SIMPLE PURSUIT OF ONE EVADER BY A GROUP

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The pursuit of one evader by a group of controlled pursuers is considered for the case of simple motion of the players in nonempty compact sets. Sufficient solvability conditions are derived. These conditions are sometimes also necessary.

The theory of differential games [1-3] contains many studies of the pursuit of one evader by a group of controlled pursuers [4-9]. These studies usually supply sufficient solvability conditions for the pursuit problem, which are sometimes also necessary. It is important to restate these conditions in a visual geometric form (for instance, zero is an interior point of the convex hull of some set). We consider this topic for the case of simple motion of the players, when the control regions are nonempty compact sets. These control regions, however, are not constrained by the traditional condition that the evader control region can be fitted by translation inside the control region of any pursuer. The results of this study are the most complete so far for the problem of simple group pursuit [4, 6, 7, 9].

1. Consider the differential game

$$\dot{z}_i = u_i - v, \quad z_i \in R^k, \quad u_i \in U_i, \quad v \in V, \quad z_i(0) = z_i^0, \quad i = \overline{1, n}. \tag{1}$$

Here R^k is the k-dimensional Euclidean space, U_i and V are nonempty compact sets. A family of nonempty convex compacta $M_1, \ldots, M_n$ are given in R^k, defining the terminal set M^*.

We say that the game (1) can be terminated from the initial state $z^0 = (z_1^0, \ldots z_n^0)$ not later than in time $T(z^0)$ if there exist Borel measurable functions u_i: $V \to U_i$, $i = 1, \ldots, n$, such that for any Lebesgue measurable function v: $[0, T(z^0)] \to V$ we have the inclusion $z_i(t) \in M$ for at least one $i \in \{1, \ldots, n\}$ for some $t = t^*$, $t^* \leq T(z^0)$, where $z_i(t)$ is the solution of the system of differential equations

$$\dot{z}_i(t) = u_i(v(t)) - v(t). \tag{2}$$

We say that the game (1) starting from the initial state $z^0 \in R^{kn} \backslash M^*$ allows evasion of the set M^* if there exists a Lebesgue measurable function v: $[0, +\infty) \to V$ such that $z_i(t) \notin M_i$ for all $i \in \{1, \ldots, n\}$, $t \in [0, +\infty)$ for any Lebesgue-measurable functions u_i: $[0, +\infty) \to U_i$, $i = 1, \ldots, n$. In this case, the evader control is programmed, i.e., it is constructed using only information about the initial state z^0.

Denote by int H, $\bar{H}$, ∂H, co H, and con H respectively the interior, the closure, the boundary, the convex hull, and the conical hull of an arbitrary subset H of the space R^k, and by $[\text{con } H]^*$ the conjugate cone of con H. By $S_r(x)$ we denote a closed ball in R^k centered at the point x with the radius $r > 0$, i.e., $S_r(x) = \{y \in R^k: \|y - x\| \leq r\}$.

Let $\Omega(R^k)$ (co $\Omega(R^k)$) be the metric space of all nonempty compact (and convex) subsets of the space R^k with the Hausdorff metric. For the set $F \in \Omega(R^k)$ we define the supporting function $c(F, \psi) = \max_{f \in F}(f, \psi)$, $\psi \in R^k$.

Let $\psi_0 \in R^k$, $\|\psi_0\| \neq 0$. The set $U(F, \psi_0) = \{f \in F: (f, \psi_0) = c(F, \psi_0)\}$ is called the supporting set to the set F in the direction ψ_0. If the supporting set $U(F, \psi_0)$ consists of a single point, we say that the set F is strictly convex in the direction $\psi_0 \in R^k$ [10]. The set $F \in \Omega(R^k)$ is strictly convex if it is strictly convex in every direction $\psi_0 \in R^k$, $\|\psi_0\| \neq 0$.

Translated from Kibernetika i Sistemnyi Analiz, No. 3, pp. 131-137, May-June, 1992. Original article submitted July 12, 1990.

 1060-0396/92/2803-0438$12.50

The set $F \in \Omega(R^k)$ is called a compactum with a smooth boundary if $U(F, \psi) \cap U(F, \psi') = \varnothing\ \forall\psi, \psi' \in \partial S_1(0)$, $\psi \neq \psi'$.

The mapping C of the set $X \subset R^m$ to $\Omega(R^k)$ is called a multivalued mapping; the mapping $C: X \to \Omega(R^k)$ is measurable ($\mathcal{B}$-measurable) if the set X is Lebesgue-measurable (Borel-measurable) and for every $D \in \Omega(R^k)$ the set $\{x \in X: C(x) \subset D\}$ is Lebesgue-measurable (Borel-measurable).

We give several auxiliary propositions.

LEMMA 1. Let $X, M \in \Omega(R^k)$, $X \cap M = \varnothing$, $Y \in \Omega(R^m)$, assume that the mapping $A: Y \to \Omega(R^k)$ is upper semicontinuous, and

$$\overline{\text{con}}\,(M-x) \cap A(y) \neq \varnothing, \quad \forall x \in X, \quad y \in Y.$$

Then the function $\alpha: X \times Y \to R$ defined by the formula $\alpha(x, y) = \max\{\alpha \geq 0: \alpha(M - x) \cap A(y) \neq \varnothing\}$ is upper semicontinuous.

LEMMA 2. Let $X \in \Omega(R^k)$, $0 \notin X$, $Y \in \Omega(R^m)$, and $A: Y \to \text{co}\,\Omega(R^k)$ is a continuous strictly convex valued mapping, $-\overline{\text{con}}\ x \cap A(y) \neq \varnothing\ \forall x \in X, \forall y \in Y$.

Then the function $\alpha: X \times Y \to R$ defined by the formula $\alpha(x, y) = \max\{\alpha \geq 0: -\alpha x \in A(y)\}$ is continuous.

Proof. Assume the contrary: at some point (x_0, y_0) from the set $X \times Y$ the function $\alpha(x, y)$ is not lower semicontinuous, i.e., there exists a sequence $\{(x_r, y_r)\}$, $(x_r, y_r) \in X \times Y$, converging to the point (x_0, y_0) such that

$$\lim_{r\to\infty} \alpha(x_r, y_r) = \alpha_0' < \alpha_0, \quad \alpha_0 = \alpha(x_0, y_0).$$

From the definition of the function $\alpha(x, y)$ we have

$$-\alpha_0 x_0 \in \partial A(y_0), \quad -\alpha(x_r, y_r)\, x_r \in \partial A(y_r).$$

Since the continuous multivalued mapping $A(y)$ is convex-valued, the mapping $\partial A: Y \to \Omega(R^k)$ is also continuous [11]. Thus, $-\alpha_0' x_0 \in \partial A(y_0)$.

Let $\rho = \alpha_0 - \alpha_0'$. The sequence $\{\alpha(x_r, y_r)\}$ converges to α_0', and therefore for $\varepsilon = \rho/3$ there exists a natural N_1 such that

$$|\alpha(x_r, y_r) - \alpha_0'| \leqslant \varepsilon, \quad \forall r \geqslant N_1. \tag{3}$$

Since the set $A(y_0)$ is convex and strictly convex, the point $p = -1/3(\alpha_0' + 2\alpha_0)x_0$ is in the interior of $A(y_0)$ and there exists a natural N_2 such that $p \in \text{int}\, A(y_r)\ \forall r \geq N_2$. Thus, for $r \geq \max\{N_1, N_2\}$ we have

$$\alpha(x_r, y_r) \geqslant 1/3\alpha_0' + 2/3\alpha_0 = \alpha_0' + 2/3\rho,$$

which contradicts the inequality (3).

COROLLARY 1. Let $M \in \Omega(R^k)$, $Y \in \Omega(R^m)$, $x \in R^k \backslash M$, and $A: Y \to \text{co}\,\Omega(R^k)$ is a continuous strictly convex valued mapping, $\overline{\text{con}}\ (m - x) \cap A(y) \neq \varnothing\ \forall m \in M, y \in Y$. Then the function $\alpha: Y \to R$ defined by the formula $\alpha(y) = \max\{\alpha \geq 0: \alpha(M - x) \cap \partial A(y) \neq \varnothing\}$ is continuous.

Proof. Consider the function

$$\alpha(m, y) = \max\{\alpha \geqslant 0: \alpha(m-x) \cap \partial A(y) \neq \varnothing\}, \quad m \in M, \quad y \in Y.$$

It is easy to see that $\max\{\alpha \geq 0: \alpha(m - x) \cap \partial A(y) \neq \varnothing\} = \max(\alpha \geq 0: \alpha(m - x) \cap A(y) \neq \varnothing\}\ \forall m \in M$, $y \in Y$. Therefore the function $\alpha(m, y)$ is continuous on the set $M \times Y$.

Now, using the relationships $\alpha(y) = \max_{m\in M} \alpha(m, y)$, we conclude that the function $\alpha(y)$ is continuous on Y.

2. Let us state the most general solvability propositions for the differential game (1), which will be used in subsequent analysis.

Consider the multivalued mappings

$$W_i(z_i, v) = \overrightarrow{\text{con}}(M_i - z_i) \cap (U_i - v),$$
$$\overline{W}_i(z_i, v) = \overrightarrow{\text{con}}(M_i - z_i) \cap (\text{co}\, U_i - v), \quad z_i \in R^k \setminus M_i, \quad v \in V, \quad i = \overline{1, n}.$$

Condition 1. For a fixed point $z = (z_1, \ldots, z_n) \in R^{kn} \setminus M^*$ we have the relationships $W_i(z_i, v) \neq \varnothing$, $i = 1, \ldots, n$, for any $v \in V$.

Condition 2. For a fixed point $z = (z_1, \ldots, z_n) \in R^{kn} \setminus M^*$ we have relationships $\overline{W}_i(z_i, v) \neq \varnothing$, $i = 1, \ldots, n$, for any $v \in V$.

Take fixed points z, $\bar{z}$ which satisfy conditions 1 and 2 respectively and define the decision functions

$$\alpha_i(z_i, v) = \max\{\alpha \geqslant 0 : \alpha(M_i - z_i) \cap (U_i - v) \neq \varnothing\}, \tag{4}$$

$$\bar{\alpha}_i(\bar{z}_i, v) = \max\{\alpha \geqslant 0 : \alpha(M_i - z_i) \cap (\text{co}\, U_i - v) \neq \varnothing\}, \quad v \in V, \quad i = \overline{1, n}. \tag{5}$$

Denote

$$\lambda(z, t) = 1 - \inf_{v_t(\cdot)} \max_{i = \overline{1, n}} \int_0^t \alpha_i(z_i, v(\tau))\, d\tau. \tag{6}$$

$$T(z) = \inf\{t \geqslant 0 : \lambda(z, t) \leqslant 0\}. \tag{7}$$

The infimum in (6) is over all V-valued functions that are measurable on $[0, t]$.

Let

$$\alpha(z) = \inf_{v \in V} \max_{i = \overline{1, n}} \alpha_i(z_i, v), \quad \bar{\alpha}(\bar{z}) = \inf_{v \in V} \max_{i = \overline{1, n}} \bar{\alpha}_i(\bar{z}_i, v). \tag{8}$$

THEOREM 1. Suppose that the point $z^0 = (z_1^0, \ldots, z_n^0) \in R^{kn} \setminus M^*$ satisfies condition 1 and $\alpha(z^0) > 0$. Then the game (1) can be terminated from the initial state z^0 not later than in time $T(z^0)$ bounded by $T(z^0) \leq n/\alpha(z^0)$.

Proof. By Lemma 1, the function $\alpha_i(z_i^0, v)$: $V \to R$, $i \in \{1, \ldots, n\}$ is upper semicontinuous and thus Borel-measurable. Therefore, the multivalued mapping

$$M_i(v) = \{m_i \in M_i : \alpha_i(z_i^0, v)(m_i - z_i^0) \in (U_i - v)\} \tag{9}$$

is $\mathscr{B}$-measurable [12, 13]. Separate a Borel single-valued branch $m_i(v)$ of the multivalued mapping $M_i(v)$. Then

$$u_i(v) = v + \alpha_i(z_i^0, v)(m_i(v) - z_i^0) \tag{10}$$

is a Borel function.

Let v: $[0, T(z^0)] \to V$ be some Lebesgue-measurable function. We will show that for at least one $i \in \{1, \ldots, n\}$ the solution of the system of equations (2) hits the set M_i at $t = t^*$ when for the first time

$$1 - \max_{i = \overline{1, n}} \int_0^t \alpha_i(z_i^0, v(\tau))\, d\tau = 0.$$

This t^* obviously exists and $t^* \leq T(z^0)$.

At time t^* for some $i \in \{1, \ldots, n\}$,

$$1 - \int_0^{t^*} \alpha_i(z_i^0, v(\tau))\, d\tau = 0.$$

From (2), (10) we have

$$z_i(t^*) = z_i^0 + \int_0^{t^*} (u_i(v(\tau)) - v(\tau))\, d\tau = \int_0^{t^*} \alpha_i(z_i^0, v(\tau))\, m_i(v(\tau))\, d\tau,$$

and thus $z_i(t^*) \in M_i$.

Let us prove the upper bound for $T(z^0)$:

$$1-\inf_{v_t(\cdot)}\max_{i=\overline{1,n}}\int_0^t \alpha_i(z_i^0, v(\tau))\,d\tau \leqslant 1-\frac{1}{n}\inf_{v_t(\cdot)}\int_0^t \sum_{i=\overline{1,n}}\alpha_i(z_i^0, v(\tau))\,d\tau \leqslant$$

$$\leqslant 1-\frac{1}{n}\inf_{v_t(\cdot)}\int_0^t \max_{i=\overline{1,n}}\alpha_i(z_i^0, v(\tau))\,d\tau = 1-\frac{1}{n}\alpha(z^0)t,$$

and so $T(z^0) \leq n/\alpha(z^0)$.

THEOREM 2. Suppose that the point $z^0 = (z_1^0,\dots,z_n^0) \in R^{kn}\setminus M^*$ satisfies condition 2 and $\bar{\alpha}(z^0) = 0$, while the infimum in $\inf_{v\in V}\max_{i=1,\dots,n}\bar{\alpha}_i(z_i^0, v)$ is achieved on some vector $v_0 \in V$. Then the game (1) starting from the initial state z^0 allows evasion of the set M^*.

Proof. Take $v(t) \equiv v_0$ for $t \geq 0$. Since $\bar{\alpha}_i(z_i^0, v_0) = 0\ \forall i \in \{1,\dots,n\}$, we have $\text{con}(M_i - z_i^0) \cap (\text{co}\,U_i - v_0) = \varnothing$ $\forall i \in \{1,\dots,n\}$.

Hence we obtain $z_i^0 + t(\text{co}\,U_i - v_0) \cap M_i = \varnothing\ \forall i \in \{1,\dots,n\}$, $t > 0$. Clearly, $z_i(t) \subset z_i^0 + t(\text{co}\,U_i - v_0)$ $\forall i \in \{1,\dots,n\}$, $t > 0$, and therefore $z_i(t) \notin M_i$, $i = 1,\dots,n$, $t \geq 0$, for any Lebesgue-measurable functions u_i: $[0, +\infty) \to U_i$, $i = 1,\dots,n$.

LEMMA 3. Let $M \in \text{co}\,\Omega(R^k)$, $Y \in \Omega(R^m)$, $x \in R^k\setminus M$, and A: $Y \to \Omega(R^k)$ is a continuous strictly convex valued mapping, $\text{con}(m - x) \cap A(y) \neq \varnothing\ \forall m \in M$, $y \in Y$. If $\alpha(y) = \max\{\alpha \geq 0: \alpha(M - x) \cap A(y) \neq \varnothing\}$, then the mapping $M(y) = \{m \in M: \alpha(y)(m - x) \in A(y)\}$ is single-valued and continuous.

Proof. Since the values of the multivalued mapping $A(y)$ are strictly convex sets, we have

$$\max\{\alpha\geqslant 0:\alpha(M-x)\cap A(y)\neq\varnothing\}=\max\{\alpha\geqslant 0:\alpha(M-x)\cap$$
$$\cap\,\text{co}\,A(y)\neq\varnothing\},\quad \forall v\in V. \tag{11}$$

By Lemma 2, the function $\alpha(y)$ is continuous, and the multivalued mapping $M(y)$ is upper semicontinuous [14]. We will show that it is single-valued.

Assume the contrary: for some $y_0 \in Y$ the set $M(y_0)$ consists of more than one point, i.e., there exist $m_1, m_2 \in M(y_0)$, $m_1 \neq m_2$. From the inclusions $\alpha(y_0)(m_1 - x) \in \text{co}\,A(y_0)$, $\alpha(y_0)(m_2 - x) \in \text{co}\,A(y_0)$, we obtain $\alpha(y_0)(\lambda m_1 + (1 - \lambda)m_2 - x) \in \text{co}\,A(y_0)\ \forall\lambda \in [0, 1]$.

The set $\text{co}\,A(y_0)$ is strictly convex, and therefore $\alpha(y_0)(\lambda m_1 + (1 - \lambda)m_2 - x) \in \text{int co}\,A(y_0)\ \forall\lambda \in (0, 1)$, which contradicts the equality (11). Thus, the mapping $M(v)$ is single-valued and therefore continuous.

3. Consider the differential game

$$\dot{z}_i = u_i - v,\ z_i\in R^k,\quad u_i\in\partial\,\text{co}\,P,\quad v\in P,\quad z_i(0)=z_i^0,\quad i=\overline{1,n}. \tag{12}$$

Here $P \in \Omega(R^k)$, $z_i^0 \notin M_i$, $i = 1,\dots,n$. The nonempty convex compacta $M_1,\dots,M_n$ define the terminal set M^*, as in the game (1).

Any point $z^0 = (z_1^0,\dots,z_n^0) \in R^{kn}\setminus M^*$ obviously satisfies condition 1. Using Theorems 1 and 2, we will obtain necessary and sufficient conditions of solvability for the pursuit problem in this differential game.

LEMMA 4. Let $X_1,\dots,X_r$, $r \geq 1$, be nonempty subsets of the space R^k. The inclusion $0 \in \text{int co}(\cup_{i=1,\dots,r} X_i)$ holds if and only if $[\text{con}(\cup_{i=1,\dots,r} X_i)]^* = \{0\}$.

Proof. Let $0 \in \text{int co}(\cup_{i=1,\dots,r} X_i)$. This means that there exist points $x_1,\dots,x_m$, $m \geq k + 1$, in the set $\cup_{i=1,\dots,r} X_i$ such that $0 \in \text{int co}(\cup_{i=1,\dots,m} x_i)$. Therefore, for any nonzero $\psi \in R^k$ there exists an index $i \in \{1,\dots,m\}$ such that $(x_i, \psi) <$ 0. In other words, $[\text{con}(\cup_{i=1,\dots,m} x_i)]^* = \{0\}$.

Since

$$\Big[\text{con}\Big(\bigcup_{i=\overline{1,r}} X_i\Big)\Big]^*\subset\Big[\text{con}\Big(\bigcup_{i=\overline{1,m}} x_i\Big)\Big]^*\text{and } 0\in\Big[\text{con}\Big(\bigcup_{i=\overline{1,r}} X_i\Big)\Big]^*,$$

we obtain $[\text{con}(\cup_{i=1,\dots,r} X_i)]^* = \{0\}$.

Conversely, let $[\text{con}(\cup_{i=1,\dots,r} X_i)]^* = \{0\}$. If $0 \notin \text{int co}(\cup_{i=1,\dots,r} X_i)$, then there exists a nonzero $\psi \in R^k$ such that $(\psi, x) \geq 0$ for every $x \in \cup_{i=1,\dots,r} X_i$, i.e., $\psi \in [\text{con}(\cup_{i=1,\dots,r} X_i)]^*$, a contradiction.

It is easy to verify that $[\text{con}(\cup_{i=1,\dots,r} X_i)]^* = \cap_{i=1,\dots,r}[\text{con } X_i]^*$.

Let us state the main proposition of this section.

THEOREM 3. Let P be a strictly convex compactum with a smooth boundary. Then in the game (12) the inequality $\alpha(z^0) > 0$ is equivalent to the inclusion

$$0 \in \text{int co}\Big(\bigcup_{i=\overline{1,n}} (M_i - z_i^0)\Big). \tag{13}$$

Proof. We will show that the inequality $\alpha(z^0) > 0$ implies (13). Assume the contrary: there exists $\psi \in \partial S_1(0) \cap [\text{con}(\cup_{i=1,\dots,n}(M_i - z_i^0)]^*$. Since P is a strictly convex compactum, the set $U(\text{co } P, \psi)$ consists of a single point v_0. Obviously $\alpha_i(z_i^0, v_0) > 0$ for some $i \in \{1,\dots,n\}$ and there exists a vector $m_i \in M_i$ such that $v_0 + \alpha_i(z_i^0, v_0)(m_i - z_i^0) \in \partial \text{ co } P$. Hence we obtain $(v_0, \psi) + \alpha_i(z_i^0, v_0)(m_i - z_i^0, \psi) \leq c(P, \psi)$, and so $(m_i - z_i^0, \psi) = 0$ and $v_0 + \alpha_i(z_i^0, v_0)(m_i - z_i^0) \in U(\text{co } P, \psi)$. This contradicts the condition of strict convexity of the set P.

Conversely, assume that the inclusion (13) holds. Let $\alpha(z^0) = 0$. Since the function $\max_{i=1,\dots,n} \alpha_i(z_i^0, v)$ is continuous in v, there exists a vector $v_0 \in P$ such that $\alpha_i(z_i^0, v_0) = 0$, $i = 1,\dots,n$. Hence $v_0 \in \partial \text{ co } P$. Take an arbitrary index $i \in \{1,\dots,n\}$ and an arbitrary point $m_i \in M_i$. Consider a sequence of points $\{v_r\}$, $v_r = v_0 + \lambda_r(m_i - z_i^0)$, $\lambda_r > 0$, that are not elements of the set co P and converge to the point v_0. By the separation theorem for convex sets, for any natural r there exists a vector $\psi_r \in \partial S_1(0)$ such that

$$(v_0 + \lambda_r (m_i - z_i^0), \psi_r) \geq c(P, \psi_r), \tag{14}$$

Then

$$(m_i - z_i^0, \psi_r) \geq 0. \tag{15}$$

Since the unit sphere is a compact set, we can extract from the sequence $\{\psi_r\}$ a subsequence that converges to some point $\psi_0 \in \partial S_1(0)$. Denote this convergent subsequence again by $\{\psi_r\}$. By inequalities (14) and (15), we have $(v_0, \psi_0) = c(P, \psi_0)$, $(m_i - z_i^0, \psi_0) \geq 0$.

Since $i \in \{1,\dots,n\}$, $m_i \in M_i$ are arbitrary, and boundary of set P is smooth, we obtain $\psi_0 \in [\text{con}(\cup_{i=1,\dots,n} (M_i - z_i^0)]^*$, and thus $0 \notin \text{int co } \cup_{i=1,\dots,n} (M_i - z_i^0)$, a contradiction.

Let us consider some examples which show that the assumptions concerning the boundary of the set P are essential.

Example 1. In the game (12), let $k = 2$, $n = 3$, $P = \text{co}(\cup_{i=1,2} S_1((-1)^i, 0))$, $\cup_{i=1,2,3} M_i = \{0\}$, $z_1^0 = (1, 0)$, $z_2^0 = (-1, 0)$, $z_3^0 = (0, -1)$.

The set P has a smooth boundary but it is not strictly convex. It is easy to see that $\alpha(z^0) > 0$, although $0 \notin \text{int co } (\cup_{i=1,2,3} z_i^0)$.

Example 2. In the game (12), let $k = 2$, $n = 3$, $P = S_5((4, -3)) \cap S_5((-4, -3))$, $\cup_{i=1,2,3} M_i = \{0\}$, $z_1^0 = (0, 1)$, $z_2^0 = (-1, -1)$, $z_3^0 = (1, -1)$.

The set P is strictly convex, but its boundary is not smooth. The point $0 \in P$ and $\alpha_i(z_i^0, 0) = 0$, $i = 1, 2, 3$. At the same time, $0 \in \text{int co } (\cup_{i=1,2,3} z_i^0)$.

COROLLARY 2. Assume that P is a strictly convex compactum with a smooth boundary. If $0 \in \text{int co}(\cup_{i=1,\dots,n} (M_i - z_i^0))$, then the game (12) may be terminated from the initial state z^0 not later than in time $T(z^0)$.

If $0 \notin \text{int co}(\cup_{i=1,\dots,n} (M_i - z_i^0))$, then the game (12) starting from the initial state z^0 allows evasion of the set M^*.

4. Consider the following differential pursuit game:

$$\dot{z}_i = u_i - v, \quad z_i \in R^k, \quad u_i \in U_i(z_i^0, M_i), \quad v \in P, \quad z_i(0) = z_i^0, \quad i = \overline{1, n}. \tag{16}$$

Here P is a strictly convex compactum with a smooth boundary in R^k,

$$U_i(z_i^0, M_i) = \bigcup_{\psi \in S(z_i^0, M_i)} U(P, \psi), \; S(z_i^0, M_i) = \partial S_1(0) \cap [\mathrm{con}(M_i - z_i^0)]^*.$$

The terminal set is the same as in the game (1).

Let $z^0 = (z_1^0, \ldots, z_n^0) \in R^{kn} \setminus M^*$ and assume that the inclusion (13) holds. We will show that with these pursuer control regions the point z^0 satisfies the condition (1) and the game (16) may be terminated from the initial state z^0 not later than in time $T(z^0)$ (see (7)).

To this end, let us return to the game (12). The set P is strictly convex, and therefore the multivalued mapping

$$U_i(v) = \{u_i \in \partial\, \mathrm{co}\, P : u_i \in v + \alpha_i(z_i^0, v)(M_i - z_i^0)\}$$

is single-valued and continuous, i.e., $U_i(v) = \{u_i(v)\}$. We will show that for any fixed $i \in \{1, \ldots, n\}$ and $v_0 \in V$, we have the inclusion $u_i(v_0) \in U_i(z_i^0, M_i)$. If $\alpha_i(z_i^0, v_0) = 0$, then $u_i(v_0) = v_0$. Repeating the argument used in the second part of the proof of Theorem 3, we conclude that $v_0 \in U_i(z_i^0, M_i)$.

Now consider the case when $\alpha_i(z_i^0, v_0) > 0$. The set $\alpha_i(z_i^0, v_0)(M_i - z_i^0) \cap (\mathrm{co}\, P - v_0)$ consists of a single point $u_i(v_0) - v_0$. Therefore,

$$c(\mathrm{co}\, P - v_0, \psi) = -\alpha_i(z_i^0, v_0)\, c(M_i - z_i^0, -\psi), \tag{17}$$

ψ is a supporting vector to the set $\mathrm{co}\, P - v_0$ at the point $u_i(v_0) - v_0$. From the inequality $c(\mathrm{co}\, P - v_0, \psi) \geq 0$ and the equality (17), we obtain $c(M_i - z_i^0, -\psi) \leq 0$, i.e., $u_i(v_0) \in U_i(z_i^0, M_i)$.

By Corollary 2, if the inclusion (13) holds, then the game (12) can be terminated from the initial state z^0 not later than in time $T(z^0)$, and we have previously shown that in the process of pursuit $u_i(\tau) \in U_i(z_i^0, M_i)$ for any $i \in \{1, \ldots, n\}$, $\tau \in [0, t^*)$, where t^* is the game termination time. We thus have the following proposition.

THEOREM 4. Let $z^0 = (z_1^0, \ldots, z_n^0) \in R^{kn} \setminus M^*$ and assume that the inclusion (13) holds. Then the game (16) can be terminated from the initial state z^0 not later than in time $T(z^0)$.

Example 3. Consider the game (16) with $k = 2$, $n = 3$, $M_i = S_1(0)$, $i = 1, 2, 3$, $P = S_1(0)$, $z_1^0 = (0, -2)$, $z_2^0 = (-1, \sqrt{3})$, $z_3^0 = (1, \sqrt{3})$. The control region of pursuer i is $\partial S_1(0) \cap [\mathrm{con}(S_1(0) - z_i^0)]^*$. It is easy to see that

$$[\mathrm{con}(S_1(0) - z_1^0)]^* = \{x = (x_1, x_2) : x_2 \geq \sqrt{3}\, |x_1|\},$$
$$[\mathrm{con}(S_1(0) - z_2^0)]^* = \{x = (x_1, x_2) : x_2 \leq \min\{0, \sqrt{3}x_1\},$$
$$[\mathrm{con}(S_1(0) - z_3^0)]^* = \{x = (x_1, x_2) : x_2 \leq \min\{0, -\sqrt{3}x_1\}.$$

By Theorem 4 we conclude that in this game the pursuit problem is solvable in a finite time.

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