NEW LIE-ALGEBRAIC STRUCTURES IN NONLINEAR PROBLEMS OF QUANTUM OPTICS AND LASER PHYSICS*

V. P. Karassiov

New Lie-algebraic structures (polynomial deformations of Lie algebras) are revealed in some problems of quantum optics and laser physics. Specifically, deformations of oscillator algebras due to extensions of unitary algebras by their symmetric and skew-symmetric tensors are shown to be algebras of dynamic symmetry (ADS) in models of n-photon processes with internal symmetries. Similarly, deformed algebras $su_d(2)$ are found as ADS in the context of generalized Dicke models and frequency conversion models. We also briefly discuss some possible schemes of employing the results to solving physical problems.

1. INTRODUCTION

As is known (see, e.g., [1-7] and references therein), the mathematical apparatus of Lie groups and algebras provides powerful techniques (Wigner—Racah algebras [1, 8], generalized coherent states (CGS) [3-4] etc.) for solving various quantum-optical and laser problems with Hamiltonians given by linear combinations of some Lie algebra generators or quadratic forms in the second quantization operators a_i^+, a_i (which are easily transformed in linear forms of Lie algebra generators with the help of the Jordan—Schwinger mapping [8, 9]). But many Hamiltonians of quantum optics and laser physics do not have such a simple form (see, e g., [1, 5, 7, 10-16]) and may be represented only by certain elements of universal enveloping algebras of some dynamic symmetry (DS) Lie algebras GDS [5, 7]. In general this makes direct use of Lie-algebraic techniques to solving physical problems less efficient in comparison with linear realizations of DS Lie algebras [5, 7].

In recent years, however, some new Lie-algebraic structures (quantum groups or algebras (see, e.g., review [17]), W-algebras [18], Casimir algebras [19, 20] etc.) have been introduced for solving various nonlinear physical problems. All these objects are nonlinear generalizations (deformations) of the usual (linear) Lie algebras which are generated by finite-dimensional sets $B = \{T_a\}$ of their generators T_a satisfying commutation relations (CR) of the form

$$[T_a, T_b] = f_{ab}(\{T_c\}) \tag{1.1}$$

where $f_{ab}(...)$ are some functions of the generators T_c given by power series. Furthermore, most of the deformed algebras G_d used in physics are obtained by the so-called coset construction [19, 20]

$$G_d = G_0 + V \tag{1.2}$$

as extensions of some familiar Lie algebras $G_0 = \{E_a\}$ by certain tensor operators $V = \{V_c\}$ satisfying CRs of the form

$$\text{a)}\ [E_a, V_b] = \Sigma\, \tau^c_{ab} V_c \tag{1.3a}$$

$$\text{b)}\ [V_a, V_b] = \phi_{ab}(E_c),\ V_c \in V,\ E_a \in G_0 \tag{1.3b}$$

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where Φ_{ab} are some polynomials in the generators of the subalgebra G_0 and $\tau_{ab}{}^{c}$ are matrix elements of tensor operators V. Specifically, the most popular quantum algebras $su_q(2)$ and $su_q(1,1)$ have a coset structure with the Cartan subalgebra $G_0 = u(1) = \{X_0\}$, the step generators $X_{\pm}$ as coset generators V_c, and trigonometric functions Φ_{ab} [17].

We note that until quite recently such deformed Lie algebras G_d were examined mainly in the context of quantum field theory and staistical physics models [17-19]. However, the results of recent papers [7, 21-28] show some possibilities of applications of deformed Lie algebras in other branches of modern physics. Specifically, in [24] a perturbative scheme was developed for applications of the quantum algebra $U_q(su(1,1))$ to anharmonic oscillator problems. In [25] the quadratic Hahn algebra has been shown to be a dynamic invariance algebra for an anisotropic singular oscillator. In [22, 23, 26, 27] quantum analogs of the Weyl—Heisenberg algebras h(m) as well as $su_q(2)$, $su_q(1,1)$ have been introduced in the context of quantum optics, providing generalizations of the well-known Jaynes—Cummings model and some schemes of analysis of multiphoton squeezing. But all these studies are in a sense sporadic and (except for [24, 25]) fairly artificial to be employed for directly examining realistic physical situations.

Meanwhile in [7, 21] we introduced a new class of deformed Lie algebras $I_m^{(-)}(n)$ which arise in a natural way in studies of composite many-body models with internal su(n) symmetries. Such algebras are generated by elementary su(n) vector invariants, made up of su(n) vector boson operators $a_i^+ = (a_{ia}^+)$ and $a_i = (a_{ia})$, and satisfying CR of the type (1.1) with f_{ab} being some polynomials in the generators $E_{ij} = \Sigma_a a_{ia}^+ a_{ja}$ of the Lie algebras u(m). Futhermore, algebras $I_m^{(-)}(n)$ and su(n) form generalized Howe's dual pairs [29] and act complementarily (cf. [7, 30]) on the appropriate Fock spaces, which facilitates spectral analysis of model state spaces L_m of the physical problems under consideration. Later, in [28] we showed that these deformed Lie algebras $G_d = I_m^{(-)}(n)$ are obtained by the coset construction (1.2)-(1.3) as extensions of the Lie algebra $u(m) = \{E_{ij}\} = G_0$ by its skew-symmetric tensor operators $V = \{V_c\} = \{[a_{i_1} a_{i_2} \ldots a_{i_n}] = \Sigma\, \epsilon_{\alpha_1 \ldots \alpha_n} a_{i_1 \alpha_1} \ldots a_{i_n \alpha_n} = X_{i_1 \ldots i_n},\ X^+_{i_1 \ldots i_n} = (X_{i_1 \ldots i_n})^+\}$, where $\epsilon_{\alpha_1 \ldots \alpha_n}$ is the fully antisymmetric tensor with $\epsilon_{12 \ldots n} = 1$.

These results may be generalized in an obvious manner for other groups $\hat{G}_{int}$ or Lie algebras G_{int} of "internal" physical symmetries. Indeed, we can determine finite sets $I^G{}_{inv}$ of elementary $\hat{G}_{int}$ (or G_{int}) invariants involved in second-quantized forms of Hamiltonians of physical systems under study. Then, selecting in $I_{inv}{}^G$ only quadratic invariants, we determine Lie subalgebras G_0 of complete DS algebras of the physical problems under consideration. After that, taking other (nonquadratic) invariants as coset generators V_c, we can construct via the above coset construction some finite-dimensional deformed Lie algebras as complete DS algebras of some nonlinear Hamiltonians. Since deformed Lie algebras G_d obtained in this manner retain some properties of familiar Lie algebras [7, 17, 20, 21], one can expect that a theory of G_d representations may be useful for treating various physical problems in cases where their Hamiltonians are given by linear forms in G_d generators.

The aim of this paper is to show the efficiency of the outlined approach [7, 21] in analyzing some essentially nonlinear (both semiclassical and completely quantum) problems of laser physics and quantum optics whose Hamiltonians H_a have, certain symmetries described by groups $\hat{G}_{int}(H_a)$ or Lie algebras $G_{int}(H_a)$. By a direct inspection of the H_a forms we determine $G_{int}(H_a)$ or $G_{int}(H_a)$ and then find the deformed Lie algebras $G_d(H_a)$ as DS algebras for each Hamiltonian H_a. Since all algebras $G_d(H_a)$ have the same structure (1.2)-(1.3), with G_0 being unitary algebras u(m), we describe their properties in some detail only for one of them, namely $G_d(H_1)$. For other algebras $G_d(H_a)$ we give basic sets $B(H_a)$ of the $G_d(H_a)$ generators obtained by using generalized Jordan—Schwinger mappings and point out some peculiarities of $G_d(H_a)$ as compared to $G_d(H_1)$. We also discuss decompositions of the model state spaces $L_m(H_a)$ with respect to common actions of dual pairs $(G_{int}(\hat{G}_{int}), G_d(H_a))$ as well as some ways of using deformed Lie algebras to solving physical problems.

2. DEFORMATIONS OF OSCILLATOR ALGEBRAS IN SEMICLASSICAL MODELS OF N-PHOTON PROCESSES WITH INTERNAL SYMMETRIES

Let us consider semiclassical quantum-optical models given by Hamiltonians

$$\text{a)}\quad H_1 = \sum_{i,j=1}^{m} \omega_{ij} a_i^+ a_j + \Sigma\,[g_{i_1 \ldots i_n} a^+_{i_1} \ldots a^+_{i_n} + g^*_{i_1 \ldots i_n} a_{i_1} \ldots a_{i_n}], \tag{2.1a}$$

$$\text{b)}\quad H_2 = \sum_{j,i=1}^{m} \omega_{ij} \sum_{\alpha=1}^{n} a^+_{i\alpha} a_{j\alpha} + \Sigma\,[f_{i_1 \ldots i_n} X^+_{i_1 \ldots i_n} + f^*_{i_1 \ldots i_n} X_{i_1 \ldots i_n}], \tag{2.1b}$$

where "g" and "f" are some constants or time-dependent functions; these Hamiltonians describe some n-photon parametric processes, in particular, creation and absorption of multiboson clusters in external nonquantized fields [3, 7, 14]; besides they are relevant to studies of multiphoton squeezing as well as "exotic" states of quantum light beams (see, e.g., [14, 16, 26, 27] and references therein).

2.1. The Hamiltonian H_1 is invariant with respect to the discrete group $\hat{G}_{int}(H_1) = C_n = \{c_{kn} = \exp(2\pi ik/n): a_i^+ \rightarrow c_{kn} a_i^+, k = 0, 1, \ldots, n-1\}$. The generalized Jordan—Schwinger mapping

$$E_c = E_{ij} = a_i^+ a_j,\ V = \{Y^+_{i_1 \ldots i_n} = a^+_{i_1} \ldots a^+_{i_n},\ Y_{i_1 \ldots i_n} = a_{i_1} \ldots a_{i_n}\} \tag{2.2}$$

yields the basic set $B(H_1) = \{E_{ij}, Y_{i_1} \ldots Y_{j_1}{}^+ \ldots\}$ where the generators E_{ij} satisfy standard CRs for u(m) while the coset generators $Y^+_{i_1 \ldots i_n}$, $Y_{i_1 \ldots i_n}$ form two Hermitian conjugate sets of the symmetric u(m)-tensors and obey CRs of the form (1.3)

$$\text{a) } [E_{rj}, Y^+_{i_1 \ldots i_n}] = \delta_{i_1 j} Y^+_{r i_2 \ldots i_n} + \ldots + \delta_{i_n j} Y^+_{i_1 \ldots i_{n-1} r}, \tag{2.3a}$$

$$\text{b) } [E_{rj}, Y_{i_1 \ldots i_n}] = -(\delta_{r i_1} Y_{j i_2 \ldots i_n} + \ldots + \delta_{r i_n} Y_{i_1 \ldots i_{n-1}}) \tag{2.3b}$$

$$\text{c) } [Y_{i_1 \ldots i_n}, Y^+_{j_1 \ldots j_n}] = P^m_{n-1}(\{E_{ij}\}) \tag{2.3c}$$

where $P_{n-1}{}^m(\ldots)$ are polynomials of the $(n-1)$-th order in E_{ij} whose explicit form follows from the canonical CRs for the boson operators a_i, a_j^+. Equations (2.3) together with standard CRs for u(m) define algebras $G_d(H_1)$ as polynomial deformations Osc(m; n) of the usual oscillator algebras (OA) Osc(m) = u(m) + h(m) since in the case n = 1 algebras Osc(m; 1) are reduced to Osc(m) (the label "n" indicates the kind of the coset generators as u(m)-symmetric tensors). We note that, besides Eqs. (2.3), generators E_{ij}, $Y_{i_1 \ldots i_n}$ and $Y^+_{i_1 \ldots i_n}$ of Osc(m; n) in their boson realization (2.2) satisfy some extra polynomial relations (syzygies) of the type

$$\text{a) } Y^+_{i\, i \ldots i} Y_{ii \ldots i} = E_{ii}(E_{ii}-1)(E_{ii}-2)\ldots(E_{ii}-n+1) = E^{(n)} \tag{2.4a}$$

$$\text{b) } Y^+_{i\, i_1 i_2 \ldots i_n} Y^+_{j\, j_1 j_2 \ldots j_n} = Y^+_{j\, i_1 i_2 \ldots i_n} Y^+_{i\, j_1 j_2 \ldots j_n} \text{ etc} \tag{2.4b}$$

which determine certain features of the Osc(m; n) representations on the Fock spaces.

For further clarification of structure properties of the algebras Osc(m; n), we consider in more detail the simplest example when m = 1. Introducing the notation $Y_0 = E_{11}/n$, $Y_+ = Y^+_{1 \ldots 1}$, $Y_- = Y_{1 \ldots 1}$, we find the explicit form

$$\begin{aligned} &[Y_-, Y_+] = P^1_{n-1}(Y_0) = \Delta g_n(Y_0),\ g_n(Y_0) = (nY_0)^{(n)}, \\ &A^{(B)} = A!/(A-B)!,\ \Delta f(x) = f(x+1) - f(x) \end{aligned} \tag{2.5}$$

for the CR (2.3c) that allows us to identify Osc(1; n) as a polynomial deformation $su^{(n)}(1,1)$ of the Lie algebra su(1,1) (since $su^{(2)}(1,1) = su(1,1)$). This algebra belongs to the class of so-called Casimir algebras whose construction is intimately related to deformations of the Casimir operators of familiar Lie algebras [19, 20]. Indeed, it is easy to check that the $su^{(n)}(1,1)$ Casimir operator $C_2^{(n)}(1,1)$ is given by the expression [28]

$$C_2^{(n)}(1,1) = -Y_+ Y_- + g_n(Y_0) \tag{2.6}$$

that is a deformation of the su(1,1) Casimir operator $C_2(1,1) = -E_+E_- + E_{(0)}{}^{(2)}$. This relationship of Casimir algebras with usual Lie algebras allows us to develop a theory of representations of the first ones by analogy with that of usual Lie algebras [20].

Specifically, using the realization (2.2) we can determine all irreducible representations (IRs) of $su^{(n)}(1,1)$ which act on the one-mode Fock space $L_F(1) = L_m(H_1) = \text{Span}\,\{|s\rangle = [s!]^{-1/2}(a^+)^s|0\rangle\}$. Namely, because of the identity (2.4a) with i = 1 we find

$$C_2(\mathrm{Osc}(1;n)) = C_2^{(n)}(1,1)_b = 0 \tag{2.7}$$

where the subscript "b" indicates the boson realization. From here it follows immediately that Osc(1; n) in its natural realization on $L_F(1)$ has only "n" IRs D(r) specified by the lowest weights $l(r) = r/n$, $r = 0, 1, \ldots, n-1$, and the lowest vectors

$$|l(r)\rangle = (a_1^+)^r|0\rangle \ , \ Y_0|l(r)\rangle = l(r)\ |l(r)\rangle \ , \ Y_-|l(r)\rangle = 0 \tag{2.8}$$

Furthermore, we have the decomposition

$$L_F(1) = \sum_{r=0}^{n-1} L(r) \tag{2.9}$$

of $L_F(1)$ into a direct sum of the IR D(r) carrier-spaces L(r). This result reflects the fact that on $L_F(1)$ we have the dual pair (Osc(1; n) = $su^{(n)}(1,1)$, $\hat{G}_{int} = C_n$). Naturally, for other realizations of $su^{(n)}(1,1)$ the class of its IRs may be richer as is the case for the usual su(1,1) [3, 4]. Without dwelling on all aspects of the Osc(1; n) representation theory we only write down the explicit expressions for the orthonormalized basic vectors $|s;r\rangle$, GCS $|\lambda;r\rangle_{Y_-}$ of the eigenfunction type [4] and Glauberlike GCS $|a;r\rangle_G$ for IRs D(r):

$$\text{a)}\ \ |s;r\rangle = [(r+ns)!]^{-1/2}(Y_+)^s|l(r)\rangle. \tag{2.10a}$$

$$\text{b)}\ \ |\lambda;r\rangle_{Y_-} = N \sum_{s=0}^{\infty} \lambda^s[(r+ns)^{(ns)}]^{-1/2}\ |s;r\rangle \ ,$$

$$Y_-|\lambda;r\rangle_{Y_-} = \lambda|\lambda;r\rangle_{Y_-} \, , \ N^{-2} = \sum_{s=0}^{\infty} |\lambda|^{2s}[(r+ns)^{(ns)}]^{-1} \, , \tag{2.10b}$$

$$\text{c)}\ \ |a;r\rangle_G = \exp[-|a|^2/2] \sum_s a^s\, [s!]^{-1/2}|s;r\rangle \tag{2.10c}$$

We note that GCS $|\lambda;r\rangle_{Y_-}$ and $|a;r\rangle_G$ are well determined by series (2.10b), (2.10c) as analytical vectors for all values "n" > 2 unlike the orbit type GCS $|a;r\rangle = \exp\,(aY_+ - a^*Y_-)\ |l(r)\rangle$ [31]; at the same time GCS $|a;r\rangle_G$ can be viewed as "smoothed" orbit type GCS

$$|a;r\rangle_s = \exp[a\, Y_+f(Y_0) - a^*f(Y_0)Y_-]\ |l(r)\rangle\ [r!]^{-1/2} = \exp[-|a|^2/2] \sum_s a^s\,[s!]^{-1/2}|s;r\rangle,$$

$$f(Y_0) = [(Y_0 + 1 - r/n)/(nY_0 + n)^{(n)}]^{1/2} \tag{2.11}$$

which coincide for r = 0 with so-called "fraction-photon" GCS [16, 32].

The above analysis is easily generalized for OAs Osc(m;n) with arbitrary "m." Indeed, the OA Osc(1; n) is a subalgebra of Osc(m; n) and its generators $Y_+ = (a_1^+)^n$, $Y_- = (a_1)^n$ are the only ones from the Chevalley basis [33] of Osc(m;n) which are additional to the Chevalley basis of u(m) by analogy with the usual OA Osc(m;1) = Osc(m). Therefore, starting from Osc(1;n) and using CRs (2.3a)-(2.3b), we can repeat "*mutatis mutandiis*" the main steps of the Osc(1;n) analysis for any OAs Osc(m;n).

2.2. Similarly, for the Hamiltonian H_2 we find $\hat{G}_{int}(H_2) = SU(n) = \{[u_{\alpha\beta}]: a^+_{i\alpha} \to \sum_\beta u_{\beta\alpha} a^+_{i\beta}\}$, $G_{int} = su(n)_{int} = \{\tilde{E}^{\alpha\beta} = E^{\alpha\beta} - \frac{1}{n}\delta_{\alpha\beta}\sum_{\alpha=1}^{n} E^{\alpha\alpha},\ E^{\alpha\beta} = \sum_{i=1}^{n} a^+_{i\alpha} a_{i\beta}\}$, and $B(H_2) = u(m) + X_+ + X_-$, where

$$u(m) = \{ E_{ij} = \sum_{\alpha=0}^{n} a^+_{i\alpha} a_{j\alpha} \, , \ i,j = 1, \ldots, m\},$$

$$X_+ = \{ X^+_{i_1 \ldots i_n} \} \, , \ X_- = \{ X_{i_1 \ldots i_n} = (X^+_{i_1 \ldots i_n})^+ \} \tag{2.12}$$

The coset generators $X^+_{i_1 \ldots i_n}$, $X_{i_1 \ldots i_n}$ in (2.12) are now the skew-symmetric u(m)-tensor operators which were defined in [7] (see also Sec. 1). They satisfy CRs (2.6) and (2.12) in [7] which are like Eqs. (2.3) [or (2.5) for m = n]. Therefore, as $G_d(H_2)$ we obtain another kind of polynomial deformations Osc(m; 1^n) of OAs due to extensions of u(m) by its skew-symmetric tensors (labelled by the symbol "1^n") [28].

The theory of OAs Osc(m; 1^n) and their IRs, reproducing many general structure properties of that for the Osc(m; n), has, however, certain peculiarities owing to the availability of nontrivial Lie algebras $G_{int}(H_2) = su(n)_{int}$ [7, 28]. Specifically, the simplest deformed OA Osc(n; 1^n) is equal to the direct sum su $(n)_{inv}$ + $su_{1n}(1,1)$ while the simplest OA Osc(1; n) is equal to the deformed algebra $su^{(n)}(1,1)$ only [here $su_{1n}(1,1)$ is formed by the generators $X^+_{1\ldots n}$, $X_{1\ldots n}$, $1/n\ \Sigma_{i=1}^n E_{ii}$, and $su(n)_{inv} = \{E'_{ij} = E_{ij} - \delta_{ij}\ 1/n\ \Sigma_{i=1}^n E_{ii}\} \subset u(n)$]; we also note that the right side of the analog of Eq (2.5) for $su_{1n}(1,1)$ contains the

$su(n)_{inv}$ Casimir operators as extra parameters. Further, the Fock space $L_F(mn) = \mathrm{Span}\{\Pi(a_{ia}^+)^{s_{ia}}|0\rangle\} = L_m(H_2)$ is decomposed into an infinite [unlike (2.9) for Osc(m;n)] sum

$$L_F(mn) = \sum_{\{p_i\}} L(p_1, \dots, p_{n-1}) \tag{2.13}$$

where $L(p_1, \dots, p_{n-1})$ are carrier-spaces of IRs of both $su(n)_{int}$ and Osc(m; 1^n) which are specified by the su(n) IR $D(\langle p_i \rangle)$ signature $(p_1, \dots, p_{n-1})$ [7, 21].

In order to elucidate these general remarks we consider in more detail the case m = n = 3. Then subalgebra $su_{13}(1,1) \subset$ Osc(3;1^3) is generated by the generators $X_+ = X^+_{123}$, $X_- = X_{123}$ and $X_0 = 1/3\ \Sigma_{i=1}^3 E_{ii}$ satisfying the CRs

$$\text{a)}\ [\, X_0 , X_\mp \,] = \mp X_\mp \tag{2.14a}$$

$$\text{b)}\ [\, X_- , X_+ \,] = \tau_2(X_0 ; \{C_k(3)_{inv}\}) = 3!(\, X_0 + 1\,) + 3\,(\, X_0 + 1)^{(2)} - \tfrac{1}{2}\, C_2(3) \tag{2.14b}$$

where $C_k(3)$ is the Casimir operator of the k-th order for the $su(3)_{inv}$. The algebra $su_{13}(1,1)$ belongs to the class of Casimir algebras and has the following expression

$$C_2(1,1;1^3) = -X_+X_- + (\, X_0 + 1\,)^{(3)} + 3\,(\, X_0 + 1\,)^{(2)} - \tfrac{1}{2} X_0\, C_2(3) \tag{2.15}$$

for its Casimir operator $C_2(1,1;1^3) = C_2(su_{13}(1,1))$. The generalization of the identity (2.4a) for $su_{13}(1,1)$ has the form [28]

$$X_+X_- = B_3^3(\{E_{ij}\}) = 2E_{33} + \begin{vmatrix} E_{11} & E_{13} \\ E_{31} & E_{33} \end{vmatrix} + \begin{vmatrix} E_{22} & E_{23} \\ E_{32} & E_{32} \end{vmatrix} + \begin{vmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{vmatrix} = \tag{2.16}$$

$$= (X_0 + 1)^{(3)} + 3(X_0 + 1)^{(2)} - X_0C_2(3)/2 - C_2(3) + C_3(3)/3$$

where products of operators in determinants are taken in an ordered (from the left on the right with respect to columns) form. Then, from (2.15) and (2.16) we find that

$$C_2(1,1;1^3) = C_2(3) - \tfrac{1}{3}\, C_3(3) \tag{2.17}$$

on the space $L_F(9)$ which is decomposed into the direct sum (2.13) of the $su(3)_{int}$ + Osc(3;1^3) invariant subspaces $L(p_1p_2)$ determined by the extremal vectors [7, 21]

$$|\mathrm{extr};\, p_1\, p_2\rangle = N\, (a^+_{11})^{p_1} \begin{vmatrix} a^+_{11} & a^+_{21} \\ a^+_{12} & a^+_{22} \end{vmatrix}^{p_2} |0\rangle \tag{2.18}$$

where N is a normalized constant. (A more complete review of the Osc(m;1^n) representation properties can be found in [7].)

3. POLYNOMIAL DEFORMATIONS OF SU(2) IN GENERALIZED DICKE MODELS AND IN FREQUENCY-CONVERSION PROCESSES

3.1. Generalizations of Hamiltonians (2.1) to the case of quantum sources of radiation (with "g" and "f" being operators) also lead us to deformed Lie algebras. Without dwelling here on this topic in detail, we restrict ourselves to considering only k-photon pointlike Dicke models given by Hamiltonians (see, e.g, [1, 5, 7, 11, 12])

$$\text{a)}\ H_3 = H_D = \omega\, a^+ a + \sum_{i=1}^{N} [\epsilon\, \sigma_0(i) + g\, \sigma_+(i)\, a^k + g^*\sigma_-(i)\, (a^+)^k\,], \tag{3.1a}$$

$$\text{b)}\ H_4 = H_{Db} = \sum_{i=1}^{2} \omega_i \sum_{\alpha} a^+_{i\alpha} a_{i\alpha} + \sum_{i=1}^{N} [\epsilon\sigma_0(i) + g\, \sigma_+(i)\, X_{12} + g^*\sigma_-(i)\, X^+_{12}] \tag{3.1b}$$

where $\sigma_a(i)$ are the Pauli matrices in their boson or fermion realization and $X^+_{12} = a^+_{1+}a^+_{2-} - a^+_{1-}a^+_{2+}$ is the creation operator of "polarizationally scalar" biphotons [21]. These Hamiltonians describe interactions of ensembles of two-level atoms (or molecules) with quantum one-mode (3.1a) or four-mode (3.1b) light fields. For the Hamiltonians H_3 and H_4 we have some new specific features of the algebras $G_d(H_3)$ and $G_d(H_4)$, namely, the right-hand sides of CR (1.3b) for them depend on generators of $G_{int}(H_a)$ as extra invariant operator parameters.

Indeed, we readily find that the algebra $G_{int}(H_3)$ (the group $\hat{G}_{int}(H_3)$ contains additionally the discrete subgroup S_N [1, 5, 7]) is generated by mutually commuting operators $R = S_0 + a^+a$ and $C_2(2) = 1/2\ (S_+S_- + S_-S_+) + S_0^2$, where $S_a = \Sigma_1\ \sigma_a(i)/2$ while the algebra $G_d(H_3)$ is formed by generators $V_0 = S_0$, $V_+ = S_+a^k$, and $V_- = S_-(a^+)^k$. The operators V_a satisfy the CRs

$$\text{a)}\ [V_0, V_\pm] = \pm V_\pm , \tag{3.2a}$$

$$\text{b)}\ [V_+, V_-] = \Delta_x [x^{(2)} - C_2(2)]\,[kR - kx + k]^{(k)}\Big|_{x=V_0} = \Delta_{V_0}\gamma^k(V_0) \tag{3.2b}$$

which allows us to identify the algebra $G_d(H_3)$ as a deformed $su_d{}^k(2)$ (since it is reduced to the familiar su(2) for $k = 0$) with the invariant operators R and $C_2(2)$ as extra parameters. The Casimir operators $C_2(2;k) = C_2(su_d{}^k(2))$ of this algebra has the form

$$C_2(2;k) = V_+ V_- + (V_0^{(2)} - C_2(2))\,(kR - kV_0 + k)^{(k)} = V_+ V_- + \gamma^k(V_0) \tag{3.3}$$

It is equal to zero identically on the space $L_m(H_3) = L_M \times L_F(1)$ (L_M = Span $\{\Pi_a | \pm\rangle(a)\}$ [1, 5]) because of the easily verifiable identity: $V_+ V_- = -\gamma^k(V_0)$ on the $L_M \times L_F$ states. From here it follows that on the space $L_m(H_3)$ the algebra $su_d{}^k(2)$ has only IRs with the highest weights $h = r$ (for $-j \le r \le j$) and $h = j$ (for $r \ge j$), where $j(j + 1) \in$ Spec $C_2(2)|L_M$, $r \in$ Spec R [28]. This result generalizes and makes more exact the analysis [12].

For the Hamiltonian H_4 we obtain in the same manner another deformation $su_d{}^b(2)$ of su(2) which differs from $su_d{}^k(2)$ by its explicit expression for the function $\gamma(V_0)$ in (3.2b). Namely, for $su_d{}^b(2)$ we have

$$\gamma^b(V_0) = [V_0^{(2)} - C_2(2)][(R' + 2 - V_0)_2 - C_2(1,1)] \tag{3.4}$$

where the invariant operators $R' = S_0 + 1/2\ \Sigma_{i=1}^2\ (a^+_{i-}a^+_{i+} + a^+_{i+}a_{i+})$ and $C_2(1,1) = -X_{12}^+X_{12} + [1 + 1/2\ \Sigma_{i=1}^2\ (a_{i-}a_{i-} + a_{i+}a_{i+})]^{(2)}$ together with $C_2(2)$ generate the algebra $G_{int}(H_4)$(see [7] where a spectral analysis of $L_m(H_4)$ was also given with respect to $G_{int}(H_4)$).

3.2. Similar deformations $su_d(2)$ are also obtained for the Hamiltonians

$$\text{a)}\ H_5 = \omega a_1^+ a_1 + 2\omega a_2^+ a_2 + g\,(a_1^+)^2 a_2 + g^*\,(a_1)^2 a_2^+ , \tag{3.5a}$$

$$\text{b)}\ H_6 = \sum_{i=1}^{3} \omega_i a_i^+ a_i + g\ a_1^+ a_2^+ a_3 + g^*\ a_1 a_2 a_3^+ \tag{3.5b}$$

which describe, at the quantum level, processes of the second harmonic generation (H_5) [10, 13] and of the frequency up- and down-conversion (H_6) [10, 15].

Indeed, we easily determine $G_{int}(H_4)$ = Span$\{R_0 = a_1^+a_1 + 2a_2^+a_2\}$, $B(H_4) = \{V_0 = a_1^+a_1 + a_2^+a_2$, $V_+ = (a_1^+)^2a_2$, $V_- = (a_1)^2a_2^+\}$ and $G_{int}(H_5)$ = Span$\{R_1 = a_1^+a_1 + a_2^+a_2$, $R_2 = a_2^+a_2 + a_3^+a_3\}$, $B(H_5) = \{V_0 = a_1^+a_1 + a_2^+a_2 + a_3^+a_3$, $V_+ = a_1^+a_2^+a_3$, $V_- = a_1a_2a_3^+\}$. Then, by direct calculations we find that in both cases operators V_a satisfy CRs of the form (3.2) but with functions $\gamma(V_0)$ other than in (3.2b), namely, we have

$$\gamma(V_0) = \gamma_1(V_0) = (V_0 - R_0 - 1)(2V_0 - R_0)^{(2)} \tag{3.6a}$$

for the Hamiltonian H_5 and

$$\gamma(V_0) = \gamma_2(V_0) = (V_0 - R_1)(V_0 - R_2)(V_0 - R_1 - R_2 - 1) \tag{3.6b}$$

for the Hamiltonian H_6. We note that both functions $\gamma_i(V_0)$, i = 1, 2, do not contain any deformation parameters like "k" in (3.2) and (3.3). From the expressions (3.6) for $\gamma_i(V_0)$ we also find admissible highest v_0^M and lowest v_0^m weights of the $su_d(2)$ IRs on L_m in both cases, namely, $v_0^M = r_0$, $v_0^m = r_0 - [|r_0/2|]$ for H_4 and $v_0^M = r_1 + r_2$, $v_0^m = \max(r_1, r_2)$ for H_5, where $r_i \in$ Spec R_i, $[|x|]$ is the integral part of "x".

4. CONCLUSION

In conclusion we outline some possibilities of exploiting the results to solving physical problems governed by the above Hamiltonians H_a.

First of all we note that the above G_d-invariant decompositions of the model state spaces $L_m(H_a)$ [see, e.g., (2.13)] allow us to examine the dynamics of the systems under study independently of each G_d-invariant subspace. But for lack of simple formulas for disentangling exponents of the G_d elements [5, 8], one cannot apply their orbit GCS technique for diagonalizing H_a or for finding appropriate evolution operators by analogy with the case of familiar Lie algebras [3, 5, 7]. However, we can seek eigenvectors $|h\rangle$ of H_a in the form of expansions $|h\rangle = \Sigma_v A_v(h)|v\rangle$, where $|v\rangle$ are orthonormalized basic vectors of appropriate IRs of $G_d(H_a)$ adapted to the decomposition (1.2) and coefficients $A_v(h)$ are determined by solving some recurrence relations. For usual Lie algebras su(1,1) and su(2) in this manner one finds expressions for $A_v(h)$ in terms of classical orthogonal polynomials [34]. Therfore one can expect that for algebras G_d this method will lead to finding some new classes of orthogonal polynomials that are adequate for solving essentially nonlinear physical problems.

Another way of employing algebras G_d in solving physical problems is related to generalizations of the Holstein—Primakoff mapping [35]. Specifically, following the general approaches [36, 37] we can obtain mappings O: $G_d \to G = O(G_d)$ of $G_d(H_a)$ into familiar Lie algebras G. Then, substituting images $O(F_a)$ of the G_d generators F_a into Eqs. (2.1), (3.1), (3.5) we find "distorted" (effective) exactly solvable (with the help of Lie-algebraic techniques [3, 4]) Hamiltonians $H_a(O(F_b)) = H_a^h$ which give harmonic approximations for H_a. Using the obtained "distorted" Hamiltonians H_a^h as effective ones in original physical problems, we can obtain harmonic approximations to exact solutions of the latter ones. These approximations may describe occurrences of some coherent structures in nonlinear problems (cf. [38]) which become masked in exact solutions. Higher corrections to them may be found with the aid of an algebraic perturbative scheme based on transformation properties of Hamiltonians H_a and H_a^h with respect to algebras G_d and $O(G_d)$ (cf. [24, 12]). In this way, using as the $O(G_d)$ "limit" for G_d Lie algebras G (e.g., su(2) for $su_d(2)$), we can display some peculiarities of nonlinear physical problems in comparison with their linear prototypes (cf [24]). It is also of interest to exarnine the orbit-type GCS of algebras $O(G_d)$, which may be viewed as "smoothed" GCS of G_d [cf. (2.11)], and display certain interesting physical properties (see, e.g., [16, 26, 27]).

A more detail report on development of these topics will be given in forthcoming papers (see, in particular, issue No. 4 of this journal, which is based on materials of the International Workshop "Squeezing, Groups, and Quantum Mechanics" (Baku, September 16-21, 1991)).

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