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Wave and shock velocities in relativistic magnetohydrodynamics compared with the speed of light

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We prove that the natural thermodynamic restrictions on the constitutive equations in relativistic magnetohydrodynamics (stability of equilibrium state) are necessary and sufficient to guarantee that the normal and the radial velocity of the wave front of disturbances and the shock velocity do not exceed the light speed.

1 Introduction

In many physical applications an R^N vector field $\mathbf{u}(x^\alpha)$ satisfies a first-order quasi-linear system of N balance laws of the form

$$\partial_\alpha \mathbf{F}^\alpha(\mathbf{u}) = \mathbf{f}(\mathbf{u}). \tag{1.1}$$

x^α stand for $(x^0, x^1, x^2, x^3) = (t, x^1, x^2, x^3)$ and ∂_α means $\partial/\partial x^\alpha$, x^i will be used for (x^1, x^2, x^3) and $\partial_i = \partial/\partial x^i$.

If all solutions of (1.1) also satisfy the scalar inequality

$$\partial_\alpha h^\alpha(\mathbf{u}) = g(\mathbf{u}) \leqq 0 \tag{1.2}$$

the system may be written in a particular symmetric form [1–3]. Indeed, if we define the *main field* $\mathbf{u}'$, i.e. a set of multipliers, by the relation

$$\mathbf{u}' \cdot d\mathbf{F}^\alpha \equiv dh^\alpha, \tag{1.3}$$

we may write

$$\mathbf{A}'^\alpha(\mathbf{u}')\, \partial_\alpha \mathbf{u}' = \mathbf{f}(\mathbf{u}'), \tag{1.4}$$

where:

$$\mathbf{A}'^\alpha = \frac{\partial^2 h'^\alpha}{\partial \mathbf{u}'\, \partial \mathbf{u}'}; \qquad h'^\alpha = \mathbf{u}' \cdot \mathbf{F}^\alpha - h^\alpha. \tag{1.5}$$

The system (1.4) is symmetric hyperbolic if

$$\mathcal{Q} = \delta \mathbf{u}' \cdot \delta \mathbf{F}^\alpha \xi_\alpha > 0 \tag{1.6}$$

holds for all non vanishing variations $\delta\mathbf{u}'$ and at least for one time-like vector ξ_α. It is shown in [2] that the quadratic form $\mathcal{Q}$ is equivalent to the quadratic form introduced by Friedrichs [4].

The special structure of the system and the convexity condition lead to several interesting results in non-linear wave propagation and, in particular, for shock waves. Such results concern the boundedness of shock velocities [5], growth of entropy across the shock [6, 2], etc. ...

Without loss of generality we may choose $\mathbf{F}^0$ as the field $\mathbf{u}$. With this choice there are some physical examples, as in the present case of MHD, for which the solutions of (1.1) must also satisfy a set of M semi-linear constraints

$$\partial_i \mathbf{c}^i(\mathbf{u}) = \mathbf{c}(\mathbf{u}), \qquad \nabla \mathbf{c}^i = \text{const.}, \qquad \nabla = \partial/\partial \mathbf{u}. \tag{1.7}$$

Since these constraints must be compatible with the field equations, they must be involutive i.e. they are always true, if they are satisfied by the initial data and so there exist some $M \times M$ matrices M^i such that [7]

$$\nabla \mathbf{c}^i \, \nabla \mathbf{F}^j + \nabla \mathbf{c}^j \nabla \mathbf{F}^i = M^i \nabla \mathbf{c}^j + M^j \nabla \mathbf{c}^i. \tag{1.8}$$

In this situation it is possible to extend the previous results. The system assumes the symmetric form (1.4), where in the present case the spatial part of (1.5) becomes [7]

$$\mathbf{A}'^i = \frac{\partial^2 h'^i}{\partial \mathbf{u}' \, \partial \mathbf{u}'} - \mathbf{c}^i \frac{\partial^2 \mathbf{b}}{\partial \mathbf{u}' \, \partial \mathbf{u}'}; \qquad h'^i = \mathbf{u}' \cdot \mathbf{F}^i + \mathbf{b} \cdot \mathbf{c}^i - h^i; \tag{1.9}$$

where $\mathbf{b}$ are the new multipliers for the constraints (1.7)

$$\mathbf{u}' \cdot d\mathbf{F}^i + \mathbf{b} \cdot d\mathbf{c}^i \equiv dh^i. \tag{1.10}$$

The condition of convexity (1.6) remains unchanged. Moreover it turns out that [8]

If (1.6) *is satisfied for an arbitrary time-like covector* ξ_α *and for all* $\delta\mathbf{u}' \neq 0$ *compatible with the constraints:*

$$\xi_i \, \delta \mathbf{c}^i = 0 \tag{1.11}$$

and

$$M^i = \text{const.}, \qquad \det(M^\alpha \xi_\alpha) > 0, \qquad \det(L^\alpha \xi_\alpha) > 0, \tag{1.12}$$

where

$$L^i = M^i + \nabla \mathbf{c}^i \, \nabla' \mathbf{b}, \qquad L^0 = M^0 = I, \qquad \nabla' = \partial/\partial \mathbf{u}', \tag{1.13}$$

then the normal velocity of the wave front, the radial velocity of disturbances and the shock velocity do not exceed the light speed c.

In relativistic non-linear wave problems, in particular for MHD, it is very difficulty to prove directly that all types of disturbances and shocks propagate with velocities that are bounded with respect to the light speed in vacuo [9, 10].

In this paper we show that in the case of relativistic MHD the conditions (1.6) and (1.12) are equivalent to the usual thermodynamic conditions of stability and therefore all types of signals have speeds less than or equal to c.

4. Relativistic MHD

The system of equations governing a perfectly conducting relativistic plasma forms a set of covariant conservation laws representing the conservation of particle number, energy-momentum and the Maxwell equations

$$\begin{cases} \partial_\alpha V^\alpha = 0 \\ \partial_\alpha T^{\alpha\beta} = 0 \\ \partial_\alpha \Psi^{\alpha\beta} = 0 \end{cases} \tag{2.1}$$

where:

$$V^\alpha = r u^\alpha,$$

$$T^{\alpha\beta} = (rf + B^2)\, u^\alpha u^\beta - (p + \tfrac{1}{2} B^2)\, g^{\alpha\beta} - B^\alpha B^\beta,$$

$$\Psi^{\alpha\beta} = u^\alpha B^\beta - u^\beta B^\alpha.$$

Here r is the rest mass density, f the index of the fluid $rf = \varrho + p$, ϱ is the proper energy density, p the pressure, u^α the unit four-velocity and B^α is the magnetic field: $B_\alpha u^\alpha = 0$, $B^2 = -B_\alpha B^\alpha$. The signature of the metric $g_{\alpha\beta}$ is $(+, -, -, -)$ and the speed of light c is equal to unity.

Taking into account the Gibbs relation:

$$df = T\, dS + V\, dp, \qquad V = 1/r$$

we have

$$T\, d(-rSu^\alpha) = (G + 1)\, dV^\alpha - u_\beta\, dT^{\alpha\beta} - B_\beta\, d\Psi^{\alpha\beta}, \tag{2.2}$$

$$G = f - TS - 1.$$

All differentiable solutions of (2.1) satisfy the entropy law:

$$\partial_\alpha(rSu^\alpha) = 0. \tag{2.3}$$

The system (2.1) represents the evolution laws (1.1) and the constraints (1.7) while (2.3) coincides with (1.2). We put this in evidence by writing the following identifications

$$\mathbf{F}^\alpha = \begin{pmatrix} V^\alpha \\ T^{\alpha\beta} \\ \Psi^{\alpha i} \end{pmatrix}; \quad \mathbf{f} = 0; \quad \mathbf{c}^i = \Psi^{i0}, \quad \mathbf{c} = 0; \quad h^\alpha = -rSu^\alpha, \quad g = 0. \tag{2.4}$$

By comparison of (2.2) with (1.10) (for the spatial components) and (1.3) (for the temporal part), we deduce the form of the main field $\mathbf{u}'$ [11] and of the multiplier **b** [12] appropriate to the present case

$$\mathbf{u}' \equiv \frac{1}{T}\begin{pmatrix} G+1 \\ -u_\beta \\ -B_i \end{pmatrix}; \quad \mathbf{b} = -\frac{B_0}{T}. \tag{2.5}$$

The quadratic form (1.6) assumes the form

$$\mathcal{Q} = \delta\mathbf{u}' \cdot \delta\mathbf{F}^\alpha \xi_\alpha = \xi_\alpha\, \delta\left(\frac{G+1}{T}\right)\delta(ru^\alpha),$$

$$-\xi_\alpha\, \delta\left(\frac{u_\beta}{T}\right)\delta T^{\alpha\beta} - \xi_\alpha\, \delta\left(\frac{B_\beta}{T}\right)\delta\Psi^{\alpha\beta}$$

if we take into account (1.11), i.e.:

$$\xi_i\, \delta\Psi^{i0} = \xi_\alpha\, \delta\Psi^{\alpha 0} = 0.$$

We compute the invariant $\mathcal{Q}$ in the rest frame of the fluid. $u^0 = 1$, $u^i = 0$ imply $B^0 = 0$, $\delta u^0 = 0$, $\delta B^0 = -B^i\,\delta u_i = \boldsymbol{B}\cdot\delta\boldsymbol{u}$, $(\boldsymbol{B} \equiv (B^i)$, ecc. ...):

$$\xi_0 T\mathcal{Q} = rf(\xi_0^2 - \xi^2)\,\delta\boldsymbol{u}^2 + rf(|\boldsymbol{\xi}|\,\delta\boldsymbol{u} - \xi_0\boldsymbol{\xi}\,\delta p/rf\,|\boldsymbol{\xi}|)^2$$

$$+ (\xi_0\,\delta\boldsymbol{B} - \boldsymbol{\xi}\cdot\delta\boldsymbol{u}\,\boldsymbol{B} + \boldsymbol{B}\cdot\boldsymbol{\xi}\,\delta\boldsymbol{u})^2 + (\boldsymbol{B}\wedge\boldsymbol{\xi}\cdot\delta\boldsymbol{u})^2$$

$$+ (\xi_0^2 - \xi^2)(\boldsymbol{B}\wedge\delta\boldsymbol{u})^2 + r\xi_0^2\,\mathcal{Q}',$$

where $\mathcal{Q}'$ depends only on the thermodynamic variables:

$$\mathcal{Q}' = \delta S\,\delta T - \delta p\,\delta V - V^2\,\delta p^2/f$$

$$= -G_{TT}\,\delta T^2 - 2G_{Tp}\,\delta T\;\delta p - (G_{pp} + V^2/f)\,dp^2.$$

Therefore $\mathcal{Q}$ is positive for all non vanishing variations of the field, if we have

$$G_{TT} < 0, \qquad J + \frac{V^2}{f}G_{TT} \geqq 0, \qquad J = \frac{D(G_T, G_p)}{D(T, p)}.$$

These relations are equivalent [2, 11] to the natural conditions that the specific heat is positive and that the sound velocity is not larger than the speed of light

$$c_p > 0, \qquad 0 < \left(\frac{\partial p}{\partial \varrho}\right)_S \leqq 1. \tag{2.6}$$

The conditions (2.6) for convexity have already been deduced in [11].

Now we consider the conditions (1.12). First of all we note that the constraint is linear with respect to the field $\mathbf{u} \equiv \mathbf{F}^0 \equiv (V^0, T^{0\beta}, \Psi^{0i})^T$, because by (2.4) $\mathbf{c}^i$ is equal, except for the sign, to the last block of components of $\mathbf{F}^0$. Hence, if we substitute $\mathbf{F}^j$ for $\mathbf{u}$ in $\mathbf{c}^i$, we obtain from (2.4a): $\mathbf{c}^i(\mathbf{F}^j) = \Psi^{ij}$ and therefore

$$\mathbf{c}^i(\mathbf{F}^j) + \mathbf{c}^j(\mathbf{F}^i) = 0. \tag{2.7}$$

Differentiating (2.7) with respect to $\mathbf{u}$ we deduce that the left-hand side of (1.8) is zero and thus we have $M^i = 0$, $\det(M^\alpha\xi_\alpha) = \xi_0 > 0$.

If we take into account (2.5b), we obtain:

$$\mathbf{b} = -\frac{B_0}{T} = \frac{B_i u^i}{u^0 T} = \left(-\frac{B_i}{T}\right)\left(\frac{u^i}{T}\right)\left(-\frac{T}{u^0}\right)$$

and then

$$\nabla' \mathbf{b} = \frac{1}{u^0}\left(0, \frac{B_i u^i}{u^0}, B_i, -u^i\right).$$

From (1.13) we have $L^i = \nabla \mathbf{c}^i \, \nabla' \mathbf{b} = \mathbf{c}^i (\nabla' \mathbf{b}) = u^i/u^0$ and we conclude $\det(L^\alpha \xi_\alpha) = (u^\alpha \xi_\alpha)/u^0 > 0$.

Therefore the conditions (1.12) are always satisfied for all time-like vectors ξ_α. Thus (2.6) are necessary and sufficient conditions which guarantee the following speeds are less than or equal to c:

- the magnitude of the radial velocity $\boldsymbol{\Lambda}$ with which weak disturbances propagate along the rays,
- the normal velocity $\lambda = \boldsymbol{\Lambda} \cdot \boldsymbol{n}$, where $\boldsymbol{n}$ is unit normal to the wave front,
- the shock speeds.

Lichnerowicz [10] has assumed

$$\tau_p < 0, \quad \tau_{pp} > 0, \quad \tau(p, S) = f/r$$

in order to prove that MHD shock waves propagate with less than the speed of light. These same assumptions were also introduced by Israel [9] in an uncharged fluid. The first one is equivalent to (2.6b), the second one is an *ad hoc* condition. We consider our condition (2.6a) to be more natural

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