In principle, we can construct a computer model of the conscious, a model of our model of thinking, a kind of a "derivative" of the individual adaptation function, i.e., we can model thinking in a metric in which the conscious adequately reflects the objective reality.

If we start with two-level thinking, the subject should have at least two sources for testing the adequacy of the reflection: conscious experience and unconscious experience, the latter transmitted to the conscious through "lattice" operators as a component of the discretized models. The semantics of these models (and equally our classification of the objects) falls outside the scope of the conscious; it is the function of the "lattice" operators.

Since the computer cannot model the subject as a whole, it is devoid of a source of unconscious experience and is thus incapable of independently modeling the classification and the semantics, not to say the motivation of the actions, i.e., survival, which evidently falls beyond the conscious.

We thus assert that computers only can be used to realize conscious models of thinking, and moreover only to the extent that we can transmit to the computer our conscious sense of motivation and semantics.

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A CLASS OF "AGING" DISTRIBUTIONS

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New definitions of aging are introduced and their properties are examined. Two-sided reliability bounds are obtained for a number of shock models in classes of distribution functions with these properties.

INTRODUCTION

The study of faults in technical elements whose failure rate is time or "age" dependent and also depends on the reliability of other elements in the complex system leads to a class of distribution functions (d.f.s) of time to failure with specific "aging" properties. While knowledge of the d.f. is sufficient for testing the ordinary aging properties, the analysis of these classes requires additional information about some other d.f.s that are related with the given d.f. This constitutes a natural direction of research in reliability theory, because in practice complex systems with independent elements are an exception rather than the rule.

One of the approaches to formulating "aging" principles is by extension of existing concepts. Thus, the class of distribution functions with "increasing failure rate" (IFR) has been successfully extended to the class of distribution functions with "increasing failure rate relative to the d.f. G" (G-IFR) [1]. We will consider the extension of another traditional property, "new better than used," to "new better than used relative to the d.f. G" (G-NBU). We say that the d.f. F is G-NBU if for all $x, y \in R_+$ we have the inequality

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$$\bar F(x+y) \leqslant \bar F(x)\bar G(y), \text{ where } \bar a = 1-a.$$

This property naturally arises in the model of a standby replacement pool, in which a single operating element is backed up by a sufficiently large pool of identical standby elements ("cold" redundancy). We know (see, e.g., [2]) that the reliability of a new element in the system is not equal to the reliability of a new element from the standby replacement pool (in terms of the theory of stochastic processes, this is a renewal process "with delay"). This difference may have a substantial effect on the behavior of the system, but it is difficult to determine numerically due to the lack of good statistical data. The G-NBU property is introduced to soften this difficulty insofar as it relies on the following natural assumption: the probability of failfree operation of an element from the standby replacement pool during the time y is not less than the probability of failfree operation of a system element that has already functioned without failure during the time x.

The G-NBU property is also widely used in reliability. Let us give some examples.

1. Consider the class of d.f.s F for which $\lim[F(x)/x]$ as $x \to 0$ exists and has a fixed value $\lambda \geq 0$. Then the property $\bar F(x+y) \leq \bar F(x)\exp\{-\lambda y\}$ defines the class of "new better than used failure rate" (NBUFR) distribution functions.

2. Let the d.f. G be an NBU distribution in a traditional form [4]. Then the following assertions hold:

a) the d.f. S_G is G-NBU;

b) if the d.f. F is G-NBU, then the d.f. S_F is also G-NBU, where S_F is the residual life distribution function, i.e., $S_F = (1/\Theta)\int_0^t \bar F(u)du$, $\Theta = \int_0^\infty \bar F(u)du$;

c) a nonnegative mixture of G-NBU distributions is also a G-NBU distribution;

d) the nonstationary residual life d.f. $N(t, u)$ generated by the d.f. $F \in \{G\text{-NBU}\}$ is also G-NBU.

These simple facts are stated without proof, and only the last assertion is proved. The d.f. $N(t, u)$ is expressed in terms of the underlying d.f. F and the renewal function H that it generates:

$$\bar N(t, u) = \bar F(t+u) + \int_0^t \bar F(t-x+u)\,dH(x).$$

Since $F \in \{G\text{-NBU}\}$, we have

$$\bar N(t, y+u) \leqslant \bar G(y)\bar F(t+u) + \bar G(y)\int_0^t \bar F(t-x+u)\,dH(x) = \bar G(y)\bar N(t, u).$$

Thus, $N(t, u)$ is G-NBU.

A similar extension for other "aging" principles also can be highly fruitful. In this paper, we investigate a generalization of the property "new better than used average" (NBUA). All the results can be almost automatically extended to the class of so-called "rejuvenating" distributions (this is achieved by simply inverting the inequalities). The notion of NBUA is further generalized to multivariate distributions. Such generalizations have been attempted previously. Thus, several multivariate analogs of the NBUA property have been introduced in [5]: NBUA-A, NBUA-B, NBUA-C. So far, however, no methods have been developed that utilize these properties to obtain specific reliability bounds. In this paper, we introduce a multivariate NBUA property based on a multivariate analog of residual life and apply it to derive new reliability bounds for some known systems.

1. THE CLASS OF p-NBUA DISTRIBUTION FUNCTIONS

Definition 1. The d.f. $F(x)$, $x \in R_+$, $F(0^+) = 0$, is $p(\cdot)$-NBUA, where $p(x)$, $x \in R_+$, $p(0^-) = 0$, is a nonnegative function if:

a) there exists a unique number κ such that

$$\int_0^\infty \exp\{\kappa x\}\, p(x)\, dF(x) = 1; \tag{1}$$

b) the d.f. $F^\#(t) = \int_0^t \exp\{\kappa x\}p(x)dF(x)$ is an NBUA distribution [1], i.e., $\bar S_{F\#}(x) \leq \bar F^\#(x)$ for all $x \geq 0$, where

$$\bar S_{F\#}(x) = (1/\Theta^\#)\int_x^\infty \bar F^\#(x)\,dx,$$

$$\Theta^\# = \int_0^\infty \bar F^\#(x)\,dx = \int_0^\infty x p(x)\exp\{\kappa x\}\,dF(x).$$

Let $p(x) \equiv p = \text{const}$, $p > 0$. Consider some properties of the class $\{p\text{-NBUA}\}$:

1) the d.f. $F \in \{1-\text{NBUA}\} \Leftrightarrow$ the d.f. $F \in \{\text{NBUA}\}$, $\kappa = 0$;

2) if $p > 1$, then the d.f. $F \in \{p\text{-NBUA}\} \Leftrightarrow$ the d.f. $F^{\#} \in \{\text{NBUA}\}$, $\kappa < 0$;

3) if $p < 1$, then the d.f. $F \in \{p\text{-NBUA}\} \Leftrightarrow$ the d.f. $F^{\#} \in \{\text{NBUA}\}$, $\kappa > 0$, and $\int_0^\infty \exp\{\varepsilon t\}dF(t)$ is convergent for some $\varepsilon > 0$.

These assertions make it possible to simplify part a) of the definition. The class {NBUA} is the set of d.f.s with specific "aging" properties. In practice, the study of "aging" properties usually focuses on the original d.f. F, and it is therefore relevant to consider how the "aging" properties of the d.f. F are related with those of the d.f. $F^{\#}$. We state the following result.

THEOREM 1. Let the function $p(x)f(x)$, where f is the density of the d.f. F, be a Polya function of 2nd order (PF_2), i.e., $\ln[p(x)f(x)]$ is concave on R_+. Then $F \in \{p(\cdot)\text{-NBUA}\}$.

Proof. The conditions of the theorem imply that the density $f^{\#}$ of the d.f. $F^{\#}$ is also PF_2, and thus (by Lemma 3.5.8 [4]), $F^{\#} \in \{\text{IFR}\} \subset \{\text{NBUA}\}$, where {IFR} is the class of "increasing failure rate" distribution functions. Q.E.D.

In many important cases, the density function f is PF_2. Thus, the gamma-distribution density $r(\lambda, \alpha; t) = \lambda^\alpha t^{\alpha-1}\exp\{-\lambda t\}/\Gamma(\alpha)$ for $\alpha \geq 1$ is PF_2. By Theorem 1, a sufficient condition for the gamma-distribution to be $\{p(\cdot)\text{-NBUA}\}$ is that $p(\cdot)$ is PF_2. A similar argument applies to the following distributions:

Weibull

$$\omega(x) = \alpha\lambda(\lambda x)^{\alpha-1}\exp\{-(\lambda x)^\alpha\}, \quad x \geqslant 0, \quad \alpha \geqslant 1;$$

truncated normal

$$n_0(x) = (1/a\sqrt{2\pi\sigma})\exp\{-(x-a)^2/2\sigma^2\}, \quad x \geqslant 0, \quad \sigma > 0, \quad -\infty < a < \infty;$$

normal

$$n(x) = (1/\sqrt{2\pi})\exp\{-(x-a)^2/2\sigma^2\}, \quad -\infty < x < \infty.$$

Laplace

$$l(x) = 0.5\exp\{-|x|\}, \quad -\infty < x < \infty.$$

The assumption that the original d.f. F is $\{p(\cdot)\text{-NBUA}\}$ produces prelimit two-sided bounds of the sought characteristics in the following problems:

– bounds on the termination probability of a terminating (generalized terminating) renewal process;

– bounds on the mean number of direct descendants of a particle in a branching process (Bellman–Harris and Sevast'yanov models);

– analysis of some particular stochastic systems (e.g., the problem of ruin, the problem of waiting for a first gap of prescribed length in a renewal process, etc.).

In these problems, the sought characteristics are obtained by solving the "perturbed" renewal equation

$$V(t) = g(t) + \int_0^t p(u)V(t-u)\,dF(u), \tag{2}$$

where g is a bounded measurable function, $g(t) = 0$, $t < 0$.

Consider a scheme that generates two-sided bounds on the solution of Eq. (2) when $F \in \{p(\cdot)\text{-NBUA}\}$. To each function z we associate a new function $z^{\#}$ defined by the relationship $z^{\#}(x) = \exp\{\kappa x\}z(x)$. This reduces the "perturbed" equation (2) to an ordinary renewal equation

$$V^{\#} = g^{\#} + V^{\#}*F^{\#}. \tag{3}$$

with the proper d.f. $F^{\#}$. The canonical form of the solution of this equation is

$$V^{\#} = g^{\#}*H^{\#}, \tag{4}$$

where $H^{\#} = \sum_{n=0}^\infty F^{\#n*}$ is the renewal function of the renewal process generated by the d.f. $F^{\#}$. We obtain the following bounds for the function $H^{\#}$ [1, 6.3.14]:

$$t/\Theta^{\#} \leqslant H^{\#}(t) \leqslant 1 + t/\Theta^{\#}. \tag{5}$$

The monotone behavior of the function $g^{\#}$ and the relationships (4)-(5) lead to exact upper and lower bounds on the function $V^{\#}$ and the corresponding bounds on the original function V. Below we consider some examples of such bounds.

1.1. Bound on Termination Probability of a Terminating Renewal Process

Consider a terminating renewal process generated by a proper nonarithmetic d.f. F with its origin at zero and with termination probability q at each renewal instant (we say that the process is generated by an improper d.f. with a nonzero "defect"). For the termination probability $P(t)$ of the process in the time interval $[0, t]$ we have the following theorem.

THEOREM 2. Let $F \in \{p\text{-NBUA}\}$, where $p = 1 - q$. Then

$$(q/\Theta^{\#}\varkappa)\exp\{-\varkappa t\} + p - q/\Theta^{\#}\varkappa \leqslant \bar{P}(t) \leqslant \leqslant (q/\Theta^{\#}\varkappa)\exp\{-\varkappa t\} + 1 - q/\Theta^{\#}\varkappa. \tag{6}$$

Proof. We write the renewal equation for the function $P(t)$:

$$P(t) = q + p\int_0^t P(t-u)\,dF(u). \tag{7}$$

Let us transform Eq. (7) following the scheme outlined above. Then $\kappa > 0$ and

$$P^{\#}(t) = q\exp\{\varkappa t\} + \int_0^t P^{\#}(t-u)\,dF^{\#}(u) \tag{8}$$

with the canonical form

$$P^{\#}(t) = q\int_{0-}^{t}\exp\{\varkappa(t-u)\}\,dH^{\#}(u) = qH^{\#}(t) + q\varkappa\int_0^t H^{\#}(t-u)\exp\{\varkappa u\}\,du.$$

We can now mechanically apply inequality (5) to obtain the bound (6), because $P(t) = \exp\{-\kappa t\}P^{\#}(t)$.

COROLLARY 1. Let $F \in \{p\text{-NBUA}\}$. Then

$$q/\varkappa \leqslant \Theta^{\#} \leqslant q/\varkappa p. \tag{9}$$

Proof. Let $\Delta P(t) = \bar{P}(t) - (q/\kappa\Theta^{\#})\exp\{-\kappa t\}$. The simplest way to obtain the bound (9) is by applying the asymptotic bound for the probability $P(t)$ [6, 11.6.2]

$$\bar{P}(t) \sim (q/\varkappa\Theta^{\#})\exp\{-\varkappa t\}.$$

By Theorem 2 for all $t \geq 0$

$$p - q/\varkappa\Theta^{\#} \leqslant \Delta P(t) \leqslant 1 - q/\varkappa\Theta^{\#},$$

and thus in the limit as $t \to \infty$ we have $\Delta P(t) \to 0$ and $1 - q/\kappa\Theta^{\#} \geq 0$, $p - q/\Theta^{\#} \leq 0$. Hence we easily obtain inequality (9).

In a number of well-known problems (such as insurance, inventory management, service of patients) the determination of $R(t)$ (the probability of ruin, the out-of-stock probability, etc., in the time interval $[0, t)$) reduces to a renewal equation of the form

$$R(t) = R_0 + (\alpha/c)\int_0^t R(t-x)\,\bar{F}(x)\,dx, \tag{10}$$

where F is a proper d.f. with the mean Θ; α and c are fixed parameters, such that $\alpha\Theta/c < 1$, and $R_0 = 1 - \alpha\Theta/c$.

We write this equation in operator form

$$R = R_0 + (1 - R_0)\,R*S_F,$$

where S_F is the residual life d.f. generated by the d.f. F, i.e., $S_F(x) = (1/\Theta)\int_0^x \bar{F}(u)du$.

Now, using Theorem 2, we can easily estimate the function R.

COROLLARY 2. Let the d.f. $S_F \in \{(\alpha\Theta/c)\text{-NBUA}\}$. Then

$$(R_0/\Theta^{\#}\varkappa)\exp\{-\varkappa t\} + (1 - R_0 - R_0/\varkappa\Theta^{\#}) \leqslant$$
$$\leqslant \bar{R}(t) \leqslant (R_0/\Theta^{\#}\varkappa)\exp\{-\varkappa t\} + (1 - R_0/\varkappa\Theta^{\#}), \tag{11}$$

where κ is the solution of the equation $(\alpha\Theta/c)\int_0^\infty \exp\{\kappa t\}dS_F(t) = 1$ or $(\alpha/c)\int_0^\infty \exp\{\kappa t\}\bar{F}(t)dt = 1$, $\Theta^{\#} = (\alpha/c)\int_0^\infty t\exp\{\kappa t\}\bar{F}(t)dt$.

In this case, we can relax the sufficient condition (Theorem 1) of membership in the class $\{p\text{-NBUA}\}$, specifically: if the d.f. F has an increasing failure rate (F is IFR), which is obviously weaker than the requirement that F has a Polya density of 2nd order, then $S_F \in \{p\text{-NBUA}\}$.

Note that the function $R(t)$ also can be estimated in the framework of traditional "aging" properties. Thus, if the d.f. F has the property "harmonic new better than used average" (HNBUA) (i.e., F is from the class of HNBUA distributions [7]), then

$$R(t) \geqslant R_0 \sum_{n=0}^{\infty} (1 - R_0)^n \bar{E}_\Theta^{n*}(t) = R_0 + (1 - R_0)(1 - \exp\{-tR_0/\Theta\}),$$

where $\bar{E}_\Theta = \exp\{-t/\Theta\}$.

Here we have used the relationship for HNBUA distributions $\bar{S}_F(t) \le \bar{E}_\Theta(t)$; if the d.f. F has the property "new better than used failure rate average" (NBUFRA) (i.e., F is from the class of NBUFRA distributions) and $\lambda_F\Theta + R_0 \ge 1$, where $\lambda_F = -\lim_{t\to 0} \ln \bar{F}(t)/t$, then

$$R(t) \leqslant R_0 + (1 - R_0)[1 - \exp\{-t(\lambda_F - (1 - R_0)/\Theta)\}]\, R_0/(\lambda_F\Theta - 1 + R_0).$$

($F \in \{\text{NBUFRA}\}$ implies that $\bar{F}(t) \le \exp\{-\lambda_F t\}$ and $S_F(t) \le (1 - \exp\{-\lambda_F t\})/\Theta\lambda_F$.

Consider another example: a renewal equation that arises in models with self-repairing assemblies, models of elementary particle counters, population growth models, etc.:

$$V(t) = \bar{F}(\xi) + \int_0^{\xi+} V(t - y)\,dF(y), \quad t \geqslant \xi, \tag{12}$$
$$V(t) = 0, \quad t < \xi,$$

where $\xi > 0$ is a fixed number.

This equation reduces to an ordinary renewal equation

$$V = g + V * G,$$

with the d.f.

$$G(x) = \begin{cases} F(x), & x \leqslant \xi, \\ F(\xi), & x \geqslant \xi, \end{cases} \qquad g(x) = \begin{cases} 0, & x < \xi, \\ \bar{F}(\xi), & x \geqslant \xi. \end{cases}$$

COROLLARY 3. If the proper d.f.

$$D(x) = \begin{cases} F(x)/F(\xi), & x \leqslant \xi, \\ 0, & x \geqslant \xi \end{cases}$$

is from the class $\{F(\xi)\text{-NBUA}\}$, then

$$[\bar{F}(\xi)/\varkappa\Theta^{\#}][1 - \exp\{-\varkappa(t - \xi)\}] \leqslant V(t) \leqslant \bar{F}(\xi) +$$
$$+ [\bar{F}(\xi)/\varkappa\Theta^{\#}][1 - \exp\{-\varkappa(t - \xi)\}], \tag{13}$$

where $\kappa > 0$ is the solution of the equation $\int_0^\xi \exp\{\kappa u\}dF(u) = 1$, $\Theta^{\#} = \int_0^\xi u\exp\{\kappa u\}dF(u)$.

A sufficient condition of membership in the class $\{F(\xi)\text{-NBUA}\}$ is the existence of a PF_2 density for the original d.f. F at least on the interval $[0, \xi)$.

Proof. Since $g^{\#}(t)$ is a nondecreasing function ($g^{\#}(t) = g(t)\exp\{\kappa t\}$), we have $(t/\Theta^{\#})g^{\#}(t) \le V^{\#}(t) \le (t/\Theta^{\#} + 1)g^{\#}(t)$. Transforming the principal part in these bounds, we obtain

$$(t/\Theta^{\#})g^{\#}(t) = \int_0^t (t-u)\,[\varkappa g(u) + \bar{F}(\xi)\,\delta(u-\xi)]\exp\{\varkappa u\}\,du/\Theta^{\#},$$

where $\delta(x)$ is the Dirac function.

After some elementary simplifications we obtain the expression

$$[\bar{F}(\xi)/\varkappa\Theta^{\#}]\,[\exp\{\varkappa t\} - \exp\{\varkappa\xi\}].$$

Using the relationship $V(t) = \exp\{-\kappa t\}V^{\#}(t)$ to return to the sought bounds on the function $V(t)$, we obtain (13).

1.2. Bound on the Mean Number of Descendants in the Sevast'yanov Model

The mean number of descendants $A(t)$ during the time t in a Sevast'yanov branching process satisfies the renewal equation (for references see [8])

$$A(t) = \bar{F}(t) + \int_0^t A(t-u)\,p(u)\,dF(u), \tag{14}$$

where F is the life d.f. of a single particle, $p(u)$ is the conditional mean number of direct descendants of a single particle given that its transformation occurred at the age u.

The function $p(u)$ may be fairly complex, because Sevast'yanov's model allows an arbitrary dependence between the life of a single particle and the number of its direct descendants. The above equation is a particular case of Eq. (2) and therefore the previous scheme may be applied to obtain bounds on the function $A(t)$ for the case when $F \in \{p(\cdot)\text{-NBUA}\}$. To avoid purely formal mathematical complications, we will consider a particular example, which is also of applied value. We assume that the function $p(u)$ is arbitrary and F is the gamma distribution:

$$R(\lambda, \alpha; t) = \int_0^t r(\varkappa, \alpha; u)\,du.$$

THEOREM 3. In Sevast'yanov's model let $F(t) = R(\lambda, \alpha; t)$. If $\kappa < 0$, then

$$(\varkappa\Theta^{\#} + \varkappa + 1 - \varkappa t)\,\bar{R}(\lambda, \alpha; t)/\varkappa\Theta^{\#} + (1/\varkappa\Theta^{\#})\,\eta(t) + \exp\{-\varkappa t\} \times$$
$$\times (1 + 1/\varkappa\Theta^{\#}) \le A(t) \le -(\varkappa t - \varkappa - 1)\,\bar{R}(\lambda, \alpha; t)/\varkappa\Theta^{\#} +$$
$$+ (1/\varkappa\Theta^{\#})\,\eta(t) + (1 - 1/\varkappa\Theta^{\#})\exp\{-\varkappa t\};$$

if $0 < \kappa \le \lambda$, then

$$-(\varkappa t - \varkappa - 1)\,\bar{R}(\lambda, \alpha; t)/\varkappa\Theta^{\#} + (1/\varkappa\Theta^{\#} - 1)\,\eta(t) - \exp\{-\varkappa t\}/\varkappa\Theta^{\#} \le A(t) \le$$
$$\le (\varkappa\Theta^{\#} - \varkappa t + \varkappa + 1)\,\bar{R}(\lambda, \alpha; t)/\varkappa\Theta^{\#} + (1/\varkappa\Theta^{\#})\exp\{-\varkappa t\} + (1 + 1/\varkappa\Theta^{\#})\,\eta(t),$$

where $\eta(t) = [\lambda/(\lambda - \kappa)]^{\alpha}\exp\{-\kappa t\}R(\lambda - \kappa, \alpha; t)$.

Proof. As we have noted above, the d.f. $R(\lambda, \alpha; t)$ for $\alpha \ge 1$ is $p(\cdot)$-NBUA. Therefore for the auxiliary function $A^{\#}(t)$ we can easily establish the following inequalities:

if $\kappa < 0$, then

$$\exp\{\varkappa t\}\,\bar{F}(t) + t/\Theta^{\#} - 1 - \Phi(t) \le A^{\#}(t) \le \exp\{\varkappa t\}\,\bar{F}(t) +$$
$$+ t/\Theta^{\#} - \Phi(t) + \int_0^t \exp\{\varkappa u\}\,[dF(u) - \varkappa\bar{F}(u)\,du],$$

where $\Phi(t) = \int_0^t (t-u)\exp\{\kappa u\}[dF(u) - \kappa\bar{F}(u)du]/\Theta^{\#}$;

if $\kappa > 0$, then

$$\exp\{\varkappa t\}\,\bar{F}(t) + t/\Theta^{\#} - 1 - \Phi(t) - \varkappa\int_0^t \exp\{\varkappa u\}\,\bar{F}(u)\,du \leqslant$$

$$\leqslant A^{\#}(t) \leqslant \exp\{\varkappa t\}\,\bar{F}(t) + t/\Theta^{\#} - \Phi(t) + \int_0^t \exp\{\varkappa u\}\,dF(u).$$

The final form of the inequality is established using the following relationships:

$$\int_0^t \exp\{\varkappa u\}\,dF(u) = [\lambda/(\lambda-\varkappa)]^{\alpha} R(\lambda-\varkappa,\alpha;t);$$

$$\int_0^t u\exp\{\varkappa u\}\,dF(u) = (\alpha/\lambda)\,[\lambda/(\lambda-\varkappa)]^{\alpha+1} R(\lambda-\varkappa,\alpha+1;t);$$

$$\varkappa\int_0^t \exp\{\varkappa u\}\,\bar{F}(u)\,du = \exp\{\varkappa t\}\,\bar{R}(\lambda,\alpha;t) - 1 + [\lambda/(\lambda-\varkappa)]^{\alpha} R(\lambda-\varkappa,\alpha;t);$$

$$\varkappa^2\int_0^t u\exp\{\varkappa u\}\,\bar{F}(u)\,du = (\varkappa t-1)\exp\{\varkappa t\}\,\bar{R}(\lambda,\alpha;t) + 1 + (\varkappa\alpha/\lambda)\times$$

$$\times\,[\lambda/(\lambda-\varkappa)]^{\alpha+1} R(\lambda-\varkappa,\alpha+1;t) - [\lambda/(\lambda-\varkappa)]^{\alpha} R(\lambda-\varkappa,\alpha;t);$$

$$\varkappa\int_0^t (t-u)\exp\{\varkappa u\}\,[dF(u) - \varkappa\bar{F}(u)\,du] = \varkappa t + 1 + (t\varkappa - 1 - \varkappa)\times$$

$$\times\exp\{\varkappa t\}\,\bar{R}(\lambda,\alpha;t) - R(\lambda-\varkappa,\alpha;t)\,[\lambda/(\lambda-\varkappa)]^{\alpha}.$$

2. MULTIVARIATE p-NBUA PROPERTY (Mp-NBUA)

Definition 2. Let $s_n = \Sigma_{k=1}^{n} x_k$, where $\{x_k\}$ is a sequence of jointly independent nonnegative m-dimensional random vectors, such that the random vectors x_k, $k \geq 2$, are identically distributed with the multivariate distribution function (m.d.f.) $P(x)$, $x \in R_+^m$, and nonzero marginal means $b_i = 1/\mu_i$, $i = 1,\ldots,m$ and the random vector x_1 follows the m.d.f. $P_0(x)$, $x \in R_+^m$, with means $a_i = 1/\lambda_i$, $i = 1,\ldots,m$. The random process $\nu(t) = \mathrm{Sup}\{n \mid s_n \leq t,\ t \in R_+^m\}$ is called a generalized multivariate renewal process, or GMRP (if $P_0(x) \equiv P(x)$, then this is simply a multivariate renewal process, or MRP). It is easy to note that $\nu(t) = \Sigma_{n=1}^{\infty} I(s_n \leq t)$, where $I(A)$ is the indicator of the event A.

The renewal function $H_0(t)$ of a GMRP (resp., the renewal function $H(t)$ of an MRP) is defined as the mean number of renewals (jumps of the stochastic process $\nu(t)$) in the m-dimensional parallelepiped

$$\Delta_t = \{u \mid \bar{0} \leqslant u \leqslant t;\ \bar{0}, u, t \in R_+^m\},$$

i.e., $H_0(t) = M_\nu(t)$, $H_0(0) = 0$. $H_0(t)$ can be obtained as an m-fold convolution of P and P_0:

$$H_0(t) = \sum_{n=0}^{\infty} P_0 \overset{m}{*} P^{n(*m)}. \tag{15}$$

THEOREM 4. The renewal function $H(t)$ of a GMRP has a lower bound

$$H_0(t) \geqslant (\mu t)_{\min} - (\mu a)_{\max}, \tag{16}$$

and in particular for an MRP

$$H(t) \geqslant (\mu t)_{\min} - 1, \tag{17}$$

where $(\mu t)_{\min} = \min(\mu_i t_i,\ 1 \leq i \leq m)$, $(\mu a)_{\max} = \max(x\mu_i a_i,\ 1 \leq i \leq m)$.

Proof. Consider the coordinates of the first renewal outside the region Δ_t. This is the random vector $s_{\nu(t)+1} = (s_{1\nu(t)+1},\ldots,s_{m\nu(t)+1})$, where $s_{i\nu(t)+1} = \Sigma_{k=1}^{\nu(t)+1} x_{ik}$, $x_k = (x_{1k},\ldots,x_{mk})$. If the vector $s_{\nu(t)+1}$ is outside Δ_t, this means that the difference $s_{j\nu(t)+1} - t_j$ is nonnegative for at least one of the coordinates j, i.e., $\max_{1\le j\le m}\{s_{j\nu(t)+1} - t_j\} \ge 0$. By Wald's identity,

$$Ms_{j\nu(t)+1} = M\sum_{k=1}^{\nu(t)+1} x_{jk} = b_j M\nu(t) + a_j = b_j H(t) + a_j.$$

Now, passing to expectations, we obtain in the last condition

$$0 \leqslant \max_{1\leqslant j\leqslant m}\{b_j H(t) + a_j - t_j\} \leqslant \max_{1\leqslant j\leqslant m} b_j\{H(t) + a_j/b_j - (\mu t)_{\min}\},$$

and since $b \ge \bar{0}$, we also have $H(t) + (\mu a)_{\max} - (\mu t)_{\min} \ge 0$. Hence trivially follows inequality (16).

We introduce a multivariate analog of the residual life distribution by associating to the m.d.f. P another m.d.f. S_P with a multivariate Laplace–Stieltjes transform (MLST) of the form

$$\hat{S}_P(s) = [1 - \hat{P}(s)]/(b, s), \tag{18}$$

where (b, s) is the scalar product of the random vectors $b = (b_1,\dots,b_m)$ and $s = (s_1,\dots,s_m)$; Re $s_i \ge 0$, $i = 1,\dots,m$. It is easy to see that all univariate marginal distributions have the usual form of a residual life d.f.

Definition 3. The m.d.f. P has the multivariate NBUA property (MNBUA) if for all $t \in R_+^m$ we have the inequality

$$S_P(t) \geqslant P(t). \tag{19}$$

In this case we say that the m.d.f. $P(t)$ is MNBUA, or $P \in \{\text{MNBUA}\}$.

THEOREM 5. Let the m.d.f. $P \in \{\text{MNBUA}\}$. Then the renewal function $H(t)$ of the corresponding MRP has the bound

$$H(t) \leqslant (\mu t)_{\min}. \tag{20}$$

Proof. For $P \in \{\text{MNBUA}\}$ we have

$$P^{n(*m)}(t) = \int_{\Delta_t} P(t-u)\,dP^{(n-1)(*m)}(u) \leqslant \int_{\Delta_t} S_P(t-u)\,dP^{(n-1)(*m)}(u) =$$

$$= S_P \overset{m}{*} P^{(n-1)(*m)}(t),$$

and the renewal function $H(t)$ thus can be bounded from above by the series $H(t) \le \Sigma_{n=0}^{\infty} S_P *^m P^{n(*m)}(t)$, which is the renewal function H_0 of the GMRP with $P_0(t) \equiv S_P(t)$. The simplest way to find $H_0(t)$ is by using the MLST. In operator form, (15) is rewritten as $\hat{H}_0(s) = \hat{P}_0(s)/[1 - \hat{P}(s)]$. Substituting the MLST of the m.d.f. $S_P(t)$, we obtain $\hat{H}_0(s) = 1/(b, s)$ and a direct check shows that the inverse MLST of $\hat{H}_0(s)$ is the function $(\mu t)_{\min}$.

From Theorems 5 and 4 for the m.d.f. $P \in \{\text{MNBUA}\}$ we obtain the two-sided bound on the renewal function:

$$(\mu t)_{\min} - 1 \leqslant H(t) \leqslant (\mu t)_{\min}. \tag{21}$$

Example. A shock model with damage accumulation. Consider the following functional over an MRP:

$$B(t) = \sum_{n=0}^{\infty} P\{s_{1n} \leqslant t_1, s_{1n+1} \geqslant t_1, s_{2n} \leqslant t_2, \dots, s_{mn} \leqslant t_m\}.$$

It can be interpreted as the reliability of the shock model in the time interval $[0, t_1)$. The shock model is generated by a univariate renewal process; each renewal induces damage by $(m - 1)$ parameters; for each parameter j, there is a maximum admissible level of accumulated damage t_j $(2 \le j \le m)$.

The function $B(t)$ is the solution of the multivariate renewal equation (MRE)

$$B(t) = \tilde{g}(t) + B \overset{m}{*} P, \tag{22}$$

where $\tilde{g}(t) = F\{x_{11} \ge t_1, x_{20} \le t_2, \dots, x_{m0} \le t_m\}$, or in canonical form $B(t) = \tilde{g} *^m \{H(t) + 1\}$.

Using inequality (21), we obtain a two-sided bound on the reliability of this shock model for $P \in \{\text{MNBUA}\}$:

$$(\mu t)_{\min} \overset{m}{*} \tilde{g} - \tilde{g}(t) \leqslant B(t) \leqslant (\mu t)_{\min} \overset{m}{*} \tilde{g}. \tag{23}$$

A few comments concerning MNBUA distributions.

1. We have not used the explicit form of the residual life m.d.f., but it may be useful in some cases.

THEOREM 6. The d.f. defined by the MLST (18) for any k ($1 \leq k \leq m$) has the form

$$S_P(t) = (\mu t)_{\min} - \mu_k \int_0^{t_k} P(\tilde{x}_{1k}(\omega), \dots, \tilde{x}_{mk}(\omega))\, d\omega, \tag{24}$$

where $\tilde{x}_{ik}(\omega) = t_i - b_i\mu_k t_k + b_i\mu_k\omega$, $i = 1,\dots,m$.

Proof. It is helpful to use the apparatus of generalized functions. The equality (18) contains two factors, $1 - \hat{P}(s)$ and (b, s). For definiteness, fix the integer k ($1 \leq k \leq m$). It remains to note that in the region $y \leq t_j - b_j\mu_k t_k$,

$$\int P(x_{1j}(y), \dots, x_{mj}(y))\, dy = 0,$$

since for the k-th argument $\max\{t_k - \beta_k\mu_j t_j + b_k\lambda_j y\} = t_k - b_k\mu_j t_j + b_k\mu_j(t_j - b_j\mu_k t_k) = 0$.

2. Let us consider an example of a distribution with the MNBUA property: $E(x) = \Pi_{i=1}^{m} (1 - \exp\{-\mu_i x_i\})$. This distribution plays an important role in the analysis of complex technical systems.

THEOREM 7. The m.d.f. $E \in \{\text{MNBUA}\}$.

Proof. The inequality $S_E \geq E$ is proved by induction on the dimension m of the space R_+^m. For $m = 1$, the inequality holds trivially ($S_E \equiv E$). Assuming that the inequality holds in the space R_+^k for any $k < m$, consider the residual time m.d.f.

$$S_E = \mu_\eta x_\eta - \mu_\eta \int_0^{x_\eta} \prod_{i=1}^{m} (1 - \exp\{-\mu_i x_i + \mu_\eta x_\eta - \mu_\eta\omega\})\, d\omega,$$

where $\eta = \arg(\mu x)_{\min}$ (this representation is also feasible by the previous result). Now fix the index $j \neq \eta$:

$$S_E = \mu_\eta x_\eta - \mu_\eta \int_0^{x_\eta} \prod_{i \neq j} (1 - \exp\{-\mu_i x_i + \mu_\eta x_\eta - \mu_\eta\omega)\, d\omega +$$
$$+ \mu_\eta \exp\{-\mu_j x_j + \mu_\eta x_\eta\} \int_0^{x_\eta} \prod_{i \neq j} (1 - \exp\{-\mu_i x_i + \mu_\eta x_\eta - \mu_\eta\omega) \times \tag{25}$$
$$\times \exp\{-\mu_\eta\omega\}\, d\omega.$$

The first two terms may be considered as the residual life m.d.f. in the space R_+^{m-1}, the last term is nonnegative, and therefore $S_E(x) \geq \Pi_{i\neq j} (1 - \exp\{-\mu_i x_i\}) - \Pi_{i\neq j} (1 - \exp\{-\mu_i x_i\})\exp\{-\mu_j x_j\} = E(x)$.

Definition 4. The m.d.f. $P(x)$, $x \in R_+^m$, is Mp-NBUA with $p > 0$ if:

a) there are numbers κ, j ($1 \leq j \leq m$) such that $\int_{R_+^m} \exp\{\kappa x_j\} dP(x) = 1/p$;

b) the m.d.f. $P^{\#}(t) = p \int_{\Delta_t} \exp\{\kappa x_j\} dP(x)$ is MNBUA, i.e., $S_{P^{\#}}(t) \geq P^{\#}(t)$ for all $t \in R_+^m$.

The properties of the class Mp-NBUA are similar to the properties 1-3 previously introduced for the univariate class p-NBUA. The Mp-NBUA property may be extended by treating p as the function $p(x)$, $x \in R_+^m$, as in the univariate case. The class Mp-NBUA is nonempty. An example of an m.d.f. with the Mp-NBUA property is the m.d.f. E (from Theorem 6), because $E^{\#}(x) = \Pi_{i\neq j} (1 - \exp\{-\mu_i x_i\})(1 - \exp\{-\mu_j p x_j\})$, $\kappa = \mu_j(1 - p)$.

Consider the thinned multivariate renewal process generated by the m.d.f. $P(x)$ and the "imaginary" renewal probability p ($q = 1 - p$). Such processes arise in the analysis of shock models where some shocks do not necessarily cause damage. The m.d.f. G between neighboring renewal coordinates of the thinned process is the solution of the multivariate renewal equation (MRE)

$$G = q + pG \overset{m}{*} P. \tag{26}$$

The scheme that generates two-sided bounds on the solution of the MRE in the multivariate case remains as before. To each function of the MRE we associate a new function superscripted $\#$, which is defined as

$$z^{\#}(x) = \exp\{\kappa x_j\}\, z(x).$$

This reduces the original MRE to the ordinary form

$$G^{\#} = q\exp\{\kappa x_j\} + G^{\#} \overset{m}{*} P^{\#}.$$

If $P \in \{Mp\text{-NBUA}\}$, then with an appropriate κ function $P^{\#}$ is an m.d.f.; canonical representation is $G^{\#} = (q\exp\{\kappa x_j\}) \overset{m}{*_j} (H^{\#} + 1)$, where $H^{\#}$ is the multivariate renewal function of the MRP generated by the m.d.f. $P^{\#}$. Since $P^{\#} \in \{\text{MNBUA}\}$, the bound (21) holds for $H^{\#}$ and for the m.d.f. $G^{\#}$ we have the bound

$$-q\exp\{\varkappa t_j\} \leqslant G^{\#}(t) - q\int_{\Delta_t} \exp\{\varkappa(t_j - u_j)\}\,\mu_j^{\#}\prod_{i\neq j}\delta(u_i - a_i^{\#}\mu_j^{\#}u_j)\,du \leqslant 0.$$

To simplify the integral in both bounds, we introduce the new means $a_i^0 = a_i^{\#}\mu_j^{\#}/\kappa$ and the corresponding $\mu_j^0 = 1/a_i^0$, $i = 1,\ldots,m$. Then

$$q\mu_j^{\#}\int_{\Delta_t}\exp\{-\varkappa u_j\}\prod_{i\neq j}\delta(u_i - a_i^{\#}\mu_j^{\#}u_j)\,du = (q\mu_j^{\#}/\varkappa)\times$$
$$\times\mu_j^0\int_{\Delta_t}\exp\{\mu_j^0 u_j\} - \prod_{i\neq j}\delta(u_i - a_i^0\mu_j^0 u_j)\,du.$$

The integrand in braces is the multivariate exponential density. Thus, passing back from the m.d.f. $G^{\#}$ to G, we obtain the following result.

THEOREM 8. Assume that the thinned MRP is generated by the m.d.f. $P \in \{Mp\text{-NBUA}\}$ and the probability p $(0 < p < 1,\ q = 1 - p)$. Then the m.d.f. G of the events in the thinned process satisfies the bound

$$-q \leqslant G(t) - (q\mu_j^{\#}/\varkappa)(1 - \exp\{-(\mu^0 t)_{\min}\}) \leqslant 0, \tag{27}$$

where $\mu^0 = (\mu_1^0,\ldots,\mu_m^0)$, $a_i^0 = a_i^{\#}\mu_j^{\#}/\kappa$, $\mu_i^0 = 1/a_i^0$, $a_i^{\#} = p\int_{R_+^m} u_i\exp\{\kappa u_j\}dP(u)$, $i = 1,\ldots,m$; the number κ is the solution of the equation $\int_{R_+^m}\exp\{\kappa u_j\}dP(u) = 1/p$.

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