of the control system passing through P. Switching errors arise when the path crosses the axes $\sigma = 0$ and $\dot{\sigma} = 0$. It is easy to prove that the incorrect switching zone is a priori contained in the set of points reachable as a result of incorrect switchings in time τ from the axis $\sigma = 0$ and in time 2τ from the axis $\dot{\sigma} = 0$.

For paths originating from the interval $\sigma = 0$, $|\dot{\sigma}| \leq a\tau$ in the σ, $\dot{\sigma}$ plane, the incorrect switching zone is covered by the union S of sets defined by the inequalities $|\sigma| \leq (a + K_M\alpha_M + C_0)\tau^2$ and $|\dot{\sigma}| \leq 2(K_M\alpha_M + C_0)\tau$. The partition of the σ, $\dot{\sigma}$ plane into S and its complement $\bar{S}$ corresponds to an upper semicontinuous differential inclusions, whose solutions are a priori the paths of the system (σ, $\dot{\sigma}$) starting from the instant t' when they reach the interval $\sigma = 0$, $\dot{\sigma} \in [-a\tau, a\tau]$. The boundary curves of the set $g_\tau\Omega(a)$ are majorizing curves of this differential inclusion in the half-planes $\sigma \geq 0$ and $\sigma \leq 0$, respectively.

Inequality (7.4) guarantees that the path ($\sigma(t)$, $\dot{\sigma}(t)$) does not reach $G_\tau(a)$ in the process of real 1st order sliding on the constraint $u = 1$ or $u = -1$, and does not go into this sliding mode from $G_\tau (a)$.

7.4. Fix the value a_0 of the parameter a when (7.3) is satisfied. To prove the theorem, it remains to show that any system path reaches in finite time the manifold $\sigma = 0$ for $|\dot{\sigma}| \leq a_0\tau$, and that this time can be upper bounded by a constant independent of τ, and in some neighborhood of the set $\sigma\dot{\sigma} = 0$ by a constant independent of both τ and the initial conditions.

LEMMA 7.4. Let $\gamma \in (0, K_m(1 - u_0))$. Then there exists a constant T such that for sufficiently small τ any path (σ(t), $\dot{\sigma}$(t)) that crosses at the instant t' the interval $\sigma = 0$, $|\dot{\sigma}| < g$ will cross by the time $t' + T$ the interval $\dot{\sigma} = 0$, $|\dot{\sigma}| \leq a_0\tau$.

Proof. Let $a = \gamma/\tau$. Then for small τ we have (7.3), (7.4), and the set $G_\tau (\gamma/\tau)$ is invariant in the sense of Lemma 7.3. All the paths lying in $g_\tau\Omega(\gamma/\tau)$ have the following property: the ratio of the absolute values of the ordinates $\dot{\sigma}_j$ of successive crossing points of the path with the axis $\sigma = 0$ does not exceed $q_0 < 1$ as long as $|\dot{\sigma}| \geq a_0\tau$. Therefore, the intersection of $|\dot{\sigma}_{j0}| \leq a_0\tau$ is observed after finitely many turns of the path around the origin $\sigma = \dot{\sigma} = 0$. The existence of a uniform bound (for small τ) of the time to reach the interval $\sigma = 0$, $|\dot{\sigma}| \leq a_0\tau$ is proved exactly like finiteness of the convergence time of the ideal sliding algorithm in subsec. 6.2. Q.E.D.

LEMMA 7.5. Let $\gamma > 0$. Then there exists a constant T_1 such that for sufficiently small τ any path of the system ($\sigma(t)$, $\dot{\sigma}(t)$) in time T_1 after crossing the set $\sigma = 0$, $|\dot{\sigma}| \geq \gamma$ will cross the interval $\sigma = 0$, $|\dot{\sigma}| \leq \gamma$.

Proof. As in subsec. 6.2 we show that for small τ the system path does not leave the linear zone $|\sigma| < \sigma_0$. As $\tau \to 0$, the integral funnels of the real sliding differential inclusions corresponding to the discrete switching algorithm (subsec. 5.3) and the ideal winding algorithm (subsec. 5.1) are close to each other [12]. Therefore, the functions of the successive crossings of the majorizing curves with the axis $\sigma = 0$ are also close to one another. Thus, since the set $\sigma = 0$, $|\dot{\sigma}| \geq \gamma$, $|\dot{\sigma}| \leq 2K_M$ is compact for sufficiently small τ there exists a uniform bound $\tilde{q}_0 < 1$ of the ratio $|\dot{\sigma}_1|/|\dot{\sigma}_0|$ of the absolute values of the ordinates of the intersection points of the system path with the axis $\sigma = 0$ for $|\dot{\sigma}_0| \geq \gamma$. Q.E.D.

Since Lemmas 7.3-7.5 are true, it remains to prove uniform convergence of the algorithm on the set of initial conditions in the neighborhood of a 2nd order sliding set. As an appropriate set we may take the set $|\sigma| \leq \delta$, $|u| \leq 1$ for a sufficiently small $\delta > 0$.

8. ADDITIONAL RESULTS ABOUT THE WINDING ALGORITHM

We introduce some notation that will simplify subsequent statements. Let $\xi: \Lambda_1 \times \Lambda_2 \mapsto R^s$ be a locally Lipschitzian function, Λ_1, Λ_2 smooth manifolds. Denote by $\ell_{\lambda_1}\xi(\lambda_1, \lambda_2)$ the local Lipschitz constant of the function ξ by the coordinate λ_1 in some local coordinates (in the Euclidean metric). We assume that the expression containing the notation $\ell_{\lambda_1}\xi$ satisfies some inequality, implying the existence of an atlas of local charts on $\Lambda_1 \times \Lambda_2$ and choice of local Lipschitz constants of the function ξ at the points $(\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2$ ensuring that this inequality is satisfied.

8.1. Assume that the equation

$$\dot{x} = f(t, x, u) \tag{8.1}$$

and the constraint functions $\sigma(t, x)$ satisfy assumptions 3.1'-3.5'. The following condition is assumed to hold for some locally Lipschitzian real function $\xi(t, x)$:

α) $\xi(t, x) > 0$ for all $\xi(t, x) > 0$. There exists a positive constant $\mathcal{D}$ so that for all $u/u \leq x$ and (t, x) such that $|\sigma(t, x)| < \sigma_0$ we have the inequality

$$\ell_t\xi(t,x) + \ell_x\xi(t,x)\|f(t,x,u)\| \leq \mathcal{D}\xi(t,x)\,.$$

If the function ξ is smooth, then condition α implies boundedness of $\dot{\xi}/\xi$.

Let $\sigma_1(t, x) = \sigma(t, x)\xi(t, x)$. By (8.1) the derivative $\dot{\sigma}_1$ exists almost everywhere. Let

$$\dot{u} = \begin{cases} -u & \text{for } |u| > 1\,, \\ -\alpha_M \operatorname{sign}\sigma_1 & \text{for } \sigma_1\dot{\sigma}_1 > 0,\ |u| \leq 1\,, \\ -\alpha_m \operatorname{sign}\sigma_1 & \text{for } \sigma_1\dot{\sigma}_1 \leq 0,\ |u| \leq 1\,. \end{cases} \tag{8.2}$$

THEOREM 8.1. Assume that Eq. (8.1) and the constraint function σ satisfy assumptions 3.1'-3.5', and the function $\xi(t, x)$ is positive and has a bounded logarithmic derivative $\dot{\xi}/\xi$ by (8.1) or, if it is nondifferentiable, then satisfies property α.

Let $\sigma_1 = \sigma(t, x)\cdot\xi(t, x)$. Then for sufficiently large $\alpha_m > 0$ and $\alpha_M/\alpha_m > 1$ algorithm (8.2) is a 2nd order sliding algorithm on the constraint $\sigma = 0$.

Outline of Proof. It is easy to show that within the linear zone $|\sigma| < \sigma_0$ the absolutely continuous function $\sigma_1(t, x(t))$ satisfies the differential inclusion

$$\dot{\sigma}_1 \in \xi(\dot{\sigma} + [-\mathcal{D}, \mathcal{D}]\sigma)\,.$$

Then we apply Lemma 6.1 on majorizing curves. The proof compares the winding algorithm with observations of the constraint σ and algorithm (8.2). The incorrect switching zone is covered by the interior of the parabola $|\dot{\sigma}| \leq \lambda\sqrt{|\sigma|}$, which is invariant under the transformations $g(\nu)$ (see subsec. 7.1). A detailed proof leading to the same result is given in [22].

We can also show that replacement of $\dot{\sigma}_1$ with $\delta_i\sigma$ and of sign σ_1 with sign $\sigma_1(t_1)$ in (8.2) reduces it to a real 2nd order sliding algorithm on σ [22].

The theorem shows that there is a whole family of winding algorithms. It is easy to show that the set of functions ξ satisfying condition α is closed under the operations of addition, multiplication, division, and a number of other transformations.

8.2. Consider the dynamic system linear in control

$$\dot{x} = a(t,x) + b(t,x)u\,, \tag{8.3}$$

where $x \in \mathbf{R}^n$, a, b are locally Lipschitzian functions, $u \in \mathbf{R}$. Assume that the constraint function $\sigma(t, x)$ has locally Lipschitzian partial derivatives. Suppose that we have found a function $\Phi(x)$ such that for all x we have $\Phi(x) \geq \text{const}_1 > 0$, $\ell_x\Phi(x) \leq \text{const}_1$, and the following conditions A and B hold.

A. $\sigma'_x, \ell_x\sigma'_x \cdot \Phi, \ell_x\sigma'_t, \ell_t\sigma'_x, \sigma'_t/\Phi, \ell_t\sigma'_t/\Phi, a/\Phi, \ell_t a/\Phi, \ell_x a, b, \ell_t b, \ell_x b \cdot \Phi$. are bounded uniformly in t, x.

B. There exists a constant $\delta_0 > 0$ such that $\sigma_x' b \geq \delta_0$ for all t, x.

Let

$$u = \mu k \cdot \Phi(x), \tag{8.4}$$

where $k > 0$.

Assume that $\Phi_1(x)$ is a positive locally Lipschitzian function separated from zero, and the ratio $\ell_x \Phi_1 \cdot \Phi/\Phi_1$ is uniformly bounded. Then for sufficiently large k the binary algorithm formed by the feedback (8.4) and the equation

$$\dot{\mu} = \begin{cases} -\mu & \text{for } |\mu| > 1, \\ -\alpha_M \operatorname{sign} \sigma & \text{for } \sigma \cdot (\sigma\Phi_1)^{\cdot} > 0, \ |\mu| \leq 1, \\ -\alpha_m \operatorname{sign} \sigma & \text{for } \sigma \cdot (\sigma\Phi_1)^{\cdot} \leq 0, \ |\mu| \leq 1. \end{cases}$$

is a 2nd order sliding algorithm on the constraint $\sigma = 0$ for sufficiently large $\alpha_m > 0$ and $\alpha_M/\alpha_m > 1$.

The set of functions Φ_1 is closed under addition, multiplication, division, and positive exponentiation.

In many practically important cases of Φ and Φ_1, this result can be obtained as a consequence of Theorem 8.1. In general, it is proved in [23], where parameter bounds are also given. The fact that [23] deals with directionally differentiable, and not locally Lipschitzian, functions does not affect the proof.

Discretized observations transform all the algorithms of this section into real sliding algorithms that track relationships of the form $|\sigma| \leq c\Phi(x)\tau^2$, where $c = \text{const} > 0$, and τ is the interval between observations.

8.3. Stability of the Algorithm to Nonidealities. Consider the case when the controlled plant is described by the equation

$$\dot{x} = f(t, x, u) + \Delta f(t, x, u), \tag{8.5}$$

where the small continuous "noise" Δf violates assumptions 3.3′-3.5′ (but does not violate 3.2′), although all the assumptions 3.1′-3.5′ are satisfied for the unperturbed equation (8.1) and the constraint function σ.

Recall that the operator L_u is defined by the equality

$$L_u(\cdot) = \frac{\partial}{\partial t}(\cdot) + \frac{\partial}{\partial x}(\cdot) f(t, x, u).$$

Assume that in the linear zone we have

$$|\sigma'_x \Delta f| \leq d_1, \qquad \ell_x L_u \sigma \cdot \|\Delta f\| \leq d_2,$$

where d_1, d_2 are positive constants, and the conditions of Theorem 5.1 hold. Then for sufficiently small d_1, d_2 the winding algorithm

$$\dot{u} = \begin{cases} -u & \text{for } |u| > 1, \\ -\alpha_M \operatorname{sign} \sigma & \text{for } \sigma\dot{\sigma} > 0, \ |u| \leq 1, \\ -\alpha_m \operatorname{sign} \sigma & \text{for } \sigma\dot{\sigma} \leq 0, \ |u| \leq 1. \end{cases}$$

takes in a finite time the point (t, x, u) into a set of the form $\{(t, x, u)/|\sigma| < C_1(d_2)(d_1{}^2, |\dot{\sigma}| < C_2(d_2)d_1\}$ and holds it there. For $\alpha_2 \to 0$, C_1 and C_2 tend to constants:

$$\lim_{d_2 \to 0} C_1 = \frac{(1 + q_M)^2}{2(K_m\alpha_m - C_0)(1 - q_M)^2},$$

$$\lim_{d_2 \to 0} C_2 = \frac{1 + q_M}{1 - q_M},$$

where $q_M = \sqrt{(K_M\alpha_m + C_0) / (K_m\alpha_M - C_0)}$.

This assertion is true for arbitrary $\alpha_1 > 0$, $\alpha_2 > 0$ (specific formulas exist for C_1 and C_2), but in this case the restrictions on the parameters α_M, α_m should be satisfied with a "safety margin". The corresponding formulas are quite cumbersome and are not given here.

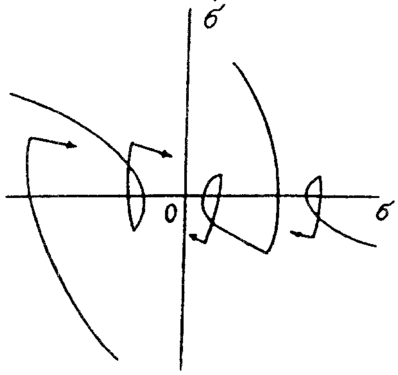

Fig. 13

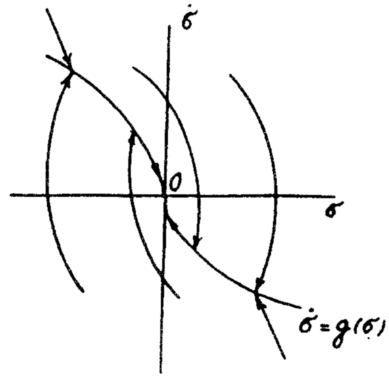

Fig. 14

Another form of nonideality to which the winding algorithm is sensitive are observation errors in $\sigma(t)$. Such errors may render the sign of σ totally unreliable. As a result, the accuracy of the algorithm becomes equal to the accuracy of the $\mathcal{A}_\mu$-algorithm: the inequality $|\sigma| \le 4\ K_M/\alpha_m$ is ensured. For a fixed observation increment τ, small observation errors do not interfere with the operation of the discrete-observation algorithm. The set covering the incorrect switching zone (subsec. 7.3) does not change.

We can show that the use of an adaptive observation increment that depends on observations (for instance, in the simplest form $t_{i+1} = t_i = \lambda\ \|\sigma(t_i)\|$ for $\lambda|\sigma(t_i)| \in [\tau_m, \tau_M]$, $\lambda > 0$, τ_m, τ_M is the minimum and the maximum increment) ensures accuracy of the same order as the observation error accuracy or of order τ_m^2 with exact observations.

9. OTHER SLIDING ALGORITHMS

The controlled plant is described by Eq. (8.1), σ is the constraint function, and assumptions 3.1′-3.5′ are satisfied.

9.1. Drift Algorithm. Assume that the constraint function $\sigma(t)$ is observed at discrete time instants $t_0, t_1, t_2, \ldots$. Using the notation of subsec. 5.3, let

$$u = \begin{cases} -u(t_i) & \text{for } |u(t_i)| > 1, \\ \alpha_M \operatorname{sign} \delta_i\sigma & \text{for } \delta_i\sigma \cdot \sigma(t_i) > 0,\ |u(t_i)| \le 1, \\ \alpha_m \operatorname{sign} \delta_i\sigma & \text{for } \delta_i\sigma \cdot \sigma(t_i) \le 0,\ |u(t_i)| \le 1, \end{cases} \tag{9.1}$$

where $\alpha_M > \alpha_m$, $t \in [t_i, t_{i+1})$.

$$t_{i+1} - t_i = \begin{cases} \tau_M & \text{for } \nu|\sigma(t_i)|^\rho > \tau_M, \\ \nu|\sigma(t_i)|^\rho & \text{for } \tau_m \le \nu|\sigma(t_i)|^\rho \le \tau_M, \\ \tau_m & \text{for } \nu|\sigma(t_i)|^\rho < \tau_m, \end{cases} \tag{9.2}$$

where $\tau_M > \tau_m > 0$, τ_M, τ_m are constants, $\nu > 0$, $0.5 \le \rho < 1$, τ_m is a small parameter.

Assume that the initial conditions are within the linear zone, and

$$|\sigma(t_0, x(t_0))| \le \delta_0 < \sigma_0 \qquad \delta_0 = \text{const} > 0\,.$$

Then for sufficiently large α_m, α_M/α_m and sufficiently small ν the drift algorithm (9.1) with a variable increment (9.2) is a real 2nd order sliding algorithm on σ.

Figure 13 is a typical path $(\sigma(t), \dot\sigma(t))$ of the drift algorithm. An important feature of the algorithm is that it converges without overshooting on σ. The corresponding theorems are proved in [19].

9.2. Algorithm with Specified Variation of the Constraint Function. This algorithm has the form

$$\dot u = \begin{cases} -u & \text{for } |u| > 1, \\ -\alpha \operatorname{sign}(\dot\sigma - g(\sigma)) & \text{for } |u| \le 1, \end{cases} \tag{9.3}$$

where $\alpha > 0$. The function g is chosen so that the solution of the equation $\dot{\sigma} = g(\sigma)$ reaches in finite time the point $\sigma = 0$, and $g_\sigma{}' g(\sigma)$ is bounded. For instance, we may take

$$g = -\lambda \operatorname{sign} \sigma \cdot |\sigma|^\gamma \text{ , where } \lambda > 0, \; 0{,}5 \leq \gamma < 1$$

Algorithm (9.3) for sufficiently large α is a 2nd order sliding algorithm on σ (with $\sigma(t_0) < \delta < \sigma_0$). Replacement of $\dot{\sigma} - g(\sigma)$ with $\delta_i \sigma - g(\sigma(t_i))(t_i - t_{i-1})$ reduces (9.3) to a real sliding algorithm on σ of order not higher than 2. Real sliding is of 2nd order, in particular, when $g(\sigma) = -\lambda \operatorname{sign} \sigma \cdot |\sigma|^{1/2}$. The introduction of a variable increment (9.2) endows the real sliding algorithm with stability to observation errors. Figure 14 shows a typical curve $(\sigma(t), \dot{\sigma}(t))$.

9.3. Asymptotic Winding Algorithm. Let

$$u = u^0 + u^1 \; ; \tag{9.4}$$

$$u^0 = \begin{cases} -\varkappa_1 \operatorname{sign} \sigma & \text{for } \eta|\sigma| > \varkappa_1 \, , \\ -\eta\sigma & \text{for } \eta|\sigma| \leq \varkappa_1 \; ; \end{cases} \tag{9.5}$$

$$\dot{u}^1 = \begin{cases} -u^1 & \text{for } |u^1| > 1 \, , \\ -\alpha \operatorname{sign} \sigma(t,x) & \text{for } |u^1| \leq 1 \, . \end{cases} \tag{9.6}$$

Here α, η, $\varkappa_1$ are positive constants, $1 + \varkappa_1 < \varkappa$.

THEOREM 9.1. Let assumptions 3.1-3.4 be true and assume that within the linear zone $|\sigma| < \sigma_0$ the quantities $L_u L_v \sigma$; $L_u L_v L_v \sigma$, $(\partial/\partial u) L_u L_u \sigma$, $L_u (\partial/\partial u) L_u \sigma$, $(\partial^2/\partial u^2) L_u \sigma$ are bounded uniformly in t, x, $|u| \leq \kappa$, $|v| \leq \varkappa$. Then for sufficiently large α and η/α, algorithm (9.4)-(9.6) ensures exponential convergence of $\sigma(t)$ and $\dot{\sigma}(t)$ to zero.

By increasing α and η/α, the damping ratio can be made arbitrarily large. If the controlled system is linear in control, i.e., $(\partial^2/\partial u^2) L_u \sigma = 0$, the assertion of the theorem is true for all sufficiently large α and η. The control component (9.5) ensures asymptotic stability (in the deviation metric $|\sigma| + |\dot{\sigma}|$) of the 2nd order sliding mode that arises, according to Theorem 4.1, when we apply the $\mathcal{A}_\mu$-algorithm (9.6).

The asymptotic winding algorithm, denoted as $\mathcal{A}_{\mu\nu}$ -algorithm, is applied in the theory of binary systems for control of homogeneous dynamic systems linear in x and u with a linear constraint function $\sigma = cx$ [9].

Proof is by choosing a Lyapunov function in the form

$$H = (\alpha \dot{\sigma}'_u - \operatorname{sign} \sigma \, [L_u L_v \sigma]|_{u=u_{eq}}) |\sigma| + \frac{1}{2} (\dot{\sigma} + \lambda \sigma)^2 \; ,$$

where $\lambda > 0$. At the same time, we derive sufficient stability conditions for the sliding mode of the S_μ-system. We also use Lemma 6.1.

A system linear in control is controlled by the same technique as in subsec. 3.3, and to satisfy the conditions of the theorem we have to augment conditions a and b of Sec. 3 with additional condition c, given below:

c) for some constant $\varphi_0 > 0$ the inequality $|\sigma/\Phi| < \varphi_0$ implies uniform boundedness in t, x of the quantities $\Phi \sigma^{(3)}_{tx^2}$, $\Phi^2 \sigma^{(3)}_{x^3}$, $\sigma^{(3)}_{t^2 x}$, $\sigma^{(3)}_{t^3} / \Phi$, $\Phi a''_{xx}$, a''_{tx}, a''_{tt} / Φ, b''_{tt}, $\Phi b''_{tx}$, $\Phi^2 b''_{xx}$.

Here, as in condition a of Sec. 3, the function $\Phi(x)$ satisfies the conditions of boundedness of $\Phi_z{}'$ and $\Phi_{xx}{}'' \Phi$, and in addition it also ensures boundedness of $\Phi_{x0}{}^{(3)} \Phi^2$. An example of such a function is

$$\Phi(x) = \sqrt{x^t \mathcal{D} x + h_0} \; ,$$

where $\mathcal{D}$ is a positive semidefinite matrix, $h_0 = \text{const} > 0$.

9.4. All the algorithms described in this paper can be applied to control the smooth dynamic system

$$\dot{x} = f(t, x, u) \; ,$$

whose state space is a Banach space. The proofs remain unchanged. The theorems are restated either by modifying the concept of 2nd order sliding mode for the infinite dimensional case or by simply assuming that the proposed algorithms ensure that the functions σ and $\dot{\sigma}$ are exactly zero or satisfy the following inequalities:

$$|\sigma| \leq C_1 \tau^2 \; , \qquad |\dot{\sigma}| \leq C_2 \tau \; .$$

10. COMPUTER SIMULATION RESULTS

Let us demonstrate the application of the various algorithms to a randomly chosen controlled dynamic system. Simulation results for the winding algorithm are also given in [14]. Simulation of the drift algorithm is reported in [19].

10.1. We used the dynamic system

$$\begin{aligned}\dot{x}_1 &= -5x_1 + 10x_2 + 4x_3 + x_1 \sin t\,,\\ \dot{x}_2 &= 6x_1 - 3x_2 - 2x_3 + 3(x_1 + x_2 + x_3)\cos t\,,\\ x_3 &= x_1 + 3x_3 + 4x_2 \cos 5t + 4 \sin 5t + 10(1 + 0,5 \cos 10t)\mu(u)\Phi(x)\end{aligned} \quad (10.1)$$

where

$$\mu(u) = 3u - \cos 30t \cdot \sin u - u^2/4\,. \quad (10.2)$$

Here u is the control, $\Phi(x) = |x_1| + |x_2| + |x_3|$. The constraint function is the coordinate x_3. All the assumptions of subsec. 8.2 are satisfied, and we may thus take

$$\dot{\mu} = \begin{cases} -\mu(t_i) & \text{for } \mu(t_i) > 1\,,\\ -\alpha_M \operatorname{sign} x_3(t_i) & \text{for } x_3(t_i)\cdot\delta_i(x_3/\Phi_1) > 0,\ |u(t_i)| \le 1\,,\\ -\alpha_m \operatorname{sign} x_3(t_i) & \text{for } x_3(t_i)\cdot\delta_i(x_3/\Phi_1) \le 0,\ |u(t_i)| \le 1\,,\end{cases} \quad (10.3)$$

where $\alpha_M > \alpha_m > 0$, $\Phi_1 = \Phi^2 - \Phi + 1 + |x_1 - x_2|$.

We used the initial values $t_0 = 0$, $x_1(0) = 2$, $x_2(0) = -2$, $x_3(0) = 10$, $u(0) = 0$ and the parameter values $\alpha_M = 40$, $\alpha_m = 8$. The spacing between observations $\tau = 5 \cdot 10^{-4}$, the integration increment $\Delta t = 10^{-4}$. Integration was by Euler's method.

Solving the equation $x_3 = 0$ for μ, we obtain the function

$$\mu_{eq} = -\frac{x_1 + 3x_3 + 4x_2 \cos 5t + 4 \sin 5t}{10(1 + 0,5 \cos 10t)\Phi(x)}.$$

We can show that for almost all t

$$\dot{\sigma}_1 \in [\tilde{K}_m, \tilde{K}_M](\mu(u(t)) - \mu_{eq}(t, x(t))) + \sigma_1[-\lambda, \lambda]\,,$$

where $0 < \tilde{K}_m < \tilde{K}_M$, $\lambda > 0$ are some constants. Thus, for small σ_1, the difference $\varepsilon = \mu - \mu_{eq}$ approximately characterizes the derivative $\dot{\sigma}_1$.

For these parameter values the constraint was tracked with accuracy $|\sigma_1| = |x_3/\Phi| \le 6.61\cdot10^{-4}$. Then the observation spacing τ and the integration increment Δt were reduced to 1/100, and the algorithm tracked the constraint with accuracy $|\sigma_1| \le 7.04\cdot10^{-8}$. For comparison note that the ordinary 1st order sliding algorithm $u = -\operatorname{sign}\sigma_1(t_i)$ tracked the constraint in these simulations with accuracy $|\sigma_1| \le 1.20\cdot10^{-2}$, and increment reduction to 1/100 improved the tracking accuracy to $|\sigma_1| \le 1.04\cdot10^{-4}$. While ensuring exact tracking of the constraint $\sigma_1 = 0$, the relay algorithm $u = -\operatorname{sign} x_3$ in general does not produce a single-valued description of system operation in the sliding mode for vanishingly small switching nonidealities because the system (1), (2) is nonlinear in the control u [3, 21].

10.2. Now let

$$\Phi(x) = \sqrt{x_1^2 + x_2^2 + x_3^2 + 1}\,. \quad (10.4)$$

When the function $\Phi(x)$ is chosen in the form (4), the system (1), (2) and the constraint function $\sigma = x_3/\Phi(x)$ satisfy assumptions 3.1-3.5 (see Proposition 3.1) and the conditions of Theorem 9.1.

Simulation results for the winding algorithm in its basic form, i.e., with $\Phi_1 = \Phi$ (see Sec. 5), are similar to the results described in subsec. 10.1. The initial values and the constants of the algorithm are the same as in subsec. 10.1.

Drift Algorithm

Let

$$\dot{u} = \begin{cases} -u(t_i) & \text{for } |u(t_i)| > 1 , \\ \alpha_M \operatorname{sign} \delta_i\sigma & \text{for } \delta_i\sigma \cdot \sigma(t_i) > 0, |u(t_i)| \le 1 , \\ \alpha_m \operatorname{sign} \delta_i\sigma & \text{for } \delta_i\sigma \cdot \sigma(t_i) \le 0, |u(t_i)| \le 1 , \end{cases}$$

where $\alpha_M > \alpha_m > 0$, $t \in [t_i, t_{i+1}]$; τ_M for $\eta\sqrt{|\sigma(t_i)|} > \tau_M$, $t_{i+1} - t_i = \eta\sqrt{|\sigma(t_i)|}$ for $\tau_m \le \eta\sqrt{|\sigma(t_i)|} \le \tau_M$, τ_m for $\eta\sqrt{|\sigma(t_i)|} < \tau_m$, where $\tau_M > \tau_m > 0$, $\eta > 0$.

We used the same initial values and the same values for the parameters α_M, α_m as previously. We took $\eta = 0.02$, $\tau_M = 0.05$, $\tau_m = 0.0005$ integration increment $\Delta t = 0.0001$.

Sliding was tracked with accuracy $|\sigma| \le 1.30 \cdot 10^{-3}$, and after reducing τ_m and Δt to 1/100 the accuracy improved to $|\sigma| \le 7.12 \cdot 10^{-8}$.

Algorithm with Specified Variation of the Constraint Function

Let

$$\dot{u} = \begin{cases} -u(t_i) & \text{for } |u(t_i)| > 1 , \\ -\alpha \operatorname{sign}(\delta_i\sigma - 10|\sigma(t_i)|^{\frac{3}{4}}\tau) & \text{for } |u(t_i)| \le 1 , \end{cases}$$

where $\alpha > 0$, $t \in [t_i, t_{i+1})$, $t_i - t_{i-1} = \tau > 0$. We used the same initial conditions: $\alpha = 16$, $\tau = 0.0005$. Tracking was accurate to within $|\sigma| \le 1.32 \cdot 10^{-3}$, and after reduction of τ and Δt to 1/100 the accuracy improved to $|\sigma| \le 1.10 \cdot 10^{-7}$.

Asymptotic Winding Algorithm

Let $u = u^0 + u^1$, where

$$u^0 = \begin{cases} -\operatorname{sign} \sigma(t_i) & \text{for } \eta|\sigma(t_i)| > 1 , \\ -\eta\sigma(t_i) & \text{for } \eta|\sigma(t_i)| \le 1 , \end{cases}$$
$$\eta > 0, \quad t \in [t_i, t_{i+1}), \quad t_{i+1} - t_i = \tau > 0;$$
$$\dot{u}^1 = \begin{cases} -u^1(t_i) & \text{for } |u^1(t_i)| > 1 , \\ -\alpha \operatorname{sign} \sigma(t_i) & \text{for } |u^1(t_i)| \le 1 , \end{cases}$$

$\alpha > 0$.

The same initial conditions were used: $\alpha = 16$, $\eta = 2$, $\tau = 0.0005$, $\Delta t = 0.0001$. The achieved accuracy was $|\sigma| \le 1.37 \cdot 10^{-3}$.

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