STOCK OPTIMIZATION IN TRANSPORTATION/STORAGE SYSTEMS

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Optimal stock reorder policies are determined for transportation/storage systems with queueing of user requests. Exact and approximate solution methods are proposed.

Optimal inventory control is an important element in optimization of complex systems. The inventory control problem in the global setting cannot be solved in the framework of a single model that simultaneously allows for all features of the procurement process. Various authors have accordingly examined different aspects of this problem for different inventory system models.

A fairly broad class of systems is described by models in which the stock level continuously diminishes over time (as long as the warehouse is not empty) until it reaches the lower admissible bound; then the stocks are replenished by a Poisson stream of random deliveries with a known distribution function. The operation of such systems is adequately described by a random walk between two retaining walls [1]. The results of [1] can be applied to solve a number of problems relevant for analysis and optimization of such systems.

Various aspects of this problem are the subject of extensive literature. Further references can be found in [2, 3].

Most inventory control studies assume that the stock-diminishing user requests are satisfied instantaneously, i.e., the user requests are not enqueued if there are stocks in the system. Request queues form only in a stock-out situation, which requires introduction of negative inventory levels. The operation of a stock system is thus optimized ignoring the waiting costs of the enqueued user requests and the costs due to the limited number of waiting places in queue. Such situations occur in transportation/storage systems (TSS), for instance, in petroleum-product supply systems (petroleum depots, service stations, etc.) [4, 5].

In this paper, we consider the stock process in a TSS and solve the corresponding optimization problem. A single-commodity model is analyzed, although it can be easily generalized to multicommodity TSS.

The TSS includes a storage capacity Q. Its input is a Poisson stream of user requests of unit magnitude with rate λ_p. The resource is issued in response to user requests through a single channel. The issue time of the resource to user requests (i.e., the service time of one user request) is an exponential random variable with mean μ_p^{-1}, $\mu_p^{-1} > 0$. We thus see that the stock level diminishes by one unit only after the current user request has been processed. The resource is issued from stock until the stock level drops to the value q, $0 \le q \le Q-1$, after which the warehouse stops issuing the resource regardless of the queue length of user requests (q may be interpreted as safety stock). At this point (i.e., when the stock level is q) the system can place orders of a different size, depending on transport capacity.

Here we consider a fairly general case. Assume that the system orders a quantity i, $i \in \{1,2,\ldots,Q-q\}$, with probability $\alpha_i(qn)$, where n is the queue length of user requests; here $\Sigma_{i=1}^{Q-q}\,\alpha_i(qn) = 1\ \forall n$. The order is delivered with a certain delay, due to shipping and unloading in the warehouse (the warehouse does not fulfill any user requests during this time). The delay is an exponential random variable with mean γ.

User requests do not have information about the stock level in the warehouse and are absolutely "patient", i.e., they wait in queue until the stock is replenished. The maximum number of user requests in the system is N. In other words, a user request that arrives when a system already contains N such requests is lost.

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We use the following notation: mn is the two-dimensional state vector of the system, where m is the stock level and n is the number of user requests in the system; $E := \{mn\colon m = q,\ldots,Q,\ n = 0,\ldots,N\}$ is the phase state space of the system; $\pi(mn)$ is the stationary probability of state mn; c_p^1 is the cost associated with the waiting of one user request for a unit time; c_p^2 is the cost associated with the loss of one user request; d is the cost of storing a unit quantity of stock in the warehouse for a unit time; $\theta(x;\, y)$ is the waiting time for the transition $x \to y$, where x and y are in the same phase state space.

The optimization problem for this TSS is stated as follows. Find the (optimal) values $\alpha_i(qn)$, $i = 1,\ldots,Q - q$, $qn \in E$, that minimize under stationary conditions the total cost per unit time G associated with waiting and loss of user requests and the holding of stock in the warehouse.

We solve the problem by a Markovian approach.

The elements of the generating matrix of the Markov chain that describes the operation of this system are defined as

$$\theta(mn;\ m'n') = \begin{cases} \lambda_p, & \text{if} \quad m' = m,\ n' = n+1,\ n \leqslant N-1, \\ \mu_p, & \text{if} \quad m > q,\ m' = m-1,\ n' = n-1,\ n > 0, \\ \gamma\alpha_i(qn), & \text{if} \quad m = q,\ m' = q+i,\ n' = n, \\ 0 & \text{otherwise}. \end{cases} \tag{1}$$

From expressions (1), we form a system of equilibrium equations for $\pi(mn)$, $mn \in E$:

$$\begin{gathered} \sum_{m'n' \in E^-_{mn}} \theta(mn;\ m'n')\,\pi(mn) = \sum_{m'n' \in E^+_{mn}} \theta(m'n';\ mn)\,\pi(m'n'), \\ \sum_{mn \in E} \pi(mn) = 1, \end{gathered} \tag{2}$$

where E_{mn}^- is the subset of states $m'n' \in E$ that the system can reach from mn in one step; E_{mn}^+ is the subset of states $m'n' \in E$ from which the system can reach the state mn in one step.

The total cost in the system is calculated as

$$G = \sum_{mn \in E} (nc_p^1 + \lambda_p c_p^2 + dm)\,\pi(mn). \tag{3}$$

The optimization problem for this system thus reduces to minimizing (3) subject to the constraints (2). This is a Markov programming problem and it can be solved using linear programming (LP) or dynamic programming (DP) methods [6]. The solution of the problem produces the minimum total cost G^* and at the same time determines the Markovian controls $\alpha_i(qn)$, $i = 1,\ldots,Q - q$, for which G^* is achieved. The optimal values of $\alpha_i(qn)$, $i = 1,\ldots,Q - q$, are either 0 or 1 in each state $qn \in E$, and the solution of the problem thus produces a deterministic inventory control policy for the system: if in state qn we have $\alpha_{i_0}(qn) = 1$ for some i_0, $i_0 \in \{1,\ldots,Q - q\}$, then the quantity i_0 must be ordered in this state.

Let us consider an approximate method for the solution of this problem. An approximate method is needed when the dimension of the phase state space of the system $|E| = (N + 1)(Q - q + 1)$ is too high, so that exact computer algorithms run into insurmountable difficulties when attempting to solve the problem. The proposed approximation method is based on the idea of phase-space aggregation of the states of stochastic systems [7].

Assume that the system operates under high load conditions (i.e., $\lambda_p \gg \mu_p$) and stock replenishment is a rare event compared with the arrival of user requests (i.e., $\lambda_p \gg \nu$). We also assume that when the queue of user requests reaches a certain critical (limit) length, all user requests leave the system after the processing of the request is completed, and the initial stock level almost does not change.

Under these assumptions, we consider the following decomposition of the phase state space:

$$E = \bigcup_{m=q}^{Q} E_m, \quad E_m \cap E_{m'} = \varnothing, \quad m \neq m', \tag{4}$$

where $E_m := \{mn \in E\colon n = 0,\ldots,N\}$.

The class of states E_m of the initial chain is called an aggregated state $\langle m\rangle$, $m = q,\dots,Q$. The decomposition (4) is used to construct the aggregation function $U\colon E \to \hat{E}$, $\hat{E} = \{\langle q\rangle, \langle q+1\rangle, \dots, \langle Q\rangle\}$, which is defined as

$$U(mn) = \langle m\rangle, \quad mn \in E_m, \quad m = \overline{q, Q}. \tag{5}$$

The aggregation function (5) thus defines an aggregated model (aggregated relative to the original model), which is also a Markov chain with the phase state space $\hat{E}$.

By our assumptions, the classes E_m, $m = q,\dots,Q$, are ergodic classes of the supporting chain with the stationary distributions $\rho^m = \{\rho^m(mn)\colon mn \in E_m\}$:

$$\rho^m(m0) = \mu_p/(\mu_p + N\lambda_p); \quad \rho^m(mn) = \lambda_p/(\mu_p + N\lambda_p), \qquad n = \overline{1, N}. \tag{6}$$

Using the relationships (6), we conclude that the generating matrix elements of the aggregated model $\hat{\theta}(x; y)$, $x, y \in \hat{E}$, are given by

$$\hat{\theta}(\langle m\rangle; \langle m'\rangle) = \begin{cases} \hat{\theta}_{mm'}, & \text{if } q+1 \leqslant m \leqslant Q, \quad m' = m-1, \\ \hat{\theta}_{qm'}, & \text{if } m = q, \; q+1 \leqslant m' \leqslant Q, \\ 0 & \text{otherwise.} \end{cases}$$

Here we use the following notation:

$$\hat{\theta}_{mm'} := \sum_{mn \in E_m} \sum_{m'n' \in E_{m'}} \rho^m(mn)\, \theta(mn; m'n'), \quad q+1 \leqslant m \leqslant Q, \quad m' = m-1;$$

$$\hat{\theta}_{qm'} := \sum_{qn \in E_q} \sum_{m'n' \in E_{m'}} \rho^q(qn)\, \theta(qn; m'n'), \qquad q+1 \leqslant m' \leqslant Q.$$

Using (3), we obtain after simple manipulations

$$\hat{\theta}_{mm'} = N\lambda_p\mu_p/(\mu_p + N\lambda_p), \qquad q+1 \leqslant m \leqslant Q, \quad m' = m-1;$$

$$\hat{\theta}_{qm'} = \gamma \sum_{qn \in E_q} \rho^q(qn)\, \alpha_{m'-q}(qn), \qquad q+1 \leqslant m' \leqslant Q.$$

The total cost in the aggregated model is given by

$$\hat{G} := \sum_{\langle m\rangle \in \hat{E}} \left[c_p^1 \frac{N(N+1)}{2} + \lambda_p c_p^2 + dm \right] \pi(\langle m\rangle),$$

where $\pi(\langle m\rangle)$ is the stationary probability of the aggregated state $\langle m\rangle$, $\langle m\rangle \in \hat{E}$.

Then the optimization problem for the aggregated model is posed as

$$\hat{G} \to \min \tag{7}$$

subject to

$$\pi(\langle m\rangle) = \left[\frac{\gamma}{N\lambda_p} \alpha_{m-q}(q0) + \frac{1}{N\mu_p} \sum_{n=1}^{N} \alpha_{m-q}(qn) \right] \pi(\langle q\rangle), \qquad m = \overline{q+1, Q}; \tag{8}$$

$$\sum_{\langle m\rangle \in \hat{E}} \pi(\langle m\rangle) = 1; \tag{9}$$

$$\sum_{i=1}^{Q-q} \alpha_i(qn) = 1 \qquad \forall n = \overline{0, N}. \tag{10}$$

Solving the problem (7)-(10), we obtain the approximate optimal values $\alpha_i(qn)$, $i = 1,\dots,Q-q$, $qn \in E$.

The dimension of the phase state space E, as noted above, is $|E| = (N+1)(Q-q+1)$, while the dimension of the phase state space $\hat{E}$ of the aggregated model is $|\hat{E}| = Q - q + 1$. In real systems, the parameters N and Q are usually of the same order of magnitude, so that the size of the optimization problem for the initial model increases as Q^2 while the size of the corresponding problem for the aggregated model increases linearly.

It is important to note that the proposed algorithm for the construction of the aggregated model is not the only possible one. The phase state space E of the initial model can be decomposed in different ways, and as we know [7] the phase aggregation algorithm is determined uniquely for each phase state space decomposition of the original model. We thus need to consider the choice of the particular phase state space decomposition scheme in order to construct an aggregated model from the original model. This choice should be based on practical considerations, primarily allowing for the required accuracy of the end result. It is well known that deeper aggregation of the original model increases the errors in the results, while on the other hand the aggregation procedure involves certain methodological and computational difficulties. The proposed decomposition (4) is successful in the sense that it produces a sufficiently simple structure of the constraint system in the optimization problem of the aggregated model (7)-(10). As a result, special-purpose software allows completely automatic realization of the problem (7)-(10) by computer.

In conclusion, we would like to give some arguments in favor of using the LP method for the solution of the given optimization problem. Real TSS operate under uncertainty with respect to the arrival rate of user requests λ_p and stock deliveries ν. In practice, it is therefore important to consider the stability of the optimal values of the Markovian controls $\alpha_i(qn)$ relative to changes in the parameters λ_p and ν. In other words, we need to determine the ranges of the parameters λ_p and ν where the optimal values $\alpha_i(qn)$ obtained for fixed values of these parameters remain unchanged. Special application software is available to answer this question. Moreover, the LP method permits introducing additional linear constraints. The DP method unfortunately does not have these features. Note that the inventory model considered in this paper can be generalized to TSS with user requests of an arbitrary size.

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