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$\mathbb{Z}_n$ Baxter Model: Symmetries and the Belavin Parametrization

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The $\mathbb{Z}_n$ Baxter model is an exactly solvable lattice model in the special case of the Belavin parametrization. For this parametrization we calculate the partition function, κ, in an antiferromagnetic region and the order parameter in a ferromagnetic region. We find that the order parameter is expressible in terms of a modular function of level n which for $n=2$ is the Onsager–Yang–Baxter result. In addition we determine the symmetry group of the finite lattice partition function for the general $\mathbb{Z}_n$ Baxter model.

KEY WORDS: Statistical mechanics; lattice statistics; Baxter model; inversion relations; Heisenberg group; elliptic modular functions of level n.

1. INTRODUCTION

One possible n-state generalization of Baxter's eight vertex model[1] is the completely $\mathbb{Z}_n$ symmetric vertex model first introduced by Belavin[6] and by Chudnovsky and Chudnovsky.[9] If S_{ij}^{kl} denotes the Boltzmann weight for a single vertex with bond states $i, j, k, l \in \mathbb{Z}_n$ (see Fig. 1), then the vertex model is said to be completely $\mathbb{Z}_n$ symmetric if

$$\text{(i)} \quad S_{ij}^{kl} = 0 \qquad \text{unless } i+j = k+l \bmod n$$

and

$$\text{(ii)} \quad S_{i+p\ j+p}^{k+p\ l+p} = S_{ij}^{kl} \qquad \text{for every } i, j, k, l, p \in \mathbb{Z}_n$$
$$\text{and addition is defined mod } n$$

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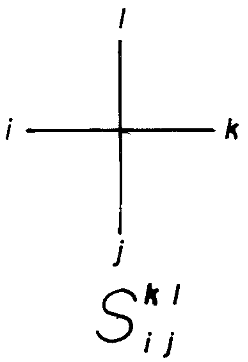

Fig. 1. The vertex configuration $i, j, k, l \in \mathbb{Z}_n$ with Boltzmann weight S^{kl}_{ij}.

For $n=2$ this is Baxter's eight-vertex model; thus, we call the completely $\mathbb{Z}_n$ symmetric vertex model the $\mathbb{Z}_n$ Baxter model. Observe that (for $n>2$) rotation by 90° violates $\mathbb{Z}_n$ symmetry.

Belavin[6] introduced a parametrization of S^{kl}_{ij} which for $n=2$ reduces to Baxter's parametrization of the Boltzmann weights in terms of Jacobi theta functions. Belavin conjectured that his parametrization satisfies the Yang–Baxter equations[1,3,22] which was subsequently verified in Refs. 7, 8, and 20. Thus one might reasonably expect that this special $\mathbb{Z}_n$ Baxter model is exactly solvable in the sense that the free energy per site and the order parameters are exactly computable. It is the purpose of this paper to begin this program.

In Section 2 we summarize the relevant results needed from the Heisenberg group and its relation to Jacobi theta functions. We believe that this point of view provides a natural understanding of the role of theta functions in the $\mathbb{Z}_n$ Baxter model.[9,20]

In Section 3 we analyze the Belavin parametrization of the Boltzmann weights. In particular we give a product representation which gives a clearer picture of some of the properties of this parametrization. From this representation we are unable to find any region (for $n>2$) where all the Boltzmann weights are positive, nor can a simple "gauge transformation" on S^{kl}_{ij} that preserves the $\mathbb{Z}_n$ symmetry lead to positive regions. We should say that this nonpositivity has been checked on various lines but that does not rule out some "nonstandard" region having the desired physical property. In any case our calculations are performed in unphysical regions.

In Section 4 we determine the symmetry group for the partition function for the $\mathbb{Z}_n$ model. Our proof is a generalization of the Johnson, Krinsky, McCoy proof[12] of the Fan and Wu[10] symmetries for $n=2$. Our results for $n=2$ along with the 90° rotational symmetry valid for $n=2$ reduce to the results of Fan and Wu.[10] This section should have independent interest since the Belavin restriction is not assumed.

In Section 5 we derive inversion relations for the Baxter face operators U_i and V_i assuming the Belavin parametrization. We stress that the V_i inversion relation must be independently derived since the lack of rotational symmetry through 90° does not allow one to obtain the V_i inversion relation from the U_i inversion relation (except, of course, for $n=2$).

In Section 6 the partition function per site in the thermodynamic limit, κ, is derived using the inversion relations of Section 5 and Baxter's matrix inversion techniques for the region where $S_0^{n-1}{}_{n-1}^{0}$ is dominant. For $n=2$ our results reduce to Baxter's partition function for the eight-vertex model, and for general n and $q \to 0$ our κ reduces to the κ derived by Babelon, de Vega, and Viallet[5] using the method of nested Bethe ansatz (see Schultz[19] for further analysis of the nested Bethe ansatz).

In Section 7 the order parameters, $\langle \sigma^k \rangle$, in the ferromagnetic region are computed using Baxter's method of corner transfer matrices.[2,3] Here σ is the dual spin variable whose values are nth roots of unity. Our principal result is that these $\langle \sigma^k \rangle$, $k=1,\ldots,n-1$, are expressible in terms of elliptic modular functions of level n which for $n=2$ reduces to the Onsager–Yang–Baxter spontaneous magnetization.[17,21,2,3] From our formulas for $\langle \sigma^k \rangle$ we derive for $k=0,\ldots,n-1$ (assuming $\langle \sigma^k \rangle$ is an expected value)

$$\mathrm{Prob}(\sigma=\omega^k)=\varphi(q) \sum_{\substack{m \equiv k(n) \\ m \geqslant 0}} p(m)\, q^m$$

where $\omega=\exp(2\pi i/n)$, $\varphi(q)=\prod_{l=1}^{\infty}(1-q^l)$, and $p(m)$ is the number of partitions of m. For $0<q<1$ we show that $\sum_{k=0}^{n-1} \mathrm{Prob}(\sigma=\omega^k)=1$, $\mathrm{Prob}(\sigma=\omega^k) \to \delta_{k,0}$ as $q \to 0$, $\mathrm{Prob}(\sigma=\omega^k) \to 1/n$ as $q \to 1$, and $0<\mathrm{Prob}(\sigma=\omega^k)<1$. Thus, even though our calculations are performed in an unphysical region, the quantities $\mathrm{Prob}(\sigma=\omega^k)$ do have the interpretation as probabilities. For special values of n and k the above partition theoretic sum has been extensively studied by Ramanujan and others (see, e.g., Knopp[13]).

2. HEISENBERG GROUP AND THETA FUNCTIONS

To establish notation and for the convenience of the reader, we summarize those results needed in the following sections concerning the Heisenberg group and theta functions. Our principal references are Mumford[16] and Krazer.[14]

Let $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ be the upper half-plane, $\Lambda_\tau = \{\xi_1 \tau + \xi_2 \mid \xi_1, \xi_2 \in \mathbb{Z}, \tau \in \mathbb{H}\}$ the lattice generated by 1 and τ, and $E_\tau = \mathbb{C}/\Lambda_\tau$ the complex torus which can be identified with an elliptic curve. Let $f: \mathbb{C} \to \mathbb{C}$ be an entire function and define

$$\begin{aligned}(S_b f)(z) &= f(z+b)\\ (T_a f)(z) &= \exp(\pi i a^2 \tau + 2\pi i a z)\, f(z + a\tau)\end{aligned} \tag{2.1}$$

where $a, b \in \mathbb{R}$, $\tau \in \mathbb{H}$. Then

$$\begin{aligned}S_{b_1} S_{b_2} &= S_{b_1 + b_2}\\ T_{a_1} T_{a_2} &= T_{a_1 + a_2}\end{aligned}$$

and

$$S_b T_a = \exp(2\pi i a b)\, T_a S_b$$

The group generated by the T_a's and S_b's is the Heisenberg group

$$\mathscr{G} = C_1^* \times \mathbb{R} \times \mathbb{R} \qquad (C_1^* = \{z \in \mathbb{C} \mid |z| = 1\})$$

where (λ, a, b) stands for the transformation

$$(U_{(\lambda, a, b)} f)(z) = \lambda (T_a S_b f)(z)$$

The group law is given by

$$(\lambda, a, b)(\lambda', a', b') = (\lambda\lambda' \exp(2\pi i b a'), a + a', b + b')$$

We will be particularly interested in the subgroups

$$\begin{aligned}\Gamma &= \{(1, a, b) \in \mathscr{G} \mid a, b \in \mathbb{Z}\}\\ n\Gamma &= \{(1, na, nb) \in \mathscr{G} \mid a, b \in \mathbb{Z}\}\\ \mathscr{G}_n &= \{(\lambda, a, b) \in \mathscr{G} \mid \lambda \in \mathscr{M}_{n^2}, a, b \in (1/n)\mathbb{Z}/n\mathbb{Z}\}\end{aligned}$$

where $\mathscr{M}_m$ is the group of mth roots of 1.

The Jacobi theta function

$$\begin{aligned}\vartheta(z, \tau) &= \sum_{m \in \mathbb{Z}} \exp(\pi i m^2 \tau + 2\pi i m z)\\ &= \prod_{m=1}^{\infty} (1 - q^{2m})(1 + e^{-2\pi i z} q^{2m-1})(1 + e^{2\pi i z} q^{2m-1})\end{aligned} \tag{2.2}$$

with $q = e^{\pi i \tau}$, $z \in \mathbb{C}$, $\tau \in \mathbb{H}$ is, up to scalars, the unique entire function invariant under Γ. Explicitly,

$$\vartheta(z + \xi_1 \tau + \xi_2, \tau) = u_{\xi_1}(z, \tau)\, \vartheta(z, \tau), \qquad \xi_1, \xi_2 \in \mathbb{Z} \tag{2.3}$$

where

$$u_{\xi_1}(z, \tau) = \exp(-\pi i \xi_1^2 \tau - 2\pi i \xi_1 z) \tag{2.4}$$

The Jacobi theta functions of rational characteristics $a, b \in (1/n)\mathbb{Z}$ are defined by

$$\begin{aligned} \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z, \tau) &= (S_b T_a \vartheta)(z) \\ &= \exp[\pi i a^2 \tau + 2\pi i a(z + b)]\, \vartheta(z + a\tau + b) \\ &= \sum_{m \in \mathbb{Z}} \exp[\pi i (m + a)^2 \tau + 2\pi i (m + a)(z + b)] \end{aligned} \tag{2.5}$$

The functions $\vartheta[{a \atop b}](z, \tau)$, a, $b \in (1/n)\mathbb{Z}/\mathbb{Z}$ form a basis for the complex vector space V_n of entire functions invariant under the subgroup $n\Gamma$. The action of the Heisenberg group $\mathscr{G}_n$ is summarized by

$$\text{(i)} \quad \left(S_\beta \vartheta\begin{bmatrix} a \\ b \end{bmatrix}\right)(z) = \vartheta\begin{bmatrix} a \\ b + \beta \end{bmatrix}(z, \tau), \qquad a, b, \beta \in \frac{1}{n}\mathbb{Z}$$

$$\text{(ii)} \quad \left(T_\alpha \vartheta\begin{bmatrix} a \\ b \end{bmatrix}\right)(z) = \exp(-2\pi i \alpha b)\, \vartheta\begin{bmatrix} a + \alpha \\ b \end{bmatrix}(z, \tau), \quad a, b, \alpha \in \frac{1}{n}\mathbb{Z}$$

$$\text{(iii)} \quad \vartheta\begin{bmatrix} a + p \\ b + q \end{bmatrix}(z, \tau) = \exp(2\pi i a q) \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z, \tau), \qquad \text{for } a, b \in \frac{1}{n}\mathbb{Z},\ p, q \in \mathbb{Z} \tag{2.6}$$

the zeros are

$$\text{(iv)} \quad \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z, \tau) = 0 \text{ at } z = (\tfrac{1}{2} - a)\tau + (\tfrac{1}{2} - b) \bmod \Lambda_\tau$$

In our analysis below on the $\mathbb{Z}_n$ model we find there is an interplay between the above Heisenberg groups and associated theta functions and the Heisenberg group that arises when the factor of automorphy $u_{\xi_1}(z, \tau)$ is replaced by $[u_{\xi_1}(z, \tau)]^n$ [geometrically this corresponds to tensoring the line bundle defined by $u_{\xi_1}(z, \tau)$ with itself n times]. We define, therefore,

$$(J_{a,b} f)(z) = \exp(\pi i n a^2 \tau + 2\pi i n a z)\, f(z + a\tau + b)$$

with a, $b \in \mathbb{R}$, f an entire function. Such $J_{a,b}$ generate the Heisenberg group

$$\mathscr{G}^{(n)} = C_1^* \times \mathbb{R} \times \mathbb{R}$$

where the group law is now

$$(\lambda, a, b)(\lambda', a', b') = (\lambda\lambda' \exp(2\pi inba'), a + a', b + b')$$

Define the subgroups

$$\Gamma^{(n)} = \{(1, a, b) \in \mathscr{G}^{(n)} \mid a, b \in \mathbb{Z}\}$$

and

$$\mathscr{H}_n = \left\{(\lambda, a, b) \in \mathscr{G}^{(n)} \;\middle|\; \lambda \in \mathscr{M}_n, a, b \in \frac{1}{n} \mathbb{Z}/\mathbb{Z}\right\}$$

Then an entire function θ is said to be an nth order theta function of characteristic 0, 0 if it is invariant under $\Gamma^{(n)}$, i.e.,

$$\theta(z + \xi_1 \tau + \xi_2) = (u_{\xi_1}(z, \tau))^n \, \theta(z), \qquad \xi_1, \xi_2 \in \mathbb{Z}$$

If M_n is the complex vector space of nth order theta functions, then it is well known that dim $M_n = n$ and a basis is given by $\vartheta[{}^{j/n}_{0}](nz, n\tau)$. In what follows a certain finite-dimensional representation of $\mathscr{H}_n$ acting on M_n will play a central role. First observe that $J_{a,b}$: $M_n \to M_n$ if $a, b \in (1/n)\mathbb{Z}$. It can be shown that this representation of $\mathscr{H}_n$ is irreducible.

There exists a vector $\Theta \in M_n$ such that

$$J_{0,1/n} \Theta = \Theta$$

[take $\Theta(z) = \prod_{j=0}^{n-1} \vartheta[{}^{0}_{j/n}](z, \tau)$]. The existence of such Θ permits a canonical construction of a basis for M_n. Define

$$\theta_j = J_{j/n,0} \Theta, \qquad j = 0, 1,..., n-1$$

and observe that

$$J_{1/n,0} \theta_j = \theta_{j+1}$$
$$J_{0,1/n} \theta_j = \omega^j \theta_j$$

where $\omega = \exp(2\pi i/n)$. We call $\{\theta_j\}_{j \in \mathbb{Z}_n}$ the Bianchi basis for M_n.

In terms of coordinates we choose the standard basis $\{e_j\}_{j \in \mathbb{Z}_n}$ of $\mathbb{C}^n$ and define $n \times n$ matrices h and g by

$$he_j = e_{j+1}, \qquad ge_j = \omega^j e_j, \qquad j \in \mathbb{Z}_n \tag{2.7}$$

If we denote by $G_n = \mathbb{Z}_n \times \mathbb{Z}_n$, $\langle \alpha, \beta \rangle = \alpha_1 \beta_2 - \alpha_2 \beta_1$, and define

$$I_\alpha = h^{\alpha_1} g^{\alpha_2}, \qquad \alpha = (\alpha_1, \alpha_2) \in G_n \tag{2.8}$$

then

$$I_\alpha I_\beta = \omega^{\alpha_2 \beta_1} I_{\alpha+\beta} = \omega^{\langle \beta, \alpha \rangle} I_\beta I_\alpha \tag{2.9}$$

is a representation of the Heisenberg group

$$H_n = \{(\lambda, \alpha) \mid \lambda \in \mathscr{M}_n, \alpha \in G_n\}$$

which is essentially $\mathscr{H}_n$. Throughout this paper a complete residue system for $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ will be chosen to be $\{0, 1,..., n-1\}$.

In any discussion of E_τ it is natural to ask what happens when different generators are used to generate the lattice Λ_τ. Thus let $\alpha = \alpha_1 \tau + \alpha_2$, $\beta = \beta_1 \tau + \beta_2$ be elements of Λ_τ with $\langle \alpha, \beta \rangle = \alpha_1 \beta_2 - \alpha_2 \beta_1 = 1$. Then any point $\gamma \in \Lambda_\tau$, $\gamma = \gamma_1 \tau + \gamma_2$ has coordinates in the α, β basis given by $\gamma = \xi_1 \alpha + \xi_2 \beta$; that is,

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

For H_n we can choose as generators ω, I_α, I_β since for any $I_\gamma = h^{\gamma_1} g^{\gamma_2}$ we have $I_\gamma = (I_\alpha)^{\xi_1} (I_\beta)^{\xi_2} \times n$th root of unity. We define

$$I_\xi^A = I_\alpha^{\xi_1} I_\beta^{\xi_2}, \qquad A = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix} \in SL(2, \mathbb{Z}) \tag{2.10}$$

Then it is easy to show there exists an invertible U_A (independent of ξ) such that

$$I_\xi^A = U_A I_\xi U_A^{-1} \tag{2.11}$$

where, of course, $I_\xi = h^{\xi_1} g^{\xi_2}$. One way to demonstrate this is to choose as the cyclic vector the vector x_0 which is the eigenvector of I_β corresponding to the eigenvalue 1. Then, as above, form $x_j = (I_\alpha)^j x_0$. The matrix U_A is then the change of basis matrix $U_A e_j = x_j$.

The transformation of theta functions themselves is somewhat more involved. For any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$, extend the action to $\mathbb{C} \times \mathbb{H}$ by

$$\gamma(z, \tau) = \left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right) \tag{2.12}$$

We give the corresponding action on $\vartheta[\begin{smallmatrix} a \\ b \end{smallmatrix}](z, \tau)$ for the generators $(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$ and $(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix})$ of $SL(2, \mathbb{Z})$:

$$\vartheta\begin{bmatrix} \frac{1}{2}+a \\ \frac{1}{2}+b \end{bmatrix}(z, \tau+1) = \exp\left(\frac{\pi i}{4}\right)\exp(-\pi i a^2)\vartheta\begin{bmatrix} \frac{1}{2}+a \\ \frac{1}{2}+a+b \end{bmatrix}(z, \tau) \quad (2.13)$$

and

$$\vartheta\begin{bmatrix} \frac{1}{2}+a \\ \frac{1}{2}+b \end{bmatrix}\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = -i(-i\tau)^{1/2}\exp\left[2\pi i\left(ab+\frac{a-b}{2}\right)\right]\exp\left(\frac{\pi i z^2}{\tau}\right) \times \vartheta\begin{bmatrix} \frac{1}{2}+b \\ \frac{1}{2}-a \end{bmatrix}(z, \tau) \quad (2.14)$$

Here we used $\vartheta[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}](z, \tau)$ only for convenience. Note that in terms of the characteristics $\frac{1}{2}+a$, $\frac{1}{2}+b$ the action of the matrices takes a particularly simple form; namely, if $A \in SL(2, \mathbb{Z})$ then the transformation in the characteristics (a, b) is given by ${}^tA(\begin{smallmatrix} a \\ b \end{smallmatrix})$.

This action generalizes if we include translations by lattice points of Λ_τ. Namely, we get a semidirect product $SL(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$ which acts on $\mathbb{C} \times \mathbb{H}$ by $(z, \tau) \to ((z+m\tau+n)/(c\tau+d), (a\tau+b)/(c\tau+d))$.

3. $\mathbb{Z}_n$ BAXTER MODEL AND THE BELAVIN PARAMETRIZATION

The Boltzmann weights S_{ij}^{kl} define a matrix S in the standard basis $\{e_i \otimes e_j\}_{i,j \in \mathbb{Z}_n}$ for $\mathbb{C}^n \times \mathbb{C}^n$. This matrix is completely $\mathbb{Z}_n$ symmetric; that is, S_{ij}^{kl} satisfies conditions (i) and (ii) in the Introduction or equivalently[6,9,20]

$$S = \sum_{\alpha \in G_n} w_\alpha I_\alpha \otimes I_\alpha^{-1} \quad (3.1)$$

for some $w_\alpha \in \mathbb{C}$. Thus the general $\mathbb{Z}_n$ Baxter model has n^2 independent parameters (actually n^2-1 is a better count since one parameter is an overall normalization factor). The Belavin parametrization[6] is

$$w_\alpha(z) = \frac{\vartheta\begin{bmatrix} \alpha_1/n \\ \alpha_2/n \end{bmatrix}(z+\eta, \tau)}{\vartheta\begin{bmatrix} \alpha_1/n \\ \alpha_2/n \end{bmatrix}(\eta, \tau)}, \qquad \alpha \in G_n \quad (3.2)$$

where z, $\eta \in \mathbb{C}$, $\tau \in \mathbb{H}$. Observe that there are only three independent parameters. It will be convenient to define a new variable $w \in \mathbb{C}$ by

$$\eta = \frac{w}{n} + \frac{1}{2}\tau + \frac{1}{2} \quad (3.3)$$

For such a choice of w_α the resulting S matrix satisfies the Yang–Baxter equation[1,22] as was first conjectured by Belavin and then later confirmed in Refs. 7, 8, and 20. For $n=2$ the results reduce to the pioneering work of Baxter[1,3,4]; see Ref. 20 for further details.

Because of $\mathbb{Z}_n$ symmetry it is sufficient to consider S^{ab}_{0a+b}, or S^{ab} as we shall henceforth abbreviate. From (2.8) and (3.1) we have

$$S^{ab}(z, w, \tau) = \sum_{\alpha \in \mathbb{Z}_n} w_{(-a,\alpha)} \omega^{-b\alpha}, \qquad a, b \in \mathbb{Z}_n \tag{3.4}$$

Using the transformation properties of the theta functions, a computation shows that for $\xi \in \mathbb{Z}^2$

$$S^{ab}(z + \xi_1 \tau + \xi_2, w, \tau) = \exp\left[-i\pi \xi_1^2 \tau - 2\pi i \xi_1 \left(z + \frac{w}{n} + \frac{\tau}{2} + \frac{1}{2} \right) \right] \times \omega^{-a\xi_2} S^{a, b+\xi_1}(z, w, \tau) \tag{3.5a}$$

$$S^{ab}(z, w + \xi_1 \tau + \xi_2, \tau) = \exp\left(-2\pi i \frac{\xi_1}{n} z \right) \omega^{b\xi_2} S^{a-\xi_1, b}(z, w, \tau) \tag{3.5b}$$

We sometimes abbreviate $S^{ab}(z, w, \tau)$ to simply $S^{ab}(z)$.

We now proceed to write S^{ab} as a product of theta functions. The plan will be to find the zeros and poles of S^{ab} and use them to construct a product of theta functions with the same zero and pole structure. To begin, first note that each S^{ab} is entire and quasiperiodic in z on the $\Lambda_{n\tau}$ lattice. Specifically,

$$\begin{aligned} S^{ab}(z + n\tau, w, \tau) &= \exp\left[-\pi i n^2 \tau - 2\pi i n \left(z + \frac{w}{n} + \frac{\tau}{2} + \frac{1}{2} \right) \right] S^{ab}(z, w, \tau) \\ S^{ab}(z+1, w, \tau) &= \omega^{-a} S^{ab}(z, w, \tau) \end{aligned} \tag{3.6}$$

It is an elementary complex analysis argument that if f is entire, not identically zero, and satisfies

$$f(z+\tau) = \exp[-2\pi i (A_1 + A_2 z)]\, f(z)$$
$$f(z+1) = \exp(-2\pi i B)\, f(z)$$

then necessarily A_2 is a positive integer, and f has A_2 zeros in Λ_τ with

$$\sum \text{zeros} = \tfrac{1}{2} A_2 + B\tau - A_1 \qquad (\text{mod } \Lambda_\tau)$$

We apply this to S^{ab} to conclude there are n zeros in $\Lambda_{n\tau}$ with sum

$$\sum \text{zeros} = a\tau - w + \tfrac{1}{2} n(n-1) \qquad (\text{mod } \Lambda_{n\tau})$$

Since $w_\alpha(0)=1$ we immediately see that $S^{ab}(0)=0$ for $b\neq 0$. From (3.5) we have

$$S^{ab}(l\tau)=(\text{multiplier})\times S^{a,b+l}(0)=0 \qquad \text{for} \quad l\neq -b(n)$$

Thus we have located $n-1$ zeros of S^{ab} and from the above sum of zeros we conclude the remaining zero must be

$$z=(a-b)\tau-w \qquad (\Lambda_{n\tau})$$

Define

$$\psi_{ab}(z)=e^{-i\pi z}\vartheta\begin{bmatrix}(-a+b)/n+\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z+w,n\tau)\prod_{\substack{k=0\\ k\neq b}}^{n-1}\vartheta\begin{bmatrix}k/n+\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z,n\tau) \tag{3.7}$$

then ψ_{ab} has the same zeros (in z) as S^{ab} and has the same transformation properties (3.6). Thus

$$S^{ab}(z,w,\tau)=c_{ab}(w,\tau)\,\psi_{ab}(z) \tag{3.8}$$

To determine $c_{ab}(w,\tau)$ we evaluate S^{ab} and ψ_{ab} at $z=-b\tau$. From (3.5)

$$S^{ab}(-b\tau,w,\tau)=\exp\left[-i\pi b^2\tau+2\pi ib\left(\frac{w}{n}+\frac{\tau}{2}+\frac{1}{2}\right)\right]n \tag{3.9a}$$

and

$$\begin{aligned}\psi_{ab}(-b\tau)&=e^{i\pi b\tau}\vartheta\begin{bmatrix}(-a+b)/n+\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(w-b\tau,n\tau)\prod_{\substack{k=0\\ k\neq b}}^{n-1}\vartheta\begin{bmatrix}k/n+\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(-b\tau,n\tau)\\ &=\exp\left[-i\pi b^2\tau+2\pi ib\left(\frac{w}{n}+\frac{\tau}{2}+\frac{1}{2}\right)\right]\vartheta\begin{bmatrix}-a/n+\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(w,n\tau)\\ &\quad\times\prod_{k=1}^{n-1}\vartheta\begin{bmatrix}k/n+\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0,n\tau)\end{aligned} \tag{3.9b}$$

Comparing (3.8) with (3.9) we now evaluate $c_{ab}(w,\tau)$ to obtain

$$\begin{aligned}S^{ab}(z,w,\tau)&=n\exp(-i\pi z)\,\frac{\vartheta\begin{bmatrix}(-a+b)/n+\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z+w,n\tau)}{\vartheta\begin{bmatrix}-a/n+\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(w,n\tau)}\\ &\quad\times\frac{\prod_{k=0,k\neq b}^{n-1}\vartheta\begin{bmatrix}k/n+\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z,n\tau)}{\prod_{k=1}^{n-1}\vartheta\begin{bmatrix}k/n+\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0,n\tau)}\end{aligned} \tag{3.10}$$

This gives the factorized Belavin S matrix. Since multiplication of each Boltzmann weight by a common factor does not affect the statistical mechanics in any significant way, we will often replace S^{ab} by $f(z)\,S^{ab}(z)$ where $f(z)$ is some function independent of a and b. In particular, if we use the identity

$$\sum_{k=0}^{n-1} \vartheta\begin{bmatrix} k/n+\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(z, n\tau) = \gamma_0 \vartheta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(z, \tau) \tag{3.11}$$

(γ_0 is constant in z), then (3.10) can be written as

$$S^{ab}(z, w, \tau) = f(z)\, \frac{\vartheta\begin{bmatrix} (-a+b)/n+\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(z+w, n\tau)}{\vartheta\begin{bmatrix} -a/n+\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(w, n\tau)\vartheta\begin{bmatrix} b/n+\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(z, n\tau)} \tag{3.12}$$

where $f(z)$ has the obvious definition. This final form of the Belavin S matrix is particularly simple and it shows the dependence of the weight S^{ab} on a and b. From this product representation the identities

$$S^{ab}(-w, w, \tau) = -S^{ba}(-w, w, \tau)$$

and (3.13a)

$$S^{ab}(w, w, \tau) = S^{-b,-a}(w, w, \tau)$$

follow. If $P_{12} e_i \otimes e_j = e_j \otimes e_i$, then (3.13a) in terms of the S matrix becomes

$$S(-w, w, \tau) = -S(-w, w, \tau)\, P^{12} \quad \text{and} \quad S(w, w, \tau) = P^{12} S(w, w, \tau) \tag{3.13b}$$

An important limiting case of Baxter's eight-vertex model is the six-vertex model (see Ref. 15 and references therein) obtained by passing to the limit $\tau \to i\infty$ (or $q \to 0$). Here too, the $q \to 0$ limit of the $\mathbb{Z}_n$ Baxter model will be important as independent methods of nested Bethe ansatz are available.[5,19] We record here the $q \to 0$ limit of the Belavin parametrization:

$$\begin{aligned}
&S^{00}(z, w, \tau) \to n \exp(-\pi i z)\, \frac{\sin \pi(z+w)}{\sin \pi w} \\
&S^{a,0}(z, w, \tau) \to n \exp(-\pi i z) \exp\left[2\pi i\left(\frac{1}{2}-\frac{a}{n}\right) z\right], && a \neq 0 \\
&S^{0b}(z, w, \tau) \to n \exp(-\pi i z) \exp\left[2\pi i\left(\frac{b}{n}-\frac{1}{2}\right) w\right] \frac{\sin \pi z}{\sin \pi w}, && b \neq 0 \\
&S^{ab}(z, w, \tau) \to 0, && a \neq 0, b \neq 0
\end{aligned} \tag{3.14}$$

Let $z' = z + w/n + \frac{1}{2}\tau + \frac{1}{2}$, $\tau' = n\tau$, $a' = -a/n$, then if we define

$$S_b\begin{bmatrix} a' \\ 0 \end{bmatrix}(z', \tau') = S^{-ab}(z, w, \tau)$$

(3.6) becomes

$$\begin{aligned} S_b\begin{bmatrix} a' \\ 0 \end{bmatrix}(z' + \tau', \tau') &= \exp(-i\pi n\tau' - 2\pi i n z')\, S_b\begin{bmatrix} a' \\ 0 \end{bmatrix}(z', \tau') \\ S_b\begin{bmatrix} a' \\ 0 \end{bmatrix}(z' + 1, \tau') &= \exp(2\pi i a')\, S_b\begin{bmatrix} a' \\ 0 \end{bmatrix}(z', \tau') \end{aligned} \tag{3.15}$$

If $M_n[{a' \atop b'}]$ denotes the complex vector space of nth-order theta functions of rational characteristics a', b', then it is well known[14] that $\dim M_n[{a' \atop b'}] = n$ and a basis is given by $\vartheta[{a' \atop (b'+\alpha)/n}](z', \tau'/n)$, $\alpha = 0,..., n-1$. Hence (3.15) is the statement that $S_b[{a' \atop 0}](z', \tau') \in M_n[{a' \atop 0}]$ for every $b = 0,..., n-1$. The Belavin representation (3.4) becomes

$$S_b\begin{bmatrix} a' \\ 0 \end{bmatrix}(z', \tau') = \sum_{\alpha=0}^{n-1} c_{\alpha,b}\, \vartheta\begin{bmatrix} a' \\ \alpha/n \end{bmatrix}\left(z', \frac{\tau'}{n}\right) \tag{3.16}$$

where

$$c_{\alpha,b} = \omega^{-b\alpha}\left(\vartheta\begin{bmatrix} a' \\ \alpha/n \end{bmatrix}\left(\eta, \frac{\tau'}{n}\right)\right)^{-1} \qquad \alpha = 0,..., n-1$$

Since $\vartheta[{a' \atop \alpha/n}](z', \tau'/n)/\vartheta[{a' \atop \alpha/n}](\eta, \tau'/n)$, $\alpha = 0,..., n-1$, also form a basis for $M_n[{a' \atop 0}]$ and since the matrix $(\Omega)_{ij} = \omega^{-ij}$, $i, j = 0,..., n-1$, is invertible, it follows that the functions $S_b[{a' \atop 0}](z', \tau')$, $b = 0,..., n-1$, form a basis for $M_n[{a' \atop 0}]$.

4. SYMMETRIES OF THE PARTITION FUNCTION

Recall[1-3] that the transfer matrix T for a vertex model with M rows, N columns with periodic boundary conditions can be written as

$$T_{\boldsymbol{\alpha},\boldsymbol{\alpha}'} = \mathrm{Tr}(L(\alpha_1, \alpha_1') \cdots L(\alpha_N, \alpha_N')) \tag{4.1}$$

where $\boldsymbol{\alpha} = (\alpha_1,..., \alpha_N)$, $\boldsymbol{\alpha}' = (\alpha_1',..., \alpha_N')$, α_i, $\alpha_i' \in \mathbb{Z}_n$, and $L(\alpha, \alpha')$ is an $n \times n$ matrix given by

$$L(\alpha, \alpha')_{\lambda\lambda'} = S_{\lambda\alpha}^{\lambda'\alpha'}, \qquad \alpha, \alpha', \lambda, \lambda' \in \mathbb{Z}_n$$

Using (3.1) we have

$$L(\alpha, \alpha') = \sum_{\gamma \in G_n} w_\gamma (I_\gamma^{-1})_{\alpha\alpha'} I_\gamma, \qquad \alpha, \alpha' \in \mathbb{Z}_n \tag{4.2}$$

and, of course,

$$Z = \mathrm{Tr}(T^M) \tag{4.3}$$

We write $Z[S]$ to denote the dependence upon S.

Owing to the invariance of the trace under similarity transformations, we see that if

$$S' = (U \otimes I)\, S(U^{-1} \otimes I)$$

then the transfer matrix (4.1) is unchanged and hence $Z[S] = Z[S']$. Likewise a similarity

$$S'' = (I \otimes U)\, S(I \otimes U^{-1})$$

produces a different transfer matrix, say T'', which is related to the original T by

$$T'' = (U \otimes \cdots \otimes U)\, T(U^{-1} \otimes \cdots \otimes U^{-1})$$

and by (4.3) we see that $Z[S'] = Z[S]$. Thus in summary, we can state that for any $n \times n$ invertible matrices U_1 and U_2 that the transformation

$$S \mapsto U_1 \otimes U_2 S U_1^{-1} \otimes U_2^{-1}$$

leaves the partition function $Z[S]$ unchanged.

Now we consider some special cases of similarity transformations of the S matrix; in particular, those associated with the Heisenberg group H_n. First choose $U_1 = I_\beta$. Owing to the $\mathbb{Z}_n$ symmetry

$$S \mapsto I_\beta \otimes I S I_\beta^{-1} \otimes I = I \otimes I_\beta^{-1} S I \otimes I_\beta \tag{4.4}$$

In terms of the w_α's

$$I_\beta \otimes I S I_\beta^{-1} \otimes I = \sum_\alpha w_\alpha I_\beta I_\alpha I_\beta^{-1} \otimes I_\alpha^{-1} = \sum_\alpha \omega^{\langle \alpha, \beta \rangle} w_\alpha I_\alpha \otimes I_\alpha^{-1}$$

where we used (2.9). Hence the similarity (4.4) corresponds to $w_\alpha \mapsto \omega^{\langle \alpha, \beta \rangle} w_\alpha$.

The second type of similarity we will consider arises from the choice of new generators for the Heisenberg group H_n. Let $A \in SL(2, \mathbb{Z})$ and define

$$S^A = \sum_{\xi \in G_n} w_\xi I_\xi^A \otimes (I_\xi^A)^{-1} \tag{4.5}$$

where I_ξ^A is given by (2.10) and $A = \left(\begin{smallmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{smallmatrix}\right)$.

Using (2.11) we conclude

$$S^A = U_A \otimes U_A S U_A^{-1} \otimes U_A^{-1}$$

so that $Z[S^A] = Z[S]$. But using $I_\xi^A = \omega^{\alpha_1\alpha_2(1-\xi_1)+\beta_1\beta_2(1-\xi_2)+\xi_1\alpha_2+\xi_2\beta_1} I_{A\xi}$ it follows that $I_\xi^A \otimes I_\xi^{A-1} = I_{A\xi} \otimes I_{A\xi}^{-1}$; and hence,

$$S^A = \sum_{\xi \in G_n} w_\xi I_{A\xi} \otimes I_{A\xi}^{-1} = \sum_{\xi \in G_n} w_{A^{-1}\xi} I_\xi \otimes I_\xi^{-1}$$

This allows us to consider the similarity as an action on w_ξ; namely, $S \mapsto S^{A^{-1}}$ corresponds to $w_\xi \to w_{A\xi}$. We now have the following:

Theorem 4.1. Let S be any $\mathbb{Z}_n$ symmetric matrix, $Z[S]$ the corresponding partition function on a finite lattice with cyclic boundary conditions. Write

$$S = \sum_{\xi \in G_n} w_\xi I_\xi \otimes I_\xi^{-1}$$

then $Z[S]$ is invariant under the transformations on S given by

$$w_\xi \mapsto \omega^{\langle \xi, \gamma \rangle} w_\xi, \qquad \gamma \in \mathbb{Z}^2$$

$$w_\xi \mapsto w_{A\xi}, \qquad A \in SL(2, \mathbb{Z})$$

The γ depends only on its coset in $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n \times \mathbb{Z}_n = G_n$, and A depends only on its coset in $SL(2, \mathbb{Z})/\Gamma(n) \approx SL(2, \mathbb{Z}_n)$, where $\Gamma(n) = \{A \in SL(2, \mathbb{Z}) \mid A = I \bmod n\}$, the principal congruence subgroup. Furthermore, the action of $SL(2, \mathbb{Z})$ normalizes the action of $\mathbb{Z}^2$.

Proof. The invariance of $Z[S]$ has already been demonstrated and the first part of the second statement is obvious. The second part is almost as clear, it depends only on the observation that for any $G \in \Gamma(n)$,

$$G = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}, \qquad I_\alpha = h^{\alpha_1} g^{\alpha_2} = h, \qquad I_\beta = h^{\beta_1} g^{\beta_2} = g,$$

so $I_\xi^G = I_\xi^{id}$

As for the last statement, what we wish to show is that for any $A \in SL(2, \mathbb{Z})$, $\gamma \in \mathbb{Z}^2$ that the action of A^{-1} followed by the action of γ followed by the action of A is equivalent to the action of some γ' on w_ξ. Specifically,

$$w_\xi \mapsto w_{A^{-1}\xi} \to \omega^{\langle A^{-1}\xi, \gamma \rangle} w_{A^{-1}\xi} \mapsto \omega^{\langle A^{-1}\xi, \gamma \rangle} w_\xi$$

Since $\langle A^{-1}\xi, \gamma \rangle = \langle \xi, A\gamma \rangle$, $\gamma' = A\gamma$. ∎

As a generalization of the weak-graph duality we have the following:

Corollary 4.2. The partition function $Z[S]$ is invariant under

$$S^{ab} \mapsto \frac{1}{n} \sum_{\gamma \in G_n} S^{-\gamma_1 \gamma_2} \omega^{\langle (a,b), \gamma \rangle}$$

Proof. Choose $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ which corresponds to $\tau \to -1/\tau$ in the previous theorem. Then $Z[S]$ is invariant under

$$w_\xi = w_{(\xi_1, \xi_2)} \mapsto w_{(\xi_2, -\xi_1)}$$

In terms of the Boltzmann weights

$$\begin{aligned} S^{ab} &= \sum_{\gamma_1 \in Z_n} w_{(-a, \gamma_1)} \omega^{-b\gamma_1} \mapsto \sum_{\gamma_1 \in Z_n} w_{(\gamma_1, a)} \omega^{-b\gamma_1} \\ &= \sum_{\gamma_1 \in Z_n} \left(\frac{1}{n} \sum_{\gamma_2 \in Z_n} S^{-\gamma_1 \gamma_2} \omega^{\gamma_2 a} \right) \omega^{-b\gamma_1} \\ &= \frac{1}{n} \sum_{\gamma \in G_n} S^{-\gamma_1 \gamma_2} \omega^{-\langle \gamma, (a,b) \rangle} \quad \blacksquare \end{aligned}$$

Let A denote any I_α with $2\alpha = 0$, $\alpha \neq 0$. This is possible only if n is even and if $\alpha = (0, n/2)$, $(n/2, 0)$, or $(n/2, n/2)$. Following Ref. 12 we introduce for N even

$$P_0 = A \otimes I \otimes \cdots \otimes A \otimes I$$

and

$$P_e = I \otimes A \otimes \cdots \otimes I \otimes A$$

(N-fold tensor product). Then we have the following:

Lemma 4.3.

$$P_0 T P_e = P_e T P_0$$

Proof. Define the $n \times n$ matrices $R'(\alpha, \alpha')$, $R''(\alpha, \alpha')$, $\alpha, \alpha' \in Z_n$ by

$$R'(\alpha, \alpha')_{\lambda\lambda'} = (I \otimes ASA \otimes I)_{\lambda\alpha}^{\lambda'\alpha'}$$

and

$$R''(\alpha, \alpha')_{\lambda\lambda'} = (A \otimes ISI \otimes A)_{\lambda\alpha}^{\lambda'\alpha'}$$

Then a calculation shows

$$(P_0 T P_e)_{\boldsymbol{\alpha\alpha}'} = (\omega^{\alpha_1\alpha_2})^{N/2}\,\mathrm{Tr}(R'(\alpha_1, \alpha_1')\,R''(\alpha_2, \alpha_2')\cdots R'(\alpha_{N-1}, \alpha_{N-1}')\,R''(\alpha_N, \alpha_N'))$$

and

$$(P_e T P_0)_{\boldsymbol{\alpha\alpha}'} = (\omega^{\alpha_1\alpha_2})^{N/2}\,\mathrm{Tr}(R''(\alpha_1, \alpha_1')\,R'(\alpha_2, \alpha_2')\cdots R''(\alpha_{N-1}, \alpha_{N-1}')\,R'(\alpha_N, \alpha_N'))$$

Now

$$\begin{aligned} A\otimes ISI\otimes A &= A\otimes A^{-1}(I\otimes A)\,S(I\otimes A) \\ &= A\otimes A^{-1}(I\otimes A^{-1})\,S(I\otimes A)\,\omega^{\alpha_1\alpha_2} \\ &= A\otimes A^{-1}(A\otimes I)\,S(A^{-1}\otimes I)\,\omega^{\alpha_1\alpha_2} \\ &= (I\otimes A^{-1})\,S(A^{-1}\otimes I) \\ &= (I\otimes A)\,S(A\otimes I) \end{aligned}$$

that is, $R'(\alpha, \alpha') = R''(\alpha, \alpha')$. ∎

An important consequence of the preceding proof is the following:

Corollary 4.4. The transformation on T

$$T \mapsto P_0 T P_e$$

is equivalent to the transformation on S given by

$$S \mapsto A\otimes A^{-1}(I\otimes A^{-1})\,S(I\otimes A) = A\otimes A^{-1}(A\otimes I)\,S(A^{-1}\otimes I)$$

This leads to the following:

Proposition 4.5. The transformation on T given in the above corollary (or the equivalent transformation on S) preserves the partition function assuming M and N are even.

Proof.

$$\begin{aligned} Z[S] &= \mathrm{Tr}(T^M) \\ &= \mathrm{Tr}(P_0 T P_e P_e T P_0 \cdots P_e T P_0)(\pm 1)^M \\ &= \mathrm{Tr}((P_0 T P_e)^M) \quad ∎ \end{aligned}$$

Thus we have the additional symmetry.

Theorem 4.6. Assume n, N, and M are even. Let A be any I_α with $2\alpha = 0$. If we define $S' = (A \otimes A^{-1})S$, then $Z[S'] = Z[S]$.

Proof.

$$\begin{aligned}
Z[S'] &= Z[I \otimes AS'I \otimes A^{-1}] \\
&= Z[A \otimes ISI \otimes A^{-1}] \\
&= Z[(A \otimes A^{-1})(I \otimes ASI \otimes A^{-1})] \\
&= Z[(A \otimes A^{-1})(I \otimes A^{-1}SI \otimes A)] \\
&= Z[S](\pm 1)^{MN} \quad \blacksquare
\end{aligned}$$

We now express this symmetry transformation in terms of the w_ξ. Let I_α be such that $I_\alpha^2 = \omega^{\alpha_1 \alpha_2} I$, then

$$\begin{aligned}
I_\alpha \otimes I_\alpha^{-1} S &= I_\alpha \otimes I_\alpha^{-1} \sum_{\xi \in G_n} w_\xi I_\xi \otimes I_\xi^{-1} \\
&= \sum_{\xi \in G_n} w_\xi \omega^{\langle \xi, \alpha \rangle} I_{\alpha + \xi} \otimes I_{\alpha + \xi}^{-1} \\
&= \sum_{\xi' \in G_n} w_{\xi' - \alpha} \omega^{\langle \xi', \alpha \rangle} I_{\xi'} \otimes I_{\xi'}^{-1}
\end{aligned}$$

Thus the symmetry on the w_ξ's is

$$w_\xi \mapsto \omega^{\langle \xi, \alpha \rangle} w_{\xi + \alpha} \tag{4.6}$$

for $\alpha = (0, n/2)$, $(n/2, 0)$, and $(n/2, n/2)$, n even.

For $n = 2$ the above symmetries reduce to (i) every permutation of w_{00}, w_{01}, w_{10}, and w_{11} is achieved and (ii) any two of the w_α's may be negated. This is an easy consequence of Theorems 4.1 and 4.6 once one observes $SL(2, Z_2) \approx S_3$, the permutation group on three elements. It is a well-known result of Fan and Wu[10] that $Z[w_{00}, w_{01}, w_{10}, w_{11}]$ is invariant under any permutation of the w_α's and any choice of signs. To recover the Fan and Wu result it is sufficient to show that we are able to negate any one of the w_α's. This is done using a symmetry not available for $n \neq 2$; namely, the fact that a rotation of the lattice by 90° takes an allowable configuration into an allowable configuration. In terms of the Boltzmann weights $S^{00} \mapsto S^{00}$, $S^{01} \mapsto S^{01}$, $S^{10} \mapsto S^{11}$, and $S^{11} \mapsto S^{10}$ which in terms of the w_α's is w_{00}, w_{01}, w_{10} fixed and $w_{11} \mapsto -w_{11}$.

Up to this point the symmetries of $Z[S]$ have been for arbitrary w_α. Now we examine the above symmetries assuming the Belavin parametrization (3.2). To begin, we consider the first type of symmetry

$w_\xi \mapsto \omega^{\langle \xi, \alpha \rangle} w_\xi$, $\alpha \in \mathbb{Z}^2$. We slightly modify the Heisenberg operators T_a and S_b to

$$(U_\alpha f)(z) = \exp[\pi i \alpha_1^2 \tau + 2\pi i \alpha_1 (z + \eta)]\, f(z + \alpha_1 \tau + \alpha_2)$$

Then $(U_\alpha w_\xi)(z) = \omega^{\langle \xi, \alpha \rangle} w_\xi(z)$, and similarly

$$(U_\alpha S)(z) = I_\alpha \otimes I S(z)\, I_\alpha^{-1} \otimes I = I \otimes I_\alpha^{-1} S(z) I \otimes I_\alpha$$

We write $Z(z, w, \tau) = Z[S(z, w, \tau)]$. For $\alpha \in \mathbb{Z}^2$

$$Z(z + \alpha_1 \tau + \alpha_2, w, \tau) = \exp[-i\pi \alpha_1^2 MN\tau - 2\pi i \alpha_1 MN(z + \eta)]\; Z(z, w, \tau)$$

The $SL(2, \mathbb{Z})$ symmetry is

$$w_\xi \left(\frac{z}{\beta_1 \tau + \beta_2}, \frac{w}{\beta_1 \tau + \beta_2}, \frac{\alpha_1 \tau + \alpha_2}{\beta_1 \tau + \beta_2} \right) = \lambda_A(z, w, \tau)\, w_{A\xi}(z, w, \tau)$$

where $A = \left(\begin{smallmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$ and

$$\lambda_A(z, w, \tau) = \exp\left[i\pi \left(\frac{\beta_1 \tau + \beta_2 - 1}{\beta_1 \tau + \beta_2} z + \frac{\beta_1 [z^2 + 2z(w/n)]}{\beta_1 \tau + \beta_2} \right) \right]$$

The determination of λ_A is involved and the details can be found in the Appendix. In terms of the S matrix

$$S \left(\frac{z}{\beta_1 \tau + \beta_2}, \frac{w}{\beta_1 \tau + \beta_2}, \frac{\alpha_1 \tau + \alpha_2}{\beta_1 \tau + \beta_2} \right) = \lambda_A(z, w, \tau)\, S^{A^{-1}}(z, w, \tau)$$

so that the partition function satisfies

$$Z \left(\frac{z}{\beta_1 \tau + \beta_2}, \frac{w}{\beta_1 \tau + \beta_2}, \frac{\alpha_1 \tau + \alpha_2}{\beta_1 \tau + \beta_2} \right) = \lambda_A^{MN}(z, w, \tau)\, Z(z, w, \tau) \tag{4.7}$$

Finally, we would like to interpret the last symmetry in terms of a transformation on a variable. To do so, note for $\beta \in \mathbb{Z}^2$

$$w_\alpha(z, w + \beta_1 \tau + \beta_2, \tau) = \exp\left(-2\pi i \frac{\beta_1}{n} z \right) w_{\alpha + \beta}(z, w, \tau)$$

Thus for n even and β_1, $\beta_2 = 0$, $n/2$ we see

$$S(z, w + \beta_1 \tau + \beta_2, \tau) = \exp\left(-2\pi i \frac{\beta_1}{n} z \right) I_\beta \otimes I_\beta^{-1} S(z, w, \tau) \tag{4.8}$$

and

$$Z(z, w + \beta_1 \tau + \beta_2, \tau) = \exp\left(-2\pi i z \frac{\beta_1}{n} MN \right) Z(z, w, \tau)$$

5. INVERSION RELATIONS FOR U_i AND V_i

In this section we derive inversion relations for the Baxter face operators U_i and V_i assuming the Belavin parametrization. We assume the reader is familiar with Chapter 13 of Baxter.[3] We first shift our point of view from a vertex description to the dual spin description. The spin variables are indexed by the dual lattice sites and take values in $\mathbb{Z}_n$ or $\mathscr{M}_n$ depending on whether we use an additive or multiplicative representation for the group law. Figure 2 gives the n to 1 map from spin configurations of an "IRF model" to the $\mathbb{Z}_n$ vertex model.

It is well known that a solution to the Yang–Baxter equation,

$$U_{i+1}(z_1)\, U_i(z_1+z_2)\, U_{i+1}(z_2) = U_i(z_2)\, U_{i+1}(z_1+z_2)\, U_i(z_1) \tag{5.1}$$

implies the inversion relation $U_i(z)\, U_i(-z) = n(z)\, id$. In previous work on exactly solvable models there exists a rotational symmetry so that from a U_i inversion relation, a corresponding V_i inversion relation follows. This route is not available for the $\mathbb{Z}_n$ Baxter model (for $n > 2$).

For completeness we give a proof of both the U_i inversion relation and the V_i inversion relation. Since several authors[7,8,20] have proved (5.1) the U_i inversion relation is not new; however, to calculate the free energy and order parameters both inversion relations are required. The V_i inversion relation given below is new. Our final results in this section are given in Theorem 5.6. It should be noted that all proofs of (5.1) (for $n > 2$) as well as our proofs below are of function theoretic nature and tend to have a verification quality to them. We mention that from the U_i inversion relation and using the transformation properties of w_α we can prove the Yang–Baxter relation. This is the route of Cherednik[8] and though our

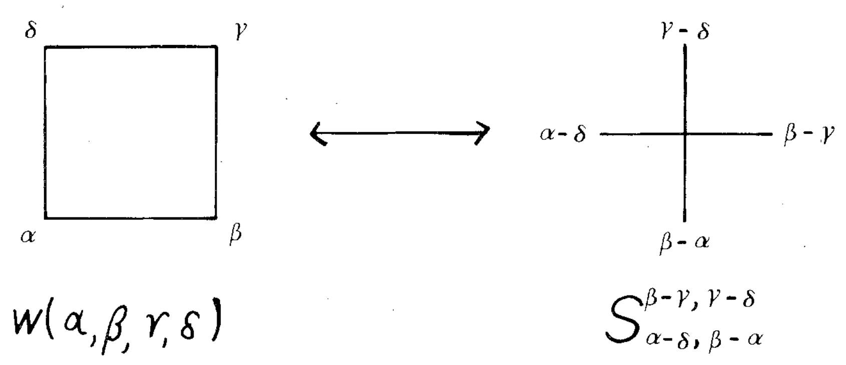

Fig. 2. The Boltzmann weight $w(\alpha, \beta, \gamma, \delta)$ for the spin configuration $\alpha, \beta, \gamma, \delta$ in terms of the corresponding vertex configuration and vertex weights S^{kl}_{ij}.

proof is technically different than that in Ref. 8, we do not include it here since it is similar in spirit; namely, function theoretic.

Recall that (see Baxter,[3] p. 369)

$$(U_i)_{\boldsymbol{\sigma},\boldsymbol{\sigma}'} = \delta(\sigma_1, \sigma_1') \cdots \delta(\sigma_{i-1}, \sigma_{i-1}')\, w(\sigma_i, \sigma_{i+1}, \sigma_i', \sigma_{i-1}) \times \delta(\sigma_{i+1}, \sigma_{i+1}') \cdots \delta(\sigma_m, \sigma_m') \tag{5.2a}$$

and

$$(V_i)_{\boldsymbol{\sigma},\boldsymbol{\sigma}'} = \delta(\sigma_1, \sigma_1') \cdots \delta(\sigma_{i-1}, \sigma_{i-1}')\, w(\sigma_{i-1}, \sigma_i, \sigma_{i+1}, \sigma_i') \times \delta(\sigma_{i+1}, \sigma_{i+1}') \cdots \delta(\sigma_m, \sigma_m') \tag{5.2b}$$

Thus U_i acts from NE to SW and V_i acts from NW to SE. We define the $n^3 \times n^3$ matrices U and V by

$$\begin{aligned}
U_{\sigma_1\sigma_2\sigma_3}^{\sigma_1'\sigma_2'\sigma_3'} &\equiv \delta_{\sigma_1}^{\sigma_1'} \delta_{\sigma_3}^{\sigma_3'} w(\sigma_2, \sigma_3, \sigma_2', \sigma_1) \\
&= \delta_{\sigma_1}^{\sigma_1'} \delta_{\sigma_3}^{\sigma_3'} S_{\sigma_2-\sigma_1,\sigma_3-\sigma_2}^{\sigma_3-\sigma_2',\sigma_2'-\sigma_1} \\
&= \delta_{\sigma_1}^{\sigma_1'} \delta_{\sigma_3}^{\sigma_3'} S^{\sigma_3+\sigma_1-(\sigma_2+\sigma_2'),\sigma_2'-\sigma_2} \\
V_{\sigma_1\sigma_2\sigma_3}^{\sigma_1'\sigma_2'\sigma_3'} &\equiv \delta_{\sigma_1}^{\sigma_1'} \delta_{\sigma_3}^{\sigma_3'} w(\sigma_1, \sigma_2, \sigma_3, \sigma_2') \\
&= \delta_{\sigma_1}^{\sigma_1'} \delta_{\sigma_3}^{\sigma_3'} S^{-(\sigma_1+\sigma_3)+(\sigma_2+\sigma_2'),\sigma_3-\sigma_1}
\end{aligned}$$

Furthermore, it is convenient to let

$$U = \bigoplus_{\sigma_1,\sigma_2 \in \mathbb{Z}_n} U_{\sigma_1}^{\sigma_2}, \qquad V = \bigoplus_{\sigma_1,\sigma_2 \in \mathbb{Z}_n} V_{\sigma_1}^{\sigma_2} \tag{5.3}$$

where $U_{\sigma_1}^{\sigma_2}$, $V_{\sigma_1}^{\sigma_2}$ are $n \times n$ matrices

$$(U_{\sigma_1}^{\sigma_2})_{\sigma\sigma'} = S^{(\sigma_1+\sigma_2)-(\sigma+\sigma'),\sigma'-\sigma} \tag{5.4a}$$

$$(V_{\sigma_1}^{\sigma_2})_{\sigma\sigma'} = S^{-(\sigma_1+\sigma_2)+(\sigma+\sigma'),\sigma_2-\sigma_1} \tag{5.4b}$$

Observe that $V_{\sigma_1}^{\sigma_2}$ is a symmetric matrix. See Fig. 3 for a geometric presentation of these operators. Thus inverting U and V is reduced to inverting $U_{\sigma_1}^{\sigma_2}$ and $V_{\sigma_1}^{\sigma_2}$ for every σ_1, $\sigma_2 \in \mathbb{Z}_n$. A further reduction is possible by using (3.5) and (5.4):

Lemma 5.1.

(i) $$U_{\sigma_1}^{\sigma_2}(z, w) = \exp\left[-2\pi i(\sigma_1+\sigma_2)\frac{z}{n}\right] U_0^0(z, w-(\sigma_1+\sigma_2)\tau)$$

(ii) $$V_{\sigma_1}^{\sigma_2}(z, w) = \lambda_1(z, w)\, V_0^0(z+(\sigma_2-\sigma_1)\tau, w+(\sigma_1+\sigma_2)\tau)$$

for every $\sigma_1, \sigma_2 \in \mathbb{Z}_n$.

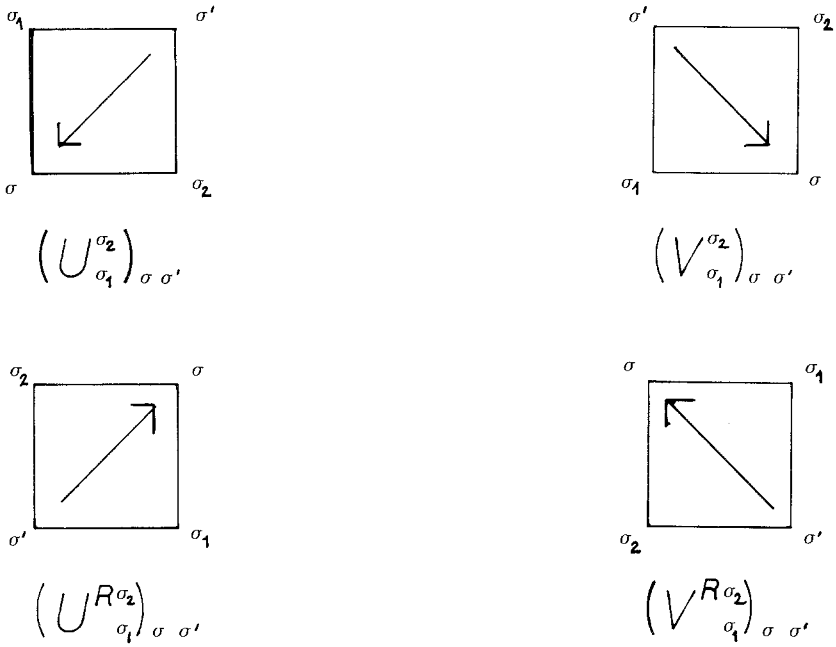

Fig. 3. The face operators $U_{\sigma_1}^{\sigma_2}$, $V_{\sigma_1}^{\sigma_2}$, $U_{\sigma_1}^{R\sigma_2}$, $V_{\sigma_1}^{R\sigma_2}$.

The multiplier $\lambda_1(z, w)$ is easily determined but it will be convenient to calculate a different overall multiplier at a later stage.

The next lemma gives the inversion relation for U_0^0.

Lemma 5.2.

$$U_0^0(z)\, U_0^0(-z) = n(z, w)\, id$$

where

$$n(z, w) = n^2 \frac{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z+w, \tau)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(-z+w, \tau)}{\vartheta^2\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(w, \tau)}$$

Proof. Using (5.4a)

$$\varphi_{\sigma\sigma'}(z) \equiv [U_0^0(z)\, U_0^0(-z)]_{\sigma\sigma'} = \sum_{a \in \mathbb{Z}_n} S^{-\sigma-a, a-\sigma}(z)\, S^{-a-\sigma', \sigma'-a}(-z) \quad (5.5)$$

From (3.5a)

$$\varphi_{\sigma\sigma'}(z+n\tau)=\exp[-2\pi i(n^2\tau+2nz)]\,\varphi_{\sigma\sigma'}(z)$$
$$\varphi_{\sigma\sigma'}(z+1)=\omega^{\sigma-\sigma'}\varphi_{\sigma\sigma'}(z)$$

which implies $\varphi_{\sigma\sigma'}$ has $2n$ zeros in $E_{n\tau}$ with sum $(\sigma'-\sigma)\tau \bmod \Lambda_{n\tau}$. Since $S^{ab}(k\tau)$ is nonzero only if $k=-b(n)$, $\varphi_{\sigma\sigma'}(k\tau)$ vanishes if $\sigma\neq\sigma'$, i.e., $z=k\tau$, $k=0,\dots,n-1$ are n zeros of $\varphi_{\sigma\sigma'}$, $\sigma\neq\sigma'$. Again using (3.5a) we see that at $z=-w+k\tau$, $\varphi_{\sigma\sigma'}$ is proportional to

$$\sum_{a\in\mathbb{Z}_n} S^{-\sigma-a,a-\sigma+k}(-w)\,S^{-a-\sigma',\sigma'-a-k}(w)$$

Recall that $S^{ab}(z)$ is zero at $z=(a-b)\tau-w$. Thus the first factor in each term in this sum is zero if $-\sigma-a=a-\sigma+k(n)$ or $-2a=k(n)$. This elementary congruence has exactly one solution if n is odd, and if n is even there are either two distinct values of a or no values at all for which this is true. In any case there are always an even number of nonzero terms in the above sum. We break the sum of the remaining nonzero terms into two parts. We do this via the correspondence $a\leftrightarrow -a-k$. Thus $\{a\in\mathbb{Z}_n \mid -2a\neq k\}=\bigcup_{i=1}^{x}\{a_i,-a_i-k\}$, where $x=[n/2]$ if n is odd or $-2a=k(n)$ has no solutions or $x=[n/2]-1$ if $-2a=k(n)$ has two solutions. Thus the sum becomes

$$\sum_i S^{-\sigma-a_i,a_i+k-\sigma}(-w)\,S^{-a_i-\sigma',\sigma'-a_i-k}(w)$$
$$+S^{a_i+k-\sigma,-a_i-\sigma}(-w)\,S^{a_i+k-\sigma',\sigma'+a_i}(w)$$

which in view of (3.13) is identically zero. Thus we have located $2n$ zeros of $\varphi_{\sigma\sigma'}$, $\sigma\neq\sigma' \bmod \Lambda_{n\tau}$; namely, $z=k\tau$, $-w+k\tau$, $k=0,\dots,n-1$. Since their sum is incorrect we conclude $\varphi_{\sigma\sigma'}\equiv 0$ for $\sigma\neq\sigma'$.

Let $\varphi_\sigma(z)\equiv\varphi_{\sigma\sigma}(z)$. Observe that $\sigma\neq\sigma'$ was not used in determining that $z=-w+k\tau$ are zeros. Thus we have located n zeros of φ_σ. We now show that the other n zeros [φ_0 is not identically zero since $\varphi_\sigma(0)=n^2$] are located at $z=w+k\tau$, $k=0,\dots,n-1$. To establish this we first show φ_0 is independent of σ. Look at $D_{\sigma\sigma'}(z)=\varphi_\sigma(z)-\varphi_{\sigma'}(z)$. $D_{\sigma\sigma'}$ has the same transformation properties as φ_σ so has $2n$ zeros in $\Lambda_{n\tau}$. Now $\varphi_\sigma(k\tau)$ can be evaluated and the result is independent of $\sigma\in Z_n$. Thus $D_{\sigma\sigma'}$ is zero at $z=k\tau$, $k=0,\dots,n-1$ and, of course, at $z=-w+k\tau$, $k=0,\dots,n-1$. This again gives $2n$ zeros with the wrong sum and hence $D_{\sigma\sigma'}\equiv 0$. Let $\varphi_\sigma(z)=n(z)$. Thus we have shown that $U_0^0(z)\,U_0^0(-z)$ is proportional to the identity. All that remains is the explicit result for $n(z)$.

Let $a \to -a-\sigma$ in the sum in (5.5) to obtain

$$\varphi_\sigma(z) = \sum_{a \in Z_n} S^{a,-a-2\sigma}(z)\, S^{a,a+2\sigma}(-z)$$

From this representation of φ_σ follows the transformation properties:

$$\varphi_\sigma(z+1) = \varphi_\sigma(z)$$
$$\varphi_\sigma(z+2\tau) = \exp[-2\pi i(4\tau+4z)]\, \varphi_\sigma(z)$$

Hence there are four zeros in $E_{2\tau}$ and they sum to zero mod $\Lambda_{2\tau}$. We already know that $-w$, $-w+\tau$ are zeros. Recalling that $S^{ab}(-w+k\tau, w, \tau) = 0$ if $k = a-b$ we see that $\varphi_\sigma(w+2\sigma\tau) = 0$ which, with independence of the σ label, implies w is a zero. From the sum condition the fourth zero is at $w+\tau$. It is now straightforward to check that the stated formula has the same transformation properties and zero set. The overall constant is obtained by setting $z=0$. ▮

Lemma 5.3.

$$U_{\sigma_1}^{\sigma_2}(z)\, U_{\sigma_1}^{\sigma_2}(-z) = n(z, w)\, id \qquad \text{for every } \sigma_1, \sigma_2 \in \mathbb{Z}_n$$

Proof. This is immediate from Lemmas 5.1 and 5.2. ▮

Lemma 5.4.

$$V_0^0(z)\, V_0^0(-z-nw) = m(z, w)\, id$$

where

$$m(z, w) = n^2 e^{\pi i n w} \frac{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z, \tau)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(-z-nw, \tau)}{\vartheta^2\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(w, \tau)}$$

Proof. Defining $\varphi_{\sigma\sigma'}(z)$ to be the $\sigma\sigma'$ matrix element of $V_0^0(z)\, V_0^0(-z-nw)$ we have

$$\varphi_{\sigma\sigma'}(z) = \sum_{a \in \mathbb{Z}_n} S^{\sigma+a,0}(z)\, S^{\sigma'+a,0}(-z-nw)$$

which has the transformation properties

$$\varphi_{\sigma\sigma'}(z+n\tau) = \exp[-2\pi i(n^2\tau + n^2 w + 2nz)]\, \varphi_{\sigma\sigma'}(z)$$
$$\varphi_{\sigma\sigma'}(z+1) = \omega^{\sigma'-\sigma}\varphi_{\sigma\sigma'}(z)$$

So again there are $2n$ zeros with sum $(\sigma-\sigma')\tau - n^2 w \bmod \Lambda_{n\tau}$. Since $S^{ab}(k\tau)$ vanishes unless $k=-b$ we obtain $2n-2$ zeros at $z=k\tau$, $k\tau-nw$, $k=1,\dots,n-1$. We claim that 0 and $-nw$ are the remaining zeros mod $\Lambda_{n\tau}$.

Setting $z=0$ in $\varphi_{\sigma\sigma'}$, using the fact that $S^{a,0}(0)$ is independent of a, and (3.12) we see the relevant identity to prove is

$$\sum_{a\in\mathbb{Z}_n} \frac{\vartheta\begin{bmatrix} -a/n+\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(-(n-1)w, n\tau)}{\vartheta\begin{bmatrix} -a/n+\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(w, n\tau)} = 0$$

or clearing denominators we must show

$$F(w) \equiv \sum_{a\in\mathbb{Z}_n} \vartheta\begin{bmatrix} -a/n+\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(-(n-1)w, n\tau) \prod_{\substack{l\in\mathbb{Z}_n \\ l\neq -a}} \vartheta\begin{bmatrix} l/n+\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(w, n\tau) = 0$$

F has the transformation properties

$$F(w+\tau) = \exp\{-2\pi i[\tfrac{1}{2}(n-1)\tau + (n-1)w]\}\, F(w)$$
$$F(w+1) = F(w)$$

implying there are $n-1$ zeros in E_τ. A simple calculation shows

$$F\left(\frac{r}{n}\right) = \sum_{a\in Z_n} \omega^{-ar}(-1)^r\, \vartheta\begin{bmatrix} -a/n+\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\left(\frac{r}{n}, n\tau\right)$$
$$\times \prod_{\substack{l\in\mathbb{Z}_n \\ l\neq -a}} \vartheta\begin{bmatrix} l/n+\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\left(\frac{r}{n}, n\tau\right), \qquad l\neq -a$$
$$= C_0\,\vartheta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\left(\frac{r}{n}, \tau\right)(-1)^r\,\delta_{r,0}$$

giving n zeros in E_τ which implies $F(w)\equiv 0$. Since $\varphi_{\sigma\sigma'}(-z-nw) = \varphi_{\sigma',\sigma}(z)$ we have that $z=-nw$ is a zero, and since $\sigma'\neq\sigma$ the $2n$ zeros of $\varphi_{\sigma\sigma'}$ have the wrong sum. Hence

$$\varphi_{\sigma\sigma'}(z) = m(z, w)\,\delta_{\sigma\sigma'}$$

where $m(z, w)$ is independent of σ as is clear from the definition of $\varphi_{\sigma\sigma}$. Thus we need only show that $m(z, w)$ has the form claimed. The zero set of $m(z, w)$ and $\varphi_{\sigma\sigma}$ coincide and they have identical transformation properties in z. Thus they agree up to a multiplicative constant. We now show this constant is 1.

In principal the multiplicative factor could be a function of w. To see that this is not the case, we examine the zero and poles of m and φ_σ as functions of w. First,

$$\varphi_\sigma(z, w+n\tau) = (-1)^{n^2} \exp[-\pi i(n^4 - 3n^2)\tau - 2\pi i n^2 z - 2\pi i(n^3 - 2n)w] \times \varphi_\sigma(z, w)$$

$$\varphi_\sigma(z, w+1) = \varphi_\sigma(z, w)$$

Hence the number of zeros minus the number of poles of φ_σ in $E_{n\tau}$ is $n^3 - 2n$. But $\varphi_\sigma(z, w)$ is zero at $z = -nw + k\tau \bmod \Lambda_{n\tau}$ which implies it is zero at $w = -z/n + (1/n)(k\tau + m_2) + m_1\tau$, $k, m_1, m_2 = 0, \ldots, n-1$ which gives n^3 zeros. $S^{a,0}(z, w, \tau)$ has an unique simple pole at $w = a\tau$. So $\varphi_\sigma(z, w)$ has poles at $w = a\tau$ (order 2), $a = 0, \ldots, n-1$. So the zeros and poles in w are exactly as those of the m which has the same transformation properties in w. Thus φ_σ and m agree up to a multiplicative factor, say $c_0(\tau)$, independent of z and w.

We now replace S^{ab} by $\tilde{S}^{ab} = \vartheta[{}^{1/2}_{1/2}](w, \tau)\, S^{ab}(z, w, \tau)$. Let γ, independent of w, be such that

$$\vartheta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(w, \tau) = \gamma \prod_{k \in \mathbb{Z}_n} \vartheta\begin{bmatrix} k/n + \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(w, n\tau) \tag{5.6}$$

So that from (3.10)

$$\tilde{S}^{a0}(z, 0, \tau) = \frac{\gamma n e^{-\pi i z}}{\prod_{k=1}^{n-1} \vartheta\begin{bmatrix} k/n + \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(0, n\tau)} \vartheta\begin{bmatrix} -a/n + \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(z, n\tau)$$
$$\times \prod_{k=1}^{n-1} \vartheta\begin{bmatrix} k/n + \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(z, n\tau) \prod_{\substack{k \in \mathbb{Z}_n \\ k \neq -a}} \vartheta\begin{bmatrix} k/n + \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(0, n\tau)$$

The last factor is zero unless $a = 0$. Hence

$$\tilde{S}^{a0}(z, 0, \tau) = n\delta_{a,0} e^{-\pi i z} \vartheta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(z, \tau)$$

Thus we have for all w

$$\sum_{a \in \mathbb{Z}_n} \tilde{S}^{a,0}(z, w, \tau)\, \tilde{S}^{a,0}(-z - nw, w, \tau)$$
$$= c_0 n^2 e^{\pi i n w} \vartheta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(z, \tau) \vartheta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(-z - nw, \tau)$$

and from the special case $w = 0$ we conclude that $c_0 = 1$. ∎

Lemma 5.5.

$$V_{\sigma_1}^{\sigma_2}(z)\, V_{\sigma_2}^{\sigma_1}(-z-nw) = m(z, w)\, id$$

Proof. By Lemmas 5.1 and 5.4

$$[V_{\sigma_1}^{\sigma_2}(z, w)]^{-1} = \lambda_1^{-1}(z, w)\{m(z+(\sigma_2-\sigma_1)\tau, w+(\sigma_1+\sigma_2)\tau)\}^{-1}$$
$$\times\; V_0^0(-z-nw-(\sigma_2-\sigma_1)\tau-n(\sigma_1+\sigma_2)\tau, w+(\sigma_1+\sigma_2)\tau)$$

But

$$m(z+(\sigma_2-\sigma_1)\tau, w+(\sigma_1+\sigma_2)\tau) = \lambda_2(z, w)\, m(z, w)$$

$$V_0^0(-z-nw+(\sigma_1-\sigma_2)\tau-n(\sigma_1+\sigma_2)\tau, w+(\sigma_1+\sigma_2)\tau$$
$$= \lambda_3'(z, w)\; V_0^0(-z-nw+(\sigma_1-\sigma_2)\tau, w+(\sigma_1+\sigma_2)\tau)$$
$$= \lambda_3(z, w)\; V_{\sigma_2}^{\sigma_1}(-z-nw, w)$$

where $\lambda_i(z, w)$ are multipliers of the form $\exp(A+Bz+Cw)$.

Putting this together we have

$$[V_{\sigma_1}^{\sigma_2}(z, w)]^{-1} = \lambda^{-1}(z, w)[m(z, w)]^{-1}\; V_{\sigma_1}^{\sigma_2}(-z-nw, w)$$

where $\lambda^{-1}(z, w) = \lambda_1^{-1}(z, w)\,\lambda_2^{-1}(z, w)\,\lambda_3(z, w)$. We must show $\lambda = 1$. The multiplier is necessarily of the form

$$\lambda(z, w) = \lambda_0 \exp(\pi i A_1 z + \pi i A_2 w)$$

If $\varphi_\sigma^{\sigma_1\sigma_2}(z, w)$ denotes the σ, σ element of $V_{\sigma_1}^{\sigma_2}(z, w)\; V_{\sigma_2}^{\sigma_1}(-z-nw, w)$, then we have established that

$$\varphi_\sigma^{\sigma_1\sigma_2}(z, w) = \lambda(z, w)\, m(z, w)$$

Now $\varphi_\sigma^{\sigma_1\sigma_2}(z, w)$ and $m(z, w)$ have the same transformation properties under $z \to z+n\tau$, $z \to z+1$, $w \to w+n\tau$, and $w \to w+1$ which implies $A_1 = A_2 = 0$. To show $\lambda_0 = 1$ proceed as in the proof that $c_0 = 1$. ∎

Thus we have

Theorem 5.6. Let $U_i(z, w)$ be the NE to SW face operator, V_i the NW to SE face operator, V_i^R the face operator rotated 180° (see Fig. 3), and define

$$h(z, w) = \frac{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z, \tau)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(w, \tau)}$$

then

$$U_i(z, w)\, U_i(-z, w) = n^2 h(z+w, w)\, h(-z+w, w)\, id$$

$$V_i(z, w)\, V_i^R(-z-nw, w) = n^2 h(z, w)\, h(-z-nw, w)\, e^{\pi i n w} id$$

Proof. In view of the previous lemmas, we have

$$V_i^{-1}(z, w) = \bigoplus_{\sigma_1, \sigma_2} [V_{\sigma_1}^{\sigma_2}(z, w)]^{-1} = \bigoplus_{\sigma_1, \sigma_2} \frac{V_{\sigma_2}^{\sigma_1}(-z-nw, w)}{m(z, w)}$$

but

$$(V_{\sigma_2}^{\sigma_1})_{\sigma\sigma'} = (V_{\sigma_2}^{\sigma_1})_{\sigma'\sigma} = (V_{\sigma_1}^{R\sigma_2})_{\sigma\sigma'}$$

(see Fig. 3). Hence

$$V_i^{-1}(z, w) = \bigoplus_{\sigma_1, \sigma_2} \frac{V_{\sigma_1}^{R\sigma_2}(-z-nw, w)}{m(z, w)} = \frac{1}{m(z, w)} V_i^R(-z-nw, w) \quad \blacksquare$$

Since $S(z+k\tau)$, $k \in n\mathbb{Z}$, is a multiple of $S(z)$ [see (3.6)], we can derive additional inversion relations by translations in $k\tau$, $k \in n\mathbb{Z}$.

For completeness, we state the following:

Theorem 5.7. If $w_\alpha(z)$ is given by the Belavin parametrization (3.2) and $U_i(z)$ is the NE to SW face operator as defined in (5.2a), then the Yang–Baxter (or star-triangle) relation (5.1) is satisfied for all $z_1, z_2 \in E_\tau$.

6. FREE ENERGY

Using the inversion relations for U_i and V_i derived in the previous section and Baxter's method of corner transfer matrices,[2,3] we can derive inversion relations for the partition function per site in the thermodynamic limit, $\kappa(z, w, \tau)$ or $\kappa(z)$ for short. The reader is reminded that the method of corner transfer matrices has several assumptions which we implicitly use when applying this technique. Since we have not improved upon this method, we refer the reader to Baxter[2-4] for a discussion of these assumptions. The two inversion relations for $\kappa(z)$ are

$$\kappa(z)\, \kappa(-z) = n(z, w) \tag{6.1a}$$

and

$$\kappa(z)\, \kappa(-z-nw) = f(w, \tau)\, m(z, w) \tag{6.1b}$$

where we are assuming that $0 < \operatorname{Im} z < \operatorname{Im}(-nw/2) < \operatorname{Im} \tau$ and $f(w, \tau) = [\kappa^2(-nw/2, w)]/[m(-nw/2, w)]$. The inversion relation (6.1a) is assumed to hold in an open strip containing $\operatorname{Im} z = 0$ and (6.1b) in an open strip containing $\operatorname{Im} z = -nw/2$.

To parallel the notation of Baxter we drop the overall term $n \exp(-i\pi z)$ in the Belavin parametrization (3.10). This changes the definitions of $n(z, w)$ and $m(z, w)$: drop the factor n^2 in $n(z, w)$ and drop the factor $n^2 \exp(\pi i n w)$ in $m(z, w)$. Using (3.4) [and removing a factor $n \exp(-i\pi z)$ from each weight] we see that in the limit $q \to 0$ and then $w \to -i\infty$ that the dominant weight is $S^{n-1,0}$ and that in this limit

$$Z \sim (S^{n-1,0})^{MN}$$

where

$$S^{a,0}(z) \sim \exp\left[2\pi i \left(\frac{1}{2} - \frac{a}{n}\right) z\right] \tag{6.2}$$

Thus if we assume $\log \kappa(z)$ has the expansion

$$\log \kappa(z) = Lz + \sum_{k \in \mathbb{Z}} C_k \exp(2\pi i k z) \tag{6.3}$$

we conclude that $L = -\pi i[(n-2)/n]$. Note that we assume (6.3) is valid slightly outside the region $0 < \operatorname{Im} z < \operatorname{Im}(-nw/2)$. This is the fundamental analyticity assumption in this method, see Baxter.[4] To solve (6.1) using (6.3) we need an expansion of $\log h(z, w)$. This is easily derived from the product expansion of theta functions:

$$\begin{aligned} \log h(z, w) = -\pi i(z+w) + \sum_{k=1}^{\infty} \frac{1}{k(1-q^{2k})} & [\exp(-2\pi i k w) + q^{2k} \exp(2\pi i k w) \\ & - \exp(2\pi i k z) - q^{2k} \exp(-2\pi i k z)] \end{aligned} \tag{6.4}$$

which is valid for $0 < \operatorname{Im} z < \operatorname{Im}(-w) < \operatorname{Im} \tau$. It is now straightforward to solve for the coefficients C_k in (6.3). We first find that $f(w, \tau) \equiv 1$ and if we set $x = \exp(-i\pi w)$, then

$$C_0 = \sum_{l=1}^{\infty} \frac{x^{2l} + q^{2l} x^{-2l}}{l(1-q^{2l})} \tag{6.5a}$$

and

$$C_k = \frac{x^{-2(n-1)k} - 1 + q^{2k}(x^{-2(n+1)k} - x^{-2nk})}{k(1-q^{2k})(1-x^{-2nk})} \tag{6.5b}$$

for $k \neq 0$. These expressions are valid for $0 < \operatorname{Im} z < \operatorname{Im}(-nw/2)$ and $0 < \operatorname{Im}(-w) < \operatorname{Im}(\tau)$.

For $n=2$ we set $z=iu$, $w=-i\lambda$, $0<u<\lambda$, so that $x=\exp(-\pi\lambda)$ and define $z_B=x^{-1}\exp(-2\pi u)$, $q_B=q^2$, then the sum $\sum_{k\in\mathbb{Z},k\neq 0} C_k \exp(2\pi ikz)$ reduces to $-\sum_{k=1}^{\infty} [(x^{3k}+q_B^k x^{-k})(z_B^k+z_B^{-k})]/[k(1-q_B^k)(1+x^{2k})]$which is precisely Baxter's result for the eight-vertex model (see, for example, Eq. (13.7.1) in Ref. 3). The term involving C_0 depends on the overall normalization of the Boltzmann weights.

To compare the $q\to 0$ limit of $\kappa(z, w, \tau)$ with the results of Babelon, de Vega, and Viallet[5] it is convenient to rewrite (6.5b) as

$$C_k=-\frac{1}{k(1-q^{2k})}\left\{q^{2k}x^{-(n+1)k}\frac{\sinh(\pi ikw)}{\sinh(\pi inkw)}+x^k\frac{\sinh[\pi i(n-1)kw]}{\sinh(\pi inkw)}\right\}$$

and

$$C_{-k}=-\frac{1}{k(1-q^{2k})}\left\{x^{(n+1)k}\frac{\sinh(\pi ikw)}{\sinh(\pi inkw)}+q^{2k}x^{-k}\frac{\sinh[\pi i(n-1)\,kw]}{\sinh(\pi inkw)}\right\}$$

for $k=1, 2,\dots$.

Now use the elementary identities

$$e^{\pi inkw}\sinh(\pi ikw)=e^{\pi ikw}\sinh(\pi inkw)-\sinh[\pi i(n-1)\,kw]$$
$$e^{-\pi inkw}\sinh(\pi ikw)=e^{-\pi ikw}\sinh(\pi inkw)-\sinh[\pi i(n-1)\,kw]$$

to write

$$\begin{aligned}\log\kappa(z, w, \tau)= &-2\pi i\left(\frac{n-1}{n}\right)z+\pi iz+\left[\sum_{k=1}^{\infty}\frac{x^{2k}+q^{2k}x^{-2k}}{k(1-q^{2k})}\right.\\ &\left.-\sum_{k=1}^{\infty}\frac{q^{2k}x^{-2k}\exp(2\pi ikz)+x^{2k}\exp(-2\pi ikz)}{k(1-q^{2k})}\right]\\ &-2\sum_{k=1}^{\infty}\frac{x^k}{k(1-q^{2k})}\frac{\sinh[\pi i(n-1)\,kw]}{\sinh(\pi inkw)}\\ &\times\sinh(2\pi ikz)(1-x^{-2k}q^{2k})\end{aligned}$$

Using the identity (6.4), the portion in square brackets is

$$\log\frac{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(w+z, \tau)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(w, \tau)}-\pi iz$$

Thus we conclude

$$\log \kappa(z, w, \tau) = -2\pi i\left(\frac{n-1}{n}\right) z + \log \frac{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z+w, \tau)}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(w, \tau)}$$

$$- 2\sum_{k=1}^{\infty} \frac{\exp(-\pi ikw)}{k(1-q^{2k})} \frac{\sinh[\pi i(n-1)\,kw]}{\sinh(\pi inkw)}$$

$$\times [1-\exp(+2\pi ikw)\, q^{2k}] \sinh(2\pi ikz) \qquad (6.6)$$

The manipulations of the above series are valid for $0 < \operatorname{Im} z < -\operatorname{Im} w < \operatorname{Im} \tau$. Now the $q \to 0$ limit of (6.6) is

$$\lim_{\tau \to i\infty} \log \kappa(z, w, \tau)$$

$$= -2\pi i\left(\frac{n-1}{n}\right) z + \log \frac{\sin \pi(z+w)}{\sin \pi w}$$

$$-2\sum_{k=1}^{\infty} \frac{\exp(-\pi ikw)}{k} \frac{\sinh[\pi i(n-1)\,kw]}{\sinh(\pi inkw)} \sinh(2\pi ikz) \qquad (6.7)$$

Identifying $w = -i(\gamma/\pi)$, $\gamma > 0$, $z = iu/\pi$, we see that (6.7) is (up to a normalization) the Babelon, de Vega, and Viallet result[5] derived using a nested Bethe ansatz. This verification of (6.5) [or (6.6)] in the $q \to 0$ limit is strong evidence that the analyticity assumptions in the inversion relation method are valid for the $\mathbb{Z}_n$ Baxter model.

7. SPONTANEOUS MAGNETIZATION

The last result we will obtain is the spontaneous magnetization, M_0, of the $\mathbb{Z}_n$ Baxter model in the ferromagnetic regime. As in Baxter,[3] the spontaneous magnetization is reduced to calculating

$$M_0 = \frac{\operatorname{Tr}(SA_d(2\lambda))}{\operatorname{Tr}(A_d(2\lambda))} \qquad (7.1)$$

where $A_d(z)$ is the diagonal corner transfer matrix and λ is an inversion point lying in the ferromagnetic region. The translationally invariant ferromagnetic ground state is the region where $S^{00}(z, w, \tau)$ is dominant. This corresponds to

$$-\operatorname{Im} \tau < \operatorname{Im}(z), \qquad \operatorname{Im}(w) < 0 \qquad (7.2)$$

so that $(w+\tau)\in\mathbb{H}$. The inversion point λ lying in this region is $\lambda = -(n/2)(w+\tau)$. We must also require $\mathrm{Im}(-(n/2)(w+\tau)) > -\mathrm{Im}\,\tau$, so w has the restriction

$$-\mathrm{Im}\,\tau < \mathrm{Im}\,w < -\left(1-\frac{2}{n}\right)\mathrm{Im}\,\tau \tag{7.3}$$

Since $A_n(0)=id$ [this follows from $U_i(0)=n^2 id$] and $S^{ab}(z+n)=S^{ab}(z)$ we have $m_r=1$ for every r and

$$A_d(z)_{r,r'} = \exp\left(2\pi i n_r \frac{z}{n}\right) \tag{7.4}$$

where $n_r\in\mathbb{Z}$.

To calculate the integers n_r we use the limits $q\to 0$ (first) and $w\to -i\infty$. Using (6.2) and the fact that $S^{0,b}\to 0$ for $b\neq 0$ we see that U_i approaches a diagonal matrix with (σ,σ) entry

$$S^{\sigma_{i-1}-2\sigma_i+\sigma_{i+1},0}(z, w, \tau)$$

which is approaching $\exp\{2\pi i[\frac{1}{2}-\overline{(\sigma_{i-1}-2\sigma_i+\sigma_{i+1})}](z/n)\}$ where we write $\bar{x}$ for the representative of $x\in\mathbb{Z}$ in $\mathbb{Z}_n$. Since A can be written as the product of the U_i matrices, A also is a diagonal matrix in the limit $q\to 0$, $w\to -i\infty$. Note that these limits are consistent with (7.2) and (7.3) since $\tau\to +i\infty$ first. $A_d(z)$ can now be found by normalizing U_i so that the maximal entry $(U_i)_{\sigma\sigma}$ is unity. For $\mathrm{Im}\,z<0$ this occurs when $\overline{\sigma_{i-1}-2\sigma_i+\sigma_{i+1}}=0$; that is, $a=0$ in (6.2). This illustrates the dominance of S^{00} in this region. Thus

$$A_d(z)_{\sigma,\sigma} = \exp\left[-2\pi i\frac{z}{n}\sum_{i=2}^{m+1}(i-1)\overline{(\sigma_{i-1}-2\sigma_i+\sigma_{i+1})}\right] \tag{7.5a}$$

so that

$$n_\sigma = -\sum_{i=2}^{m+1}(i-1)\overline{(\sigma_{i-1}-2\sigma_i+\sigma_{i+1})} \tag{7.5b}$$

Arguing as in Baxter[3] we expect that if σ is the spin set corresponding to the rth largest diagonal element of the finite A, then since there is an integer j, independent of m, such that

$$\sigma_i = s \qquad \text{for} \quad i>j \tag{7.6}$$

we expect $\lim_{m\to\infty} A_{\sigma\sigma}$ to exist and to approach

$$A_d(z)_{\sigma,\sigma} = \exp\left(-\frac{2\pi i}{n} z \sum_{i=1}^{\infty} i\mu_i\right) \tag{7.7}$$

where $\mu_i=\overline{\sigma_i-2\sigma_{i+1}+\sigma_{i+2}}$. Note that the transformation from any $\sigma=(\sigma_1, \sigma_2,...)$ satisfying (7.6) to the set of $\mu=(\mu_1, \mu_2,...)$ satisfying $\mu_i=0$ for i sufficiently large is one-to-one and onto. Equation (7.6) is valid for μ without the restriction $\mu_i=0$ for i sufficiently large if we interpret $(A_d)_{\sigma\sigma}=0$ when $\sum i\mu_i=\infty$. For the spin set labeled by μ, the ground state picked out is the one with all spins in the state $\omega^0=1$. The labeling is such that

$$\mu_k+\mu_{k+1}+\cdots+\mu_m=\overline{\sigma_k-\sigma_{k+1}}, \qquad k=1, 2,...$$

so for any m

$$\begin{aligned}\sigma_1&=(\mu_1+\mu_2+\cdots+\mu_m)+(\mu_2+\mu_3+\cdots+\mu_m)+\cdots\\&=\mu_1+2\mu_2+\cdots k\mu_k+\cdots\end{aligned}$$

Again identifying the spins as nth roots of unity we see

$$\omega^{\sigma_1}=\omega^{\mu_1+\mu_{n+1}+\cdots}(\omega^2)^{\mu_2+\mu_{n+2}+\cdots}\cdots(\omega^n)^{\mu_n+\mu_{2n}+\cdots} \tag{7.7}$$

Define

$$S=g\otimes g^2\otimes\cdots\otimes g^k\otimes\cdots \tag{7.8}$$

where g is the Heisenberg matrix (2.7). Then the (μ, μ) entry of S is exactly (7.7). Let $D(x)$ be the $n\times n$ diagonal matrix with $D_{ii}(x)=x^i$, where $x=x(z)=\exp[(-2\pi i/n)z]$. Thus $A_d(z)$ can be written as

$$A_d(z)=D(x)\otimes D(x^2)\otimes\cdots\otimes D(x^k)\otimes\cdots \tag{7.9}$$

since

$$A_d(z)_{\mu\mu}=x^{\mu_1+2\mu_2+\cdots}$$

We can now evaluate (7.1) to obtain

$$\begin{aligned}M_0&=\prod_{k=1}^{\infty}\left\{\frac{1-\exp[2\pi i(w+\tau)k]}{1-\exp[2\pi i(w+\tau+1/n)k]}\right\}\\&=\exp\left(\frac{\pi i}{12n}\right)\frac{\eta(w+\tau)}{\eta(w+\tau+1/n)}, \qquad (w+\tau)\in\mathbb{H}\end{aligned} \tag{7.10}$$

where $\eta(\tau)=\exp(\pi i\tau/12)\prod_{k=1}^{\infty}[1-\exp(2\pi ik\tau)]$, $\tau\in\mathbb{H}$.

Because we are working with an n-state model, the quantities

$$M_{0,k}=\langle\sigma_0^k\rangle \tag{7.11}$$

are also of interest. Replacing S by

$$S^k = g^k \otimes g^{2k} \otimes \cdots$$

in the above analysis we obtain

$$M_{0,k} = \exp\left(\frac{\pi i k}{12n}\right) \frac{\eta(w+\tau)}{\eta(w+\tau+k/n)}, \qquad (w+\tau) \in \mathbb{H} \tag{7.12}$$

We conclude this section with several comments concerning (7.10) and (7.12):

(i) For $n=2$, (7.10) reduces to Baxter's spontaneous magnetization M_0 as given by (13.7.21) in Ref. 3. The presence of an additional τ in our (7.10) is related to the fact that Baxter used a symmetry argument to obtain a slightly different parametrization of the Boltzmann weights in the ferromagnetic region [see (13.3.9) in Ref. 3].

(ii) Let $M \in GL(2, \mathbb{Z})$ be defined by $M = \left(\begin{smallmatrix} n & 1 \\ 0 & n \end{smallmatrix}\right)$ so that $M\tau = \tau + 1/n$. M is a transformation of order n^2. Define $\Gamma_M = \Gamma \cap M^{-1}\Gamma M$, where Γ is the modular group, i.e., $SL(2, \mathbb{Z})/\{\pm I\}$. Then we are interested in the function

$$f_M(\tau) = \frac{\eta(M\tau)}{\eta(\tau)}$$

Since $\eta^{24}(\tau)$ is a modular form of weight -12 for Γ, it follows that $[f_M(\tau)]^{24}$ is a modular function for Γ_M. More details concerning this can be found in Schoeneberg (Ref. 18, Chap. 6).

(iii) Using Schoeneberg's theory of generalized Dedekind eta functions (see Chap. 8 of Ref. 18), one can show that $M_{0,k}$ is expressible as ratios of products of $\eta_g(\tau)$. For example, for $n=3$

$$\prod_{k=1}^{\infty} \left[\frac{1-\exp(2\pi i k\tau)}{1-\omega^k \exp(2\pi i k\tau)}\right] = \frac{\eta_{10}(3\tau)}{\eta_{11}(3\tau)}, \qquad \omega = \exp\left(\frac{2\pi i}{3}\right)$$

The advantage of such a representation is that it now follows from transformation properties of $\eta_g(\tau)$ that $M_{0,k}$ raised to a sufficiently high power (this power is computable) is a modular function of level n; that is, $(M_{0,k})^p$ for some $p \in \mathbb{N}$ is invariant under $\Gamma(n)$. For $n=2$ this is the familiar result that $M_0 = (1-k^2)^{1/8}$, $k = k(\tau)$ the elliptic modulus, raised to the eight power is a modular function of level 2. Thus the $\mathbb{Z}_n$ Baxter model order parameters are expressible in terms of modular functions of level n, a result that was suggested from the analysis in Ref. 20.

(iv) The identity

$$\frac{\sum_{\mu_i=0,1,\ldots,n-1} (\omega q)^{\sum_{i=1}^{\infty} i\mu_i}}{\sum_{\mu_i=0,1,\ldots,n-1} q^{\sum_{i=1}^{\infty} i\mu_i}} = \prod_{i=1}^{\infty} \left(\frac{1-q^k}{1-\omega^k q^k}\right)$$

and other similar identities are a recurring theme in exactly solvable models. In fact, based only on an examination of allowed configurations, this identity was predicted to appear in n-state vertex models by Jimbo and Miwa.[11]

(v) Even though we are in an unphysical region, we use the elementary probability formula

$$\begin{aligned} E(\sigma^k) &= \sum_{l=0}^{n-1} \operatorname{Prob}(\sigma^k = \omega^{kl})\, \omega^{kl} \\ &= \sum_{l=0}^{n-1} \operatorname{Prob}(\sigma = \omega^k)\, \omega^{kl} \end{aligned}$$

Inverting this relation and using (7.10) and (7.12) we find

$$\begin{aligned} \operatorname{Prob}(\sigma = \omega^k) &= \frac{1}{n} \sum_{j=0}^{n-1} \omega^{-kj} \prod_{l=1}^{\infty} \left(\frac{1-q^l}{1-\omega^{jl} q^l}\right) \\ &= \frac{1}{n} + \frac{1}{n} \sum_{j=1}^{n-1} \omega^{-kj} \frac{\varphi(q)}{\varphi(\omega^j q)} \end{aligned} \tag{7.13}$$

Recalling that $1/\varphi(q) = \prod_{l=1}^{\infty} 1/(1-q^l) = \sum_{m \geqslant 0} p(m)\, q^m$, a simple calculation gives the result quoted in the Introduction. The $q \to 0$ and $q \to 1$ limits follow from (7.13).

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APPENDIX

Lemma. Let

$$w_\alpha(z, w, \tau) = \exp(-\pi i z) \frac{\vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (z + w, \tau)}{\vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (w, \tau)}$$

then

$$w_\alpha\left(\frac{z}{y}, \frac{w}{y}, \frac{x}{y}\right) = \exp\left\{\pi i\left[\left(\frac{y-1}{y} z\right) + \frac{c(z^2 + 2zw)}{y}\right]\right\} w_{A\alpha}(z, w, \tau)$$

where $A \in SL(2, \mathbb{Z})$, $A = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$, $x = a\tau + b$, and $y = c\tau + d$.

Proof. This is best done by first constructing some auxiliary functions defined by

$$F_{A,\alpha}(z, w, \tau) = \exp\left\{\pi i c y \left[z + w + \frac{1}{y}\left(\frac{\tau+1}{2}\right)\right]^2\right\} w_{A\alpha}(yz, yw, \tau)$$

$$G_{A,\alpha}(z, w, \tau) = \exp[\pi i(d - c - 1)z]\, F_{A,\alpha}(z, w, \tau)$$

We will show $G_{A,\alpha}(z, w, \tau) = \text{const} \times w_\alpha(z, w, x/y)$. As usual we check transformation properties:

$$F_{A,\alpha}(z+1, w) = \omega^{-\alpha_2} \exp[-\pi i c d]\, F_{A,\alpha}(z, w) \tag{A1}$$

For the other generator, let $z \mapsto z + x/y = z + \tau'$:

$$\begin{aligned} F_{A,\alpha}(z+\tau') = {} & \exp\left(\pi i c y \left\{2\left[z + w + \frac{1}{y}\left(\frac{\tau+1}{2}\right)\right] \cdot \frac{x}{y} + \frac{x^2}{y^2}\right\}\right) \\ & \times \exp\left\{-\pi i a^2 \tau - 2\pi i a\left[y(z+w) + \frac{\tau+1}{2}\right]\right\} \\ & \times \omega^{\langle A\alpha, x\rangle} F_{A,\alpha}(z) \end{aligned}$$

$\langle A\alpha, x\rangle = \langle \alpha, A^{-1}x\rangle = -\alpha_2$. The rest of the multiplier becomes

$$\exp\left\{\pi i[2(z+w)(cx - ay)] - a^2\tau + \frac{cx}{y}(\tau+1) + \frac{c}{y}x^2 - a(\tau+1)\right\}$$

Use $cx - ay = -1$ and simplify further to get

$$\exp\left\{\pi i\left[-2(z+w) - a^2\tau - \frac{1}{y}(\tau + 1 - cx^2)\right]\right\}$$

$$\exp\left\{-2\pi i\left[z + w + \frac{1}{y}\left(\frac{\tau+1}{2}\right)\right]\right\} \exp\left[-\pi i \frac{1}{y}(a^2 y\tau - cx^2)\right]$$

Now we rewrite the last exponential using

$$\begin{aligned}
a^2 y\tau - cx^2 &= a^2(c\tau + d)\tau - c(a\tau + b)^2 \\
&= a^2\, d\tau - 2abc\tau - cb^2 \\
&= a(ad - bc)\tau - abc\tau - cb^2 \\
&= a\tau - ab(c\tau + d) + b(ad - bc) \\
&= a\tau + b - ab(c\tau + d) \\
&= x + aby
\end{aligned}$$

Thus the net multiplier for $F_{A,\alpha}(z)$ is

$$\exp\left\{-\pi i\left(\frac{x}{y}\right) - 2\pi i\left[z + w + \frac{1}{y}\left(\frac{\tau + 1}{2}\right)\right]\right\}\omega^{-\alpha_2}\exp(-\pi iab) \tag{A2}$$

Remembering that $x/y = \tau'$, we have

$$\begin{aligned}
w_\alpha(z + \tau', \tau') &= \exp\left[-\pi i\tau' - 2\pi i\left(z + w + \frac{\tau' + 1}{2}\right)\right]\omega^{-\alpha_2} w_\alpha(z, \tau') \\
w_\alpha(z + 1) &= \omega^{\alpha_1} w_\alpha(z)
\end{aligned} \tag{A3}$$

Let us now consider three cases (I) ab, cd even, (II) ab even, cd odd, and (III) ab odd, cd even. Note that both ab and cd cannot be odd. In each case we will show $G_{A,\alpha}$ has the same multiplier as (A3).

(I) We will show

$$\frac{1}{y}\left(\frac{\tau + 1}{2}\right) = \frac{1}{2}\left(\frac{x}{y} + 1\right) + m_1\left(\frac{x}{y}\right) + m_2, \qquad m_1, m_2 \in \mathbb{Z}$$

or

$$\tau + 1 = x + y + 2m_1 x + 2m_2 y$$

or

$$\begin{aligned}
\tfrac{1}{2}(1 - a - c)\tau + \tfrac{1}{2}(1 - b - d) &= m_1 x + m_2 y \\
k_1\tau + k_2 &= m_1 x + m_2 y
\end{aligned}$$

That is, we need to solve

$$A\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

This has the solution

$$m_1 = \tfrac{1}{2}(d-c-1)$$
$$m_2 = \tfrac{1}{2}(a-b-1)$$

It is easily seen that $a-b-1$ and $d-c-1$ are even and hence we can write

$$F_{A,\alpha}(z+\tau') = \exp[-\pi i(d-c-1)\,\tau']$$
$$\times \exp\left[-\pi i\left(\frac{x}{y}\right) - 2\pi i\left(z+w+\frac{\tau'+1}{2}\right)\right] F_{A,\alpha}(z)$$

So in case (I) $G_{A,\alpha}(z, \tau)$ and $w_\alpha(z, \tau')$ have the same transformation properties in the z variable.

(II) We will show

$$\frac{1}{y}\left(\frac{\tau+1}{2}\right) = \frac{1}{2} + m_1\left(\frac{x}{y}\right) + m_2, \qquad m_1, m_2 \in \mathbb{Z}$$

This is accomplished for

$$m_1 = \tfrac{1}{2}(d-c)$$
$$m_2 = \tfrac{1}{2}(a-b-1), \qquad d-c \text{ even}, \quad a-b \text{ odd}$$

Thus

$$F_{A,\alpha}(z+\tau') = \exp[-2\pi i m_1 \tau']\exp[-\pi i\tau' - 2\pi i(z+w+\tfrac{1}{2})]\; F_{A,\alpha}(z)$$
$$F_{A,\alpha}(z+1) = -\omega^{-\alpha_2} F_{A,\alpha}(z)$$

Again we see $G_{A,\alpha} = \exp[\pi i(d-c-1)z]\; F_{A,\alpha}(z)$ has the same multipliers as $w_\alpha(z)$.

(III) This time

$$\frac{1}{y}\left(\frac{\tau+1}{2}\right) = \frac{1}{2}\frac{x}{y} + m_1\frac{x}{y} + m_2$$

where

$$m_1 = \tfrac{1}{2}(d-c-1)$$
$$m_2 = \tfrac{1}{2}(a-b), \qquad m_1, m_2 \in Z$$

As before, it is now easy to $G_{A,\alpha}$ agrees with w_α in the z variable.

It is also easy to see $G_{A,\alpha}(z)$ and $w_\alpha(z)$ have the same zeros in the $E_{\tau'}$. Thus they can only differ by some $c(w, \tau)$. Setting $z=0$, we find

$$c(w, \tau) = \exp\left\{\pi i c y\left[w + \frac{1}{y}\left(\frac{\tau+1}{2}\right)\right]^2\right\}$$

showing

$$\exp\left(\pi i\left[(d-c-1)\,z+cy\left\{z^2+2z\left[w+\frac{1}{y}\left(\frac{\tau+1}{2}\right)\right]\right\}\right]\right)w_{A\alpha}(yz, yw, \tau)$$
$$=w_\alpha\left(z, w, \frac{x}{y}\right)$$

Now simple algebra and the change of variables $z \mapsto z/y$ leads to the expression in the statement of the lemma. ∎

Remark. The above proof is motivated by the proof of the functional equation of $\vartheta(z, \tau)$; see, e.g., Mumford,[16] p. 28.

NOTE ADDED IN PROOF

In C. L. Schultz, thesis, SUNY at Stony Brook (1982, unpublished), the $q \to 0$ limit of Theorem 5.6 was conjectured.

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