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DISTRIBUTION OF A NONSTATIONARY ELECTRON BEAM IN A DENSE GAS

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The problem of the temporal and spatial dependences of the parameters of the action of a modulated fast-electron beam on a dense gas is posed on the basis of the transport equation. The problem is simplified by making it nondimensional and by transforming to the Fokker–Planck approximation. A Green's function formalism is developed for this problem and is used to express the solution of the general nonstationary problem in the form of a convolution of a nonstationary boundary flow with a stationary Green's function. The use of the derived equation is illustrated using as an example the solution of a problem with the simplest stationary Green's function corresponding to the "straight-ahead" approximation. This approximation is used to consider a general relativistic case with model scattering cross sections. The methods and results of a numerical computer solution of the nonstationary problem of electron retardation in the upper layer of the atmosphere are surveyed.

INTRODUCTION

One of the most universal methods of pumping visible band lasers with large energy output is assumed to be to supply energy to the medium from some external hard ionizer [1, 2]. In laser experiments, such an ionizer is frequently a high-power beam with initial electron energy V in the range 0.1-2 MeV and with current density $j = (0.1\text{-}10)$ kA/cm^2. These beams carry larger currents than those used to stabilize the discharge in electron-beam pumped lasers, but in the class of relativistic beams [3, 4] they are regarded as weak. The classification is based on the ratio of the energy of the intrinsic magnetic moment of the beam-current field to the electron kinetic energy. A beam is regarded as weak if the ratio $\nu/\gamma \ll 1$, where $\nu = Ir_0/ev$, I is the beam current, $r_0 = 2.818 \cdot 10^{-13}$ cm is the classical radius of the electron, e is the electron charge, v is the electron-beam velocity, and $\gamma = [1 - (v/c)^2]^{-1/2}$ is the relativistic factor. For I = 1 kA and V = 100 keV we have the ratio $\nu/\gamma \approx 0.06$; $\nu/\gamma = 1$ for subrelativistic beams at $I \approx 7$ kA.

We consider below beams with $\nu/\gamma \ll 1$, whose retardation is determined according to [3] by independent retardation of each electron, and collective effects play no role. Note that laser pumping by beams with $\nu/\gamma > 1$, whose retardation is determined by collective effects, is a moot question, since bulk ionization of the gas by the beam electrons is produced only by passage of a low-current beam. When beams with $\nu/\gamma \gg 1$ pass through a gas, the principal gas-ionization mechanism is that of breakdown on the front of the beam in the electric field induced by the rapid change of the current's magnetic field [3].

Supply of energy to a gas by bulk ionization with an electron beam offers a number of advantages. First, the beam energy goes mostly to production of plasma electrons rather than to heating of the plasma electrons, ions, and the gas. Second, in such pumping there

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is no feedback of the plasma parameters to the energy source, and hence no restrictions imposed by charge contraction, arc ignition, etc. Interest attaches also to the possibility of obtaining modulated laser emission by pumping the medium with a modulated electron beam [5]. It is just the modulation of the electron flux entering the medium which makes the problem considered below nonstationary. The spatial inhomogeneity of the problem, on the other hand, is due to the deceleration of the beam electrons by collision with the gas. The need for considering the spatial distribution of the pump over the entire beam-deceleration length follows from the following considerations: from the standpoint of obtaining maximum efficiency of conversion of the beam-electron energy into light, it is convenient to choose the length of the active zone of the laser (or its width in the case of transverse pumping) of the same order as the total beam-deceleration length. In which case the active zone can contain regions with negative gain, and to reveal these regions the spatial distribution of the pump must be known.

The greatest difficulty in the analysis of the deceleration is raised by allowance for the bending of the particle trajectories as a result of multiple scattering. No exact analytic solution of the corresponding transport equation has been obtained so far even in the simplest case of deceleration only by elastic scattering, without energy loss. What has been developed by now in sufficient detail is the so-called small-angle approximation, which is valid when the total deflection angle from the initial particle direction is small over the entire deceleration length [6, 7]. The authors of [6] present in their book valuable analytic results of the solution of the problem of passage of fast charged particles through solids. These results are quite accurate when the bending of the trajectories is small. It must be noted, to be sure, that the equations obtained in [6] are quite unwieldy and in the most general case the solution is represented in the form of the Mellin's formula for obtaining the inverse Laplace transform. It can nonetheless be stated that the mathematical formalism developed in [6] is quite adequate for the complete solution of problems in which the particle deceleration length l is much shorter than the scattering transport length l_{tr}:

$$l/l_{tr} \ll 1. \qquad (0)$$

The transport length is connected with the mean squared deviation of the particle in the course of scattering and determines in fact the length over which complete isotropization of the initially unidirectional flux takes place.

Relation (0) does not hold for electron deceleration in a gas. In fact, $l \sim (V/W)/N\sigma_r(V)$, where W is the average energy lost in one inelastic-scattering act, N is the gas density, $\sigma_r(V)$ is the total inelastic-scattering cross section; $l_{tr} \sim 1/N\sigma_{tr}(V)$, where σ_{tr} is the scattering transport cross section. If the electron is decelerated in helium, which is frequently used as a buffer gas in laser mixtures, we have at $V \sim 100$ keV, $W \sim 3$Ry, $\sigma_r[\pi a_0^2] \approx 16/V[\text{Ry}]$, $\sigma_{tr} \approx 43/(V[\text{Ry}])^2$ so that $l/l_{tr} \approx 0.9$ and is practically independent of V in the considered nonrelativistic case. In the relativistic case, the scattering cross sections tend with increasing V to constant values and l/l_{tr} increases in proportion to V. The small-angle approximation is thus all the more inapplicable to the case of deceleration of relativistic electrons in a gas. Electron deceleration can be considered with the aid of the small-angle approximation only if their penetration depth into the gas is small compared with the total deceleration length so that the electron energy losses are low. This case, however, is of little interest from the standpoint of obtaining maximum electron-energy conversion.

Problems closest to those considered above are encountered in the physics of atmospheric plasma [8-10]. The electrons produced when the neutral atmosphere is ionized by solar radiation or by particle fluxes of magnetospheric origin undergo many collisions before they slow down to thermal energies, causing thus ionization, excitation, and heating of various components of the upper atmosphere, ionization exchange between magnetically conjugate points of the ionosphere, etc. Calculation of the effectiveness of these processes has much in common with the calculation of processes that take place in the active media of plasma lasers. As a rule, the calculations of the kinetics of atmospheric plasma are based on computer solution of the corresponding kinetic equations by finite-difference methods or by the Monte Carlo method [11]. Unfortunately, the results of these studies do not yield the relations needed for specific laser mixtures. Moreover, only stationary deceleration is considered in all these papers.

We report below a nonstationary formulation, based on the transport equation, of the problem of determining the temporal and spatial dependences of the parameters of the action of a modulated fast-electron beam on a gas. The problem is simplified by nondimensionalization and by transforming to the Fokker–Planck approximation. A Green's-function formalism is developed for this problem and is used to express the solution of the general nonstationary problem in the form of a convolution of the nonstationary boundary flux with a stationary Green's function. In other words, an exact analytic solution with respect to the time variable is obtained and the general nonstationary problem is reduced in fact to finding the solution of a special stationary problem. By way of example of using the equations obtained, we solve the problem with the simplest Green's function of the stationary problem corresponding to the "straight ahead" approximation [6]. We consider in this approximation a general relativistic case with model scattering cross sections. The methods and results of a computer solution of the stationary problem of electron deceleration in a gas are surveyed.

1. Formulation of the Problem

We consider first briefly the character of formation of the energy distribution of the electrons when the gas is ionized by a hard ionizer, in this case an electron beam. On the energy axis, along which the electrons of the produced plasma are degraded, one can separate several intervals with corresponding electron groups [1]. The first group comprises the beam electrons. These are decelerated in the gas, thereby exciting and ionizing it, and their trajectories are bent also on account of the elastic collisions. The knocked-out secondary electrons of energy E, higher than the minimum gas-excitation energy (and also the threshold inelastic-interaction energy), make up the second group of the ionization-cascade electrons. The below-threshold electrons, whose energy is much higher than the gas temperature, belong to the third group. They are cooled mainly by elastic collisions with the gas. The fourth and most numerous group of plasma electrons includes the cooled slow electrons whose temperature is of the order of that of the gas. When these electrons collide with one another they acquire a Maxwellian distribution and determine the recombination flux. The electron energies are thus in the range 1-1000 eV in the first group, 10-1000 eV in the ionization cascade, 1-10 eV in the subthreshold region, and 0.05-1 eV in the plasma. Note that such a sharp demarcation of electron groups is possible only in the case of a simple atomic gas with a threshold excitation energy much higher than the temperature. In mixtures with substantially different component densities, this grouping of electrons becomes arbitrary, since, for example, electrons can be effectively cooled by both elastic and inelastic collisions in one and the same energy region.

The character of the approximations and the mathematical methods used to determine the energy distributions of the electrons of the different groups are essentially different. We consider in this paper only electrons of the first group, the beam electrons, for it is just their deceleration which makes the obtained plasma spatially inhomogeneous. The formation of the electrons of the remaining groups can be treated in the spatially local approximation, since the mean free path of these electrons is much shorter than that of the beam electrons.

Assume that a modulated electron beam with flux density $j(t, E, \mu)$ is introduced into a gas occupying the half-space $z > 0$. Here t is the time, E the energy, and $\theta = \cos \mu$, where θ is the angle between the electron-velocity vector and the z axis. Let the entering electron beam be wide enough, with width much larger than the deceleration length or let a strong external magnetic field be applied in the beam propagation direction along the z axis. Both assumptions lead to the same mathematical formulation of the problem. In fact, consider the equation for beam-electron transport in an external magnetic field $\mathbf{H} = (0, 0, H)$

$$\frac{\partial f}{\partial t} + (\mathbf{v}, \nabla f) + \frac{e}{mc}\left([\mathbf{v}, \mathbf{H}], \frac{\partial f}{\partial \mathbf{v}}\right) = I_{st}. \tag{1.1}$$

Here f is the beam-electron distribution function, e and m the electron charge and mass, c the speed of light, and I_{st} the collision integral. The action of the magnetic field on the electron is described by the third term of the left-hand side of (1.1). In a cylindrical coordinate system this term can be easily transformed, with allowance for the form of H, into

$$\frac{e}{mc}\left([\mathbf{v}, \mathbf{H}], \frac{\partial f}{\partial \mathbf{v}}\right) = -\frac{eH}{mc}\frac{\partial f}{\partial \varphi},$$

where φ is the polar angle in velocity space. It follows from this form that an external magnetic field simply twists the electron trajectory into a helix and does not influence the electron energy or its velocity components v_z and $v_\perp$. The radius R of the helix can be easily expressed in terms of the perpendicular component $v_\perp$ of the electron-velocity vector and of the strength of the magnetic field H: $R = v_\perp mc/eH$. Collisions of the beam electron with the gas change the value of $v_\perp$ and ultimately increase the radius of the helix with increasing depth of electron penetration into the gas. Thus, the true trajectories of the electron in the external magnetic field are easily found with the aid of the obtained values of the distribution functions by solving the problem of the deceleration of an infinitely wide beam without taking the external magnetic field into account. We shall therefore consider hereafter Eq. (1.1) without the third term in the left-hand side, and assume the beam to be infinitely wide. This simplifies Eq. (1.1). We rewrite this equation with a specified form of the collision integral:

$$\frac{1}{v}\frac{\partial \Phi}{\partial t}+\mu\frac{\partial \Phi}{\partial z}=-N\sigma_t(E)\Phi+N\int_{v'}d^3v'\sigma(\mathbf{v}'\to\mathbf{v})\Phi(\mathbf{v}'). \tag{1.2}$$

Here $\Phi(t, z, E, \mu) \equiv vf$, $\sigma_t(E)$ is the total scattering cross section, and $\sigma(\mathbf{v}' \to \mathbf{v})$ is the differential cross section for scattering the beam electrons by the gas particles. Equation (1.2) is integrodifferential and difficult to solve even with a computer, in view of the large number of independent variables. However, the specific features of the interaction of the beam electrons with the gas allows us to simplify this equation and reduce it to a differential equation of the Fokker–Planck type [10, 6]

$$\frac{1}{v}\frac{\partial \Phi}{\partial t}+\mu\frac{\partial \Phi}{\partial z}=\frac{\partial}{\partial E}[L(E)\Phi]+Q(E)\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial \Phi}{\partial \mu}. \tag{1.3}$$

Here $L(E) = -dE/dz$ is the electron stopping power and $Q(E) = \langle\theta_s^2\rangle/4$, $\langle\theta_s^2\rangle$ is the mean squared angle of electron deflection by the collision. We can write approximately $L(E) \approx W/l_c$, where l_c is the electron mean free path; $Q(E) \approx 1/(2l_{tr})$.

Equation (1.2) can be transformed into (1.3) because the energy loss and the deflection angle of the beam electron are small in each individual act of scattering from the gas particles. Indeed, in each collision the beam electron loses on the average an energy ΔE of the order of the electron–ion pair-production energy W, which is approximately equal to 2I, or double the gas-particle ionization potential; the deflection angle is here $\Delta\theta \sim \sqrt{I/E}$. Even at the very end of the acceleration, when the beam-electron energy drops to 1 keV, we have $\Delta E/E \sim 0.05$; $\Delta\theta \sim 0.1$. The transformation of Eq. (1.2) into (1.3) is effected by expanding the function Φ in the collision integral in powers of the small ratios $\Delta E/E$ and $\Delta\theta$. Electron collisions with gas particles, accompanied by transfer of high energies and by deflection through large angles, are rare and are effectively taken into account by the corresponding term in the expression for L(E). Besides the simplification of the form of the equation, another factor favors the change from (1.2) to (1.3). The differential cross section $\sigma(\mathbf{v}' \to \mathbf{v})$ and particularly its angular part, is known with lower accuracy than the angle-averaged quantities L(E) and Q(E). It is therefore not clear beforehand which of the equations, (1.2) or (1.3), can provide a more accurate description of the electron deceleration in the gas.

To complete the formulation of the problem we must add to (1.3) an initial and a boundary condition. We assume that the gas was not ionized prior to the entry of the electron beam

$$\Phi(t=0, z, E, \mu)=0. \tag{1.4}$$

The boundary condition is set by the flux entering through the cross section $z = 0$

$$\Phi(t, z=0, E, \mu)=j(t, E, \mu). \tag{1.5a}$$

Mathematically speaking, Eqs. (1.3)-(1.5a) constitute a mixed problem or a Cauchy problem on a semiinfinite straight line for the transport equation (1.3). Note that the flux j is specified on the boundary $z = 0$ only for $\mu > 0$, i.e., only the forward flux. The backward flux should be obtained during the solution of the problem. It is clear beforehand that at depths exceeding the deceleration length the sought solution is zero; we consider therefore hereafter the problem in a planar layer of thickness z_l smaller than or equal to the deceleration length. The boundary condition for $z = z_l$ is that the backward flux be zero

$$\Phi(t, z = z_l, E, \mu < 0) = 0. \tag{1.5b}$$

The end purpose of the problem of the electron-beam deceleration is to obtain the frequencies ν_γ of the ionization and excitation of the gas particles by the primary electrons

$$\nu_\gamma(t, z) = \int\limits_{E^* \sim 1 \text{ кэВ}}^{V} \sigma_\gamma(E) \int\limits_{-1}^{1} \Phi(t, z, E, \mu)\, d\mu\, dE, \tag{1.6}$$

where $\sigma_\gamma(E)$ is the cross section for the excitation of gas particles to state γ, including an ionized gas. We wish also to determine the integral

$$S(t, z, E) = \int\limits_{E^*+I}^{V} \sigma(E' \to E) \int\limits_{-1}^{1} \Phi(t, z, E', \mu)\, d\mu\, dE', \tag{1.7}$$

that enters as the source in the integral equation for the degradation spectrum that determines the energy distribution of the secondary electrons. The frequencies ν_γ summed with the corresponding excitation and ionization frequencies of the gas particles by the secondary electrons are used next to solve the spatially local problem of the physicochemical kinetics of the laser active medium [1].

2. Derivation of Green's Function of the Nonstationary Problem and Reduction of the Initial Problem to a Nonstationary One

2.1. Nondimensionalization. To solve Eqs. (1.3)-(1.5) it is convenient to replace the energy E by a new variable and make the problem nondimensional [6]. We consider in place of the energy the residual range

$$r(E) = \int\limits_0^E \frac{dE'}{L(E')}, \tag{2.1}$$

where r(E) is the path that must be covered by an electron of energy E. We note that since the trajectory is bent by multiple scattering, the path of the electron differs from its penetration depth. The natural dimensionless units for the problem are the range $r_0 = r(V)$ for the electron, having as it enters the gas the maximum energy V required in the problem to make z and r dimensionless and also $t_0 = r_0/v_0$, where $v_0 = v(V)$ is the velocity in the problem. We shall use t, z, and r to denote the dimensionless variables. In addition, all the quantities of the problem such as energy, velocity, flux, density, and frequencies ν_γ are assumed to be the dimensionless values of these quantities at the entry z = 0 into the gas.

We introduce a new distribution function ψ of the dimensionless variables, such that $\psi dr = \Phi dE$, i.e.,

$$\psi(t, z, r, \mu) = r_0 L(E)\Phi(t, z, E, \mu).$$

Equations (1.3)-(1.5) take now in dimensionless variables the form

$$\frac{1}{v(r)}\frac{\partial\psi}{\partial t} + \mu\frac{\partial\psi}{\partial z} = \frac{\partial\psi}{\partial r} + Q(r)\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial\psi}{\partial\mu}, \tag{2.2}$$

$$\psi(t = 0, z, r, \mu) = 0, \tag{2.3}$$

$$\psi(t, z = 0, r, \mu) = J(t, r, \mu), \tag{2.4}$$

where

$$v(r) \equiv v(E(r)),\ Q(r) \equiv r_0 Q(E(r)),\ J(t, r, \mu) = r_0 L(E(r))\, j(t, E(r), \mu).$$

Equations (1.6) and (1.7) are then transformed into

$$\nu_\gamma(t, z) = \int\limits_{r^* = r(E^*)}^{1} \sigma_\gamma(E(r)) \int\limits_{-1}^{1} \psi(t, z, r, \mu)\, d\mu\, dr, \tag{2.5}$$

$$S(t, z, E) = \int_{r^*}^{1} \sigma(E(r') \to E) \int_{-1}^{1} \psi(t, z, r', \mu)\, d\mu\, dr'. \tag{2.6}$$

We examine now the range of the variables of Eqs. (2.2)-(2.4). After nondimensionalization, the variables z and r range from 0 to 1. It is easily noted, however, that the function ψ is zero at $r > 1 - z$. Indeed, at the entrance to the gas, at $z = 0$, the most energetic electron has a residual range $r = 1$. When this electron reaches the depth z, it has covered a path longer (in view of the bending of the trajectory) or equal to z, meaning that it has a residual range $r \leq 1 - z$. Generally speaking the function $\psi(t, z, r, \mu)$ is discontinuous on the line $r = 1 - z$ and the derivatives in Eqs. (2.2) in the square $0 < z < 1$, $0 < r < 1$ are generalized. In other words, in this region Eq. (2.2) is valid not for $\psi(t, z, r, \mu)$, but for the product $\psi(t, z, r, \mu)\, \eta(1 - r - z)$, where η is the unit step function.

We consider now the range of the time variable t. The dimensionless time t ranges from 0 to T, where

$$T = \frac{1}{t_0} \int_0^V \frac{dE}{v(E) L(E)}$$

is the dimensionless deceleration time of the most energetic electron from its entry into the gas to a complete rest. Note that $T > 1$, for unit dimensionless time was taken to be $t_0 = r_0/v(V)$ and the average deceleration rate is not less than v(V).

The variable μ ranges from -1 to 1 in the problem but only from 0 to 1 in the boundary condition (2.4).

It is thus required to solve Eqs. (2.2)-(2.4) in the closed region $\{0 \leqslant t \leqslant T,\ 0 \leqslant z \leqslant 1,\ r \leqslant 1 - z,\ -1 \leqslant \mu \leqslant 1\}$.

2.2. Green's-Function Formalism. We designate the Green's function of Eqs. (2.2)-(2.4) by $\mathcal{G}(t, z, r, r', \mu, \mu')$. It satisfies in the region $\{0 < t < T,\ 0 < z < r',\ r < r' - z,\ 0 < r' < 1,\ -1 \leqslant \mu \leqslant 1,\ 0 \leqslant \mu' \leqslant 1\}$ Eq. (2.2), the zero initial condition, and the special boundary condition

$$\frac{1}{v(r)} \frac{\partial \mathcal{G}}{\partial t} + \mu \frac{\partial \mathcal{G}}{\partial z} = \frac{\partial \mathcal{G}}{\partial r} + Q(r) \frac{\partial}{\partial \mu} (1 - \mu^2) \frac{\partial \mathcal{G}}{\partial \mu}, \tag{2.7a}$$

$$\mathcal{G}(t = 0, z, r, r', \mu, \mu') = 0, \tag{2.7b}$$

$$\mathcal{G}(t, z = 0, r, r', \mu, \mu') = \delta(t)\, \delta(r' - r)\, \delta(\mu' - \mu), \tag{2.7c}$$

where δ is the Dirac delta function.

The function $\mathcal{G}(t, z, r, r', \mu, \mu')$ describes the deceleration of an electron entering the gas at the instant $t = 0$ with a residual range r' and at an angle $\theta' = \arccos \mu'$ to the z axis. Since the electron has at depth z a residual range r, the function $\mathcal{G}(t, z, r, r', \mu, \mu') = 0$ for all $r' < r + z$ but has a discontinuity on the line $r' = r + z$, $0 < r' < 1$. The solution of the Eqs. (2.2)-(2.4) is expressed with the aid of the Green's function $\mathcal{G}$ as follows:

$$\psi(t, z, r, \mu) = \int_0^t dt' \int_0^1 dr' \int_0^1 d\mu' \mathcal{G}(t - t', z, r, r', \mu, \mu')\, J(t', r', \mu'). \tag{2.8}$$

This can be verified directly.

It turns out that the function $\mathcal{G}(t, z, r, r', \mu, \mu')$ can be expressed in terms of the Green's function G(z, r, r', μ, μ') of the stationary problem by separating the time dependence and carrying out analytic integration in (2.8) with respect to the variable t'. We formulate for the stationary function G the following equation

$$\mu \frac{\partial G}{\partial z} = \frac{\partial G}{\partial r} + Q(r) \frac{\partial}{\partial \mu} (1 - \mu^2) \frac{\partial G}{\partial \mu}, \tag{2.9a}$$

$$G(z = 0, r, r', \mu, \mu') = \delta(r' - r)\, \delta(\mu' - \mu) \tag{2.9b}$$

$$\{0 \leqslant z < r',\ r < r' - z,\ 0 < r' < 1,\ -1 \leqslant \mu \leqslant 1,\ 0 \leqslant \mu' \leqslant 1\}. \tag{2.9c}$$

Just as the function $\mathscr{G}$, the function $G(z, r, r', \mu, \mu') = 0$, for all $r' < r + z$. Note that (2.9) is considerably simpler than (2.7), since it has fewer variables. This is a very important circumstance, since no analytic solutions have been found for (2.7) and (2.9), and it is much easier and "cheaper" to solve the stationary equation (2.9), from the standpoint of computer time, than the nonstationary (2.7) or the initial equations (2.2)-(2.4).

To express $\mathscr{G}$ in terms of G, we take the Laplace transform of (2.7)

$$\mathscr{G}(t, z, r, r', \mu, \mu') \overset{\mathscr{L}}{\longleftrightarrow} F(p, z, r, r', \mu, \mu').$$

To obtain the transforms F we must solve the following equation

$$\frac{p}{v(r)} F + \mu \frac{\partial F}{\partial z} = \frac{\partial F}{\partial r_i} + Q(r) \frac{\partial}{\partial \mu}(1 - \mu^2) \frac{\partial F}{\partial \mu}, \tag{2.10a}$$

$$F(p, z = 0, r, r', \mu, \mu') = \delta(r' - r)\,\delta(\mu' - \mu), \tag{2.10b}$$

where the variables change within the limits (2.9c).

We seek the solution of (2.10) in the form

$$F(p, z, r, r', \mu, \mu') = e^{\varphi(r, p)} G(z, r, r', \mu, \mu').$$

From (2.10a) we obtain for φ

$$d\varphi/dr = p/v(r).$$

Hence

$$\varphi(r, p) = p \int_c^r \frac{dx}{v(x)}, \tag{2.11}$$

where c refers to the integration constant. From (2.10b) we obtain, taking (2.9b) into account

$$e^{\varphi(r=r', p)} \delta(r - r')\,\delta(\mu - \mu') = \delta(r - r')\,\delta(\mu - \mu').$$

Hence $\varphi(r = r', p) = 0$ and $c = r'$ in (2.11). Thus,

$$F(p, z, r, r', \mu, \mu') = e^{-p\tau(r, r')} G(z, r, r', \mu, \mu').$$

We have introduced here a function of two variables

$$\tau(r, r') = \int_r^{r'} \frac{dx}{v(x)}. \tag{2.12}$$

This function has the physical meaning of the time that the electron requires to negotiate the path $r' - r$. Note that $\tau(0, 1) = T$ is the upper limit of the variation of t.

Taking the inverse transform of F, we obtain with the aid of the retardation formula

$$\mathscr{G}(t, z, r, r', \mu, \mu') = \delta(t - \tau(r, r')) G(z, r, r', \mu, \mu')\,\eta(t - \tau). \tag{2.13}$$

Substituting expression (2.13) for $\mathscr{G}$ in the solution (2.8) of the initial problem and integrating with respect to t', we obtain ultimately

$$\psi(t, z, r, \mu) = \int_{r+z}^{1} dr' \int_0^1 d\mu' G(z, r, r', \mu, \mu')\, J(t - \tau(r, r'), r', \mu')\,\eta(t - \tau). \tag{2.14}$$

The lower limit of the integration with respect to r' in (2.14) takes into account the fact that $G(z, r, r', \mu, \mu') = 0$ for $r' < r + z$. Equation (2.14) is an exact analytic solution of the initial problem (2.2)-(2.4) in terms of the time variable t.

We consider now particular cases frequently encountered in applications. If the dependence on μ can be separated in the limiting flux

$$J(t, r, \mu) = J(t, r)\,\varphi(\mu),$$

there is no need to integrate with respect to μ':

$$\psi(t, z, r, \mu) = \int_{r+z}^{1} dr' J(t - \tau(r, r'), r') G_\varphi(z, r, r', \mu),$$

where $G_\varphi(z, r, r', \mu)$ is the solution of Eq. (2.9a) with boundary condition

$$G_\varphi(z = 0, r, r', \mu) = \delta(r - r')\,\varphi(\mu).$$

If, furthermore, the flux is also monoenergetic

$$J(t, r, \mu) = J(t)\,\delta(r - 1)\,\varphi(\mu),$$

there is likewise no need to integrate with respect to r', and the solution takes the form

$$\psi(t, z, r, \mu) = J(t - \tau(r, 1))\,G_\varphi(z, r, 1, \mu).$$

In the latter case the solution of the initial problem was simply reduced to a single solution of the stationary problem. In the general case it is necessary to solve the stationary problem a number of times (on the order of 100) with different r' and μ' and evaluate a double integral. We note that the integral in (2.14) is a proper one and can be easily evaluated with a computer.

3. Solution, in the "Right-Forward" Approximation, of the Problem of Deceleration of a Modulated Electron Beam

By way of example of the use of the equations derived in the preceding section, we consider the solution of the initial problem in the simplest "right-forward" approximation, in which the bending of the trajectories is not taken into account [6], meaning that $\mu = 1$ and $\partial/\partial\mu = 0$ in all relations. The results of this approximation are good for deceleration of beams of heavy particles such as protons, in view of the additional multiplier m/M for the beam particle deflection angle in each individual scattering act. In [6] there was obtained also a more exact stationary Green's function in the small-angle approximation. It must be noted, however, that even if the small-angle approximation were valid in our initial problem, it would remain difficult to use. The equations obtained in [6] are complicated, and the specific results we need are obtainable from them by using computer calculations of approximately the same complexity as in the direct numerical solution of (2.9) by finite-difference methods without any simplifying assumptions.

In the "right-forward" approximation the equation (2.9) for the stationary Green's function G(z, r, r') is of the form

$$\partial G/\partial z = \partial G/\partial r, \tag{3.1a}$$

$$G(z = 0, r, r') = \delta(r - r'), \quad 0 < z < r', \quad r < r' - z, \quad 0 < r' < 1. \tag{3.1b}$$

Its solution is readily obtained by the method of characteristics

$$G(z, r, r') = \delta(r' - r - z). \tag{3.2}$$

The solution of the initial problem is obtained in the "right-forward" approximation from Eq. (2.14):

$$\varphi(t, r, z) = J(t - \tau(r, r + z), r + z)\,\eta(t - \tau(r, r + z)). \tag{3.3}$$

To simplify the equations we assume an entering monoenergetic beam, i.e.,

$$J(t, r) = P(t)\,\delta(r - 1),$$

then

$$\psi(t, r, z) = P(t - \tau(1 - z, 1))\,\eta(t - \tau)\,\delta(r + z - 1). \tag{3.4}$$

For the ionization and excitation frequencies of interest to us we obtain

$$\psi_\gamma(t, z) = \sigma_\gamma(E(1 - z))\,P(t - \tau(1 - z, 1))\,\eta(t - \tau). \tag{3.5}$$

To obtain more concrete results from (3.5), we must know the forms of the functions $\sigma_\gamma(E)$, P(t), L(E), v(r), E(v). We emphasize that v(r) and v(E) have different functional forms. Let the functions be of the form $L(E) = W/l_c = WN\sigma_\Pi(E)$, $\sigma_\Pi(E) = c_\Pi/v^2(E)$, $c_\Pi = \text{const}$, $v(E) = c\left|1-\left(\frac{m_0c^2}{m_0c^2+E}\right)^2\right|^{1/2}$, where m_0c^2 is the rest energy of the electron. We choose P(t) to be the step $\eta(\Delta t - t)$, corresponding to one pulse of the modulated beam. From (2.1) we obtain for r(E)

$$r(E) = \frac{c^2}{WNc_\ell}\frac{E^2}{m_0c^2+E}.$$

For the nondimensionalizing constant r_0, we get

$$r_0 = \frac{c^2}{WNc_\ell}\frac{V^2}{m_0c^2+V}.$$

The relation between the dimensionless r and E is

$$r(E) = \frac{E^2(1+E_0)}{E+E_0}. \tag{3.6}$$

The energy E and $E_0 = m_0c^2/V$ are made here nondimensional by the energy V. From (3.6) we get for E(r)

$$E(r) = \frac{r+\sqrt{r(r+4E_0(1+E_0))}}{2(1+E_0)}. \tag{3.7}$$

A family of E(r) plots for different V is shown in Fig. 1. In the limiting nonrelativistic case $E_0 \gg 1$ Eq. (3.7) goes over into $E(r) = \sqrt{r}$, and in the opposite limiting case $E_0 \ll 1$ we have E(r) = r.

We now write an expression for $\tau(r, r')$:

$$\tau(r,r') = \int_r^{r'}\frac{dx}{v(x)} = \sqrt{2E_0+1}\int_{E(r)}^{E(r')}\left[1-\left(\frac{E_0}{E_0+E'}\right)^2\right]^{1/2}dE'.$$

After integration, we obtain

$$\tau(r,r') = \sqrt{2E_0+1}\left\{[E(r')(E(r')+2E_0)]^{1/2} - [E(r)(E(r)+2E_0)]^{1/2} + E_0\left[\arcsin\frac{E_0}{E_0+E(r')} - \arcsin\frac{E_0}{E_0+E(r)}\right]\right\}. \tag{3.8}$$

It is of interest to calculate the dimensionless upper bound of the time t:

$$T = \tau(0,1) = \sqrt{2E_0+1}\left\{\sqrt{2E_0+1} - E_0\arccos\frac{E_0}{E_0+1}\right\}.$$

In the extreme relativistic case Eq. (3.8) is transformed into

$$\tau(r,r') = r' - r, \tag{3.9}$$

with $T \approx 1 + (2 - \pi/r)E_0$. The opposite nonrelativistic limiting case $E_0 \gg 1$ cannot be considered with the aid of (3.8), since it is impossible to expand arcsin x in the vicinity of x = 1, which is a branch for the arcsine. We consider this case separately. To this end we again integrate after substituting $v(r) = r^{1/4}$. As as result we get

$$\tau(r,r') = {}^4/_3[(r')^{3/4} - r^{3/4}]$$
$$\text{and } T \approx \frac{4}{3}\left[1-\frac{9}{40}\left(\arccos\frac{E_0}{E_0+1}\right)^2\right]. \tag{3.10}$$

It is also of interest to find the depth z* at which the energy of an initially relativistic electron becomes equal to m_0c^2. From (3.6) we get

$$z^* = 1 - r^* = 1 - {}^1/_2E_0(E_0+1). \tag{3.11}$$

We have thus $z = {}^5/_8$ for $V = 2m_0c^2 \approx 1$ MeV and $z^* = 0.945$ for $V = 10m_0c^2$. This shows that an electron with $V \gtrsim 1$ MeV remains relativistic over the greater part of its deceleration path.

We consider now the question of longitudinal compressibility in the space of the initial steplike pulse $\eta(\Delta t - t)$ on passing through the gas. The coordinate $z_\zeta(t)$ of the leading front is determined in accordance with (3.4) from the equation

$$t = \tau(1 - z_\zeta, 1), \tag{3.12}$$

and the coordinate $z_t(t)$ of the trailing edge from the equation

$$t - \Delta t = \tau(1 - z_t, 1), \tag{3.13}$$

where Δt is the initial temporal width of the step, i.e., the instant of time at which the trailing edge of the pulse enters the gas. The spatial width $\Delta z(t)$ of the pulse is equal to

$$\Delta z(t) = z_\zeta(t) - z_t(t).$$

The distortion of the contour of the distribution-function pulse

$$f(t, z, r) = \frac{1}{v(r)}\psi(t, r, z)$$

and the ionization frequencies (3.5) are determined by the values of their ordinates on the leading and trailing edges. By differentiating with respect to t we easily obtain from (3.12) and (3.13) the velocities and the decelerations of the fronts.

Since Eq. (3.8) is unwieldy, we consider below only limiting cases: a) purely relativistic and b) relativistic.

a) $E_0 \ll 1$. In this case the dimensionless solution is

$$\psi(t, z, r) = \eta(z - (t - \Delta t))\eta(t - z)\,\delta(r + z - 1),\ \tau(1 - z, 1) = z, \tag{3.14}$$

the beam-electron density is $n(t, z) = \int_0^1 drf(t, r, z) = \eta(z - t + \Delta t)\,\eta(t - z)$, , and the excitation frequencies are $\nu_\gamma(t, z) = n(t, z)$. Recall that all the quantities are the dimensionless entrance values at $z = 0$. It can be seen from these equations that in the purely relativistic case the excitation and ionization wave travels along the gas without changing form, with constant unit front velocities. The reason is that, notwithstanding the decrease of the beam-electron energy by deceleration, their velocity, meaning also the cross section for interaction with the gas, remains unchanged.

b) $E_0 \gg 1$. In this case

$$\begin{aligned}&\psi(t, z, r) = \eta(\Delta t - t + {}^4/_3(1 - (1 - z)^{3/4}))\,\eta(t - \tau)\,\delta(r + z - 1),\\ &\tau(1 - z, 1) = {}^4/_3[1 - (1 - z)^{3/4}],\end{aligned} \tag{3.15}$$

the density $n(t, z) = (1 - z)^{-1/4}\eta(\Delta t - t + \tau)\,\eta(t - \tau)$, and the frequencies $\nu_\gamma(t, z) = (1 - z)^{-1/2}\eta(\Delta t - t + \tau)\,\eta(t - \tau)$. The coordinates z_ζ and z_t are no longer linear but power-law functions of time:

$$z_\zeta(t) = 1 - (1 - {}^3/_4 t)^{4/3},\ z_t(t) = z_\zeta(t - \Delta t).$$

In the nonrelativistic case the excitation and ionization wave changes shape as it propagates in the gas. The spatial width $\Delta z(t)$ of the pulse decreases with time, and the height of the pulse increases, while the number of particles in the pulse

$$\int_{z_t}^{z_\zeta} n(t, z)\,dz = \Delta t = \text{const}$$

remains constant in time. It should be noted that these changes of the waveform of the excitation and ionization wave are negligible in the considered model even in the nonrelativistic case. Thus, the doubling ν_γ at the maximum, which is reached on the leading front, corresponds to the point $z_2 = {}^3/_4$ at which the beam-electron energy decreases by one-half. This is easily understood if it is recalled that the inelastic cross sections for beam-electron

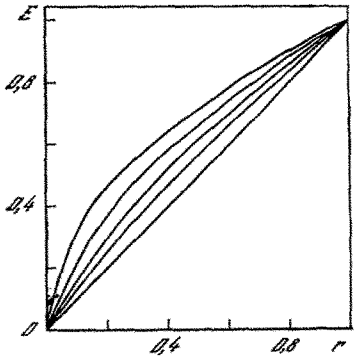

Fig. 1

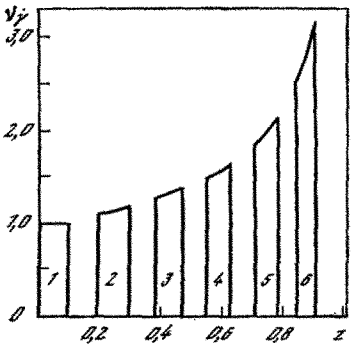

Fig. 2

Fig. 1. Family of plots of E(r) for various V. The lower limiting energy E(r) = r corresponds to the ultrarelativistic case $V >> m_0c^2$. The upper limiting curve $E(r) = \sqrt{r}$ corresponds to the nonrelativistic case $V << m_0c^2$.

Fig. 2. Shape of an ionization step pulse vs. depth at $\Delta t = 0.1$. The bars 1-6 correspond to the pulse positions at t = 0.1; 0.3; 0.5; 0.7; 0.9; 1.1.

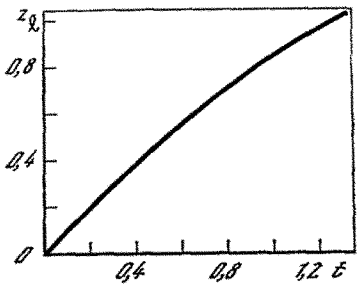

Fig. 3

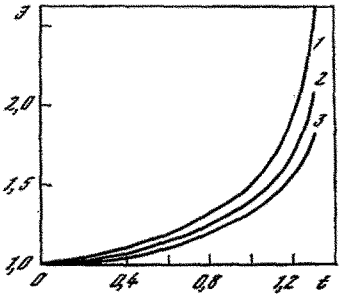

Fig. 4

Fig. 3. Time dependence of the coordinate of the leading front of the pulse.

Fig. 4. Time dependence of the relative energy input. The curves correspond to: 1) $\Delta t = 0.1$; 2) $\Delta t = 0.3$; 3) $\Delta t = 0.5$.

interaction with the gas is exactly inversely proportional to the energy in this model.

The energy input $\vartheta(t)$ to the ionized medium, referred to the initial value Δt, has the following time dependence:

$$\vartheta(t) = \frac{2}{\Delta t}\left\{\left[1 - \frac{3}{4}(t - \Delta t)\right]^{1/3} - \left[1 - \frac{3}{4}t\right]^{1/3}\right\}.$$

The leading front reaches a depth $z_2 = 3/4$ at the instant of time $t_2 = 0.862$, in which case the energy input ϑ increases only to 1.33 times the initial value at $\Delta t = 0.2$.

The change of the shape of the ionization wave is illustrated in Fig. 2. Figures 3 and 4 show the time dependence of the coordinate $z_1(t)$ of the leading front and of the relative energy input $\vartheta(t)$ for different values of Δt.

In the general case $E_0 \lesssim 1$, the solution (3.14) goes over continuously into the solution (3.15) in the vicinity of the point $z = z^*$ defined by (3.11).

It follows from the considered model in the "right-forward" approximation that relativistic beams are preferable to nonrelativistic for pumping an active medium, since they produce an ionization that is more uniform in depth. This conclusion holds apparently for beams of heavy particles, for which the "right-forward" approximation is good enough. As to electron beams, no conclusions whatever can be drawn from our foregoing analysis inasmuch, as shown

above, as the deceleration of electron beams can be treated not only by the "right-forward" approximation but also by the more accurate small-angle approximation. Moreover, the estimates show that in the case of electron beams, the conclusion that relativistic beams are superior to nonrelativistic may be reversed.

Indeed, the change of the ionization waveform in the case of electron deceleration is influenced by multiple scattering, to which the angular component of the transport operator pertains. For nonrelativistic beams the ratio of the total deceleration length r_0 to the transport scattering length l_{tr} is approximately constant and is almost independent of the initial energy V of the entering electrons. In the relativistic case, however, as shown in the Introduction, the ratio of r_0 to l_{tr} increases in proportion to V, so that multiple scattering assumes a greater role.

4. Numerical Methods of Solving the Stationary Problem

As shown in Sec. 2, the solution of the problem of gas ionization by a modulated electron beam reduces to finding the Green's function for the stationary equation (2.9a). Since no analytic expression for the Green's function has been obtained to date, numerical solutions become particularly important. It should be noted that the problem of the stationary distribution function has not been solved in the formulation (2.9a). A number of workers [8-14], however, carried out a numerical investigation of the propagation of a high-energy electron beam ($V \sim 10$ keV) in the upper layer of the atmosphere. These studies are important, since the algorithms used in them can serve as a basis for a numerical solution of the Green's function problem.

To develop the most economical and effective numerical method it is necessary to compare critically the already developed difference schemes. We attempt below to analyze the methods based on the use of difference schemes [8-10, 12] and a numerical analog of the method of variation of the constant. The Monte Carlo method in which the calculations in [11, 13, 14] are based is not considered, since it calls for much computer time. We shall concentrate hereafter on the method of variation of the constants, which is sometimes called, in analogy with [10], the "eigensolution" method. The meaning of this designation will be made clear presently. The reason for the increased interest in this algorithm is, as stated in [10], that the computer time required for the calculation by this procedure is much shorter than for other algorithms. The description of the numerical methods is based here in [10] where, besides describing the algorithms (the predictor–corrector and eigensolution methods), the results obtained by different workers are also compared.

Before proceeding to a direct exposition of the numerical methods, we write down in explicit form the expressions for L(E) and Q(E) in the model of [10]. These expressions are obtained in terms of the variables z, r, and μ by numerically determining the r(E) dependence, using Eq. (2.1).

In [10] was modeled the propagation of a beam of high-energy electrons in the upper layers of the atmosphere, with account taken of the elastic and inelastic collisions between the particles and the components of the atmosphere, as well as with allowance for the electron-energy lost to production of secondary electrons. It was assumed that after inelastic interaction (excitation or ionization) the electron does not change the direction of its motion, and merely loses energy. Under these assumptions the expressions for Q(E) and L(E) take the form

$$Q(E)=\frac{2\pi Z^2e^4}{v^2p^2}\left[\ln\left(\frac{1+\eta}{\eta}\right)-\frac{1}{1+\eta}\right], \tag{4.1}$$

$$L(E)=\sum_{\gamma}W_{\gamma}\sigma_{\gamma}(E)+I\sigma_I(E)+\int_0^{(E-I)/2}\sigma_s(E,E_s)E_s\,dE_s, \tag{4.2}$$

where Z is the atomic number of the scattering center, p the electron momentum, η the screening parameter, W_γ the excitation potential, σ_I the ionization cross section, and $\sigma_S(E, E_S)$ the differential ionization cross section.

The third term in the expression for L(E) describes the energy lost by the primary electron in the production of secondary electrons of energy E_S.

Now, having determined the expressions for the loss per unit length and the mean-squared scattering angle, we can proceed to write down the difference system.

The partial difference equation was reduced in [10] to a system of ordinary differential equations. This was done by using the following difference approximations of the angular and energy operators:

$$\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial\Phi}{\partial\mu}\Big|_{E_n,\mu_i}=[(1-\mu_{i-1/2}^2)(\Phi_{n,i-1}-\Phi_{n,i})-(1-\mu_{i+1/2}^2)(\Phi_{n,i}-\Phi_{n,i+1})](\Delta\mu)^{-2}\quad(1<i<N) \tag{4.3}$$

$$\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial\Phi}{\partial\mu}\Big|_{E_n,\mu_1}=-2\frac{\Phi_{n,1}-\Phi_{n,2}}{\Delta\mu}, \tag{4.4}$$

$$\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial\Phi}{\partial\mu}\Big|_{E_n,\mu_N}=2\frac{\Phi_{n,N-1}-\Phi_{n,N}}{\Delta\mu}, \tag{4.5}$$

where $\mu_1 = 1$, $\mu_N = -1$, $\Delta\mu = 2/(N-1)$ is the mesh of the grid $\mu_{i-1/2} = (\mu_{i-1} + \mu_i)/2$, while i and n determine respectively the angle and energy coordinates,

$$\frac{\partial}{\partial E}(L(E)\Phi)|_{E_n,\mu_i}=\frac{L_{n-1}\Phi_{n-1,i}-L_n\Phi_{n,i}}{\Delta E_{n-1}}. \tag{4.6}$$

Substituting expressions (4.3)-(4.6) in (2.9), we obtain the following system of ordinary differential equations:

$$\mu_i\frac{d\Phi_{n,i}}{dz}=\sum_j B_{ij}^n\Phi_{nj}+S_{nj}, \tag{4.7}$$

where B_{ij}^n is a three-diagonal matrix, and $S_{ni} = L_{n-1}\Phi_{n-1,i}/\Delta E_n$. In view of the unwieldy expressions, we shall not write out the explicit form of the matrix B_{ij}. It is obvious, however, that explicit expressions for B_{ij} can be easily obtained by using expressions (4.3)-(4.6).

4.1. Difference Analog of the Eigensolution Method. The obtained system of differential equations can be solved either by direct numerical integration (by the predictor–corrector method) or by constructing an algorithm based on finding the eigenvectors and eigenvalues. The latter method, as stated by the authors of [10], has the appreciable advantage of faster computation.

The gist of the eigensolution method is the following. We seek a solution of the homogeneous system (4.7) with $S_i = 0$ in the form

$$\Phi_i=\sum_{l=1}^{N}C_l\psi_{il}\exp(-\lambda_l z), \tag{4.8}$$

where λ_l is the l-th eigenvalue, ψ_{il} the l-th eigenvector at the point μ_i, and C_l a constant determined by the boundary conditions.

It is assumed next that $C_l = C_l(z)$, and the expansion of S_i in terms of the eigenvectors ψ_{il} is determined

$$S_i(z)=\mu_i\sum_{l=1}^{N}d_l(z)\psi_{il}. \tag{4.9}$$

The coefficients $d_l(z)$ are determined by inverting the matrix ψ_{il}. After substituting (4.8) and (4.9) in the system (4.7), we obtain the relation between the coefficients C_l and d_l:

$$\sum_l\left[\frac{dC_l}{dz}\exp(-\lambda_l z)-d_l\right]\psi_{il}=0. \tag{4.10}$$

If we put now

$$u_l(z)=C_l(z)\exp(-\lambda_l z), \tag{4.11}$$

we obtain the differential equation

$$\frac{du_l}{\partial z}+\lambda_l u_l=d_l. \tag{4.12}$$

The formal solutions of this differential equation, with the boundary conditions

$$u_l|_{z=0} = u_l(0), \quad u_l|_{z=z_0} = u_l(z_0)$$

take the form

$$u_l(z) = u_l(0)\exp(-\lambda_l z) + \int_0^z d_l(z')\exp[-\lambda_l(z-z')]\,dz', \tag{4.13}$$

$$u_l(z) = u_l(z_0)\exp[\lambda_l(z-z_0)] - \int_0^{z_0} d_l(z')\exp[\lambda_l(z'-z)]\,dz'. \tag{4.14}$$

Here z_0 is the maximum depth of penetration of the electron having the maximum energy. Expressions (4.13) and (4.14) for $u_\gamma(z)$ are used for positive and negative λ_γ, respectively. Calculations show that the λ_γ are real and have unity multiplicity. The number of eigenvalues is equal to the number of points of the μ grid. In [10], γ is even, one half of the eigenvalues is positive, and the other half negative. The absolute values of the positive and negative λ_γ are pairwise equal. We shall assume hereafter, in analogy with [10], that at $\gamma \leq N/2$ λ_γ they are negative and at $\gamma > N/2$ they are positive.

To find $u_\gamma(0)$ and $u_\gamma(z_0)$ it is necessary to use the boundary conditions respectively for $z = 0$ and $\mu \geq 0$ and for $z = z_0$ and $\mu < 0$. Using expression (4.8) for Φ_i, and also the relation (4.11), we can write the boundary conditions in the form of a system of algebraic equations with 2N unknowns:

$$\sum_{l\leqslant N/2} u_l(0)\,\psi_{il} + \sum_{l>N/2} u_l(0)\,\psi_{il} = \Phi_i(z=0), \quad i \leqslant N/2; \tag{4.15}$$

$$\sum_{l\leqslant N/2} u_l(z_0)\,\psi_{il} + \sum_{l>N/2} u_l(z_0)\,\psi_{il} = 0, \quad i > N/2, \tag{4.16}$$

which can be reduced, using (4.13) and (4.14), to a uniquely solvable algebraic system of N equations with N unknowns:

$$\sum_{l\leqslant N/2} u_l(z_0)\exp(\lambda_l z_0)\,\psi_{il} + \sum_{l>N/2} u_l(0)\,\psi_{il} = \Phi_i(z=0) + \sum_{l\leqslant N/2}\psi_{il}\int_{z_0}^{z} d_l\exp(\lambda_l z)\,dz, \quad i \leqslant N/2; \tag{4.17}$$

$$\sum_{l\leqslant N/2} u_l(z_0)\,\psi_{il} + \sum_{l>N/2} u_l(0)\exp(-\lambda_l z_0)\,\psi_{il} = -\sum_{l>N/2}\psi_{il}\int_0^{z_0} d_l\exp(-\lambda_l(z_0-z))\,dz, \quad i > N/2. \tag{4.18}$$

From the system (4.17) and (4.18) we determine $u_\gamma(0)$ and $u_\gamma(z_0)$, which are used next to find the electron distribution function.

To appreciate the advantages of the eigensolution method, we shall dwell in detail on the direct numerical integration of the system (4.7).

Since a fast electron cannot gain energy on the deceleration length, and only lose it, it is convenient to start the difference-scheme calculations with the maximum energy (the maximum residual range). The numerical calculation scheme does not depend on the form of the numerical method, be it the predictor–corrector method [10] or the Gauss–Seidel iteration procedure [9]. We shall therefore not specify hereafter the numerical method, and dwell only on the sequence of the operations that lead to the solution.

For each specified energy we determine the flux at $\mu > 0$ from $z = 0$ to $z = z_0$. The backward flux is assumed to be zero. The forward electron flux is memorized in the z-grid sites. After obtaining the forward flux, we calculate the backward flux from $z = z_0$ to $z = 0$.

The electron flux for $\mu < 0$ is calculated on the same spatial grid as the forward flux. To determine the backscatter, we use the previously memorized values of the forward flux. The procedure described is the first iteration. In the next iteration, we obtain the forward flux by using the backscatter values found in the first iteration, etc. Thus, only a few iterations are needed to calculate the electron flux with the required accuracy. The number of iterations depends on the accuracy selected. We note by way of example that three

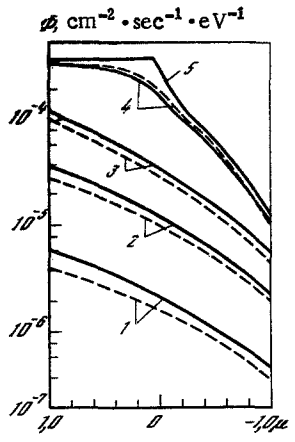

Fig. 5

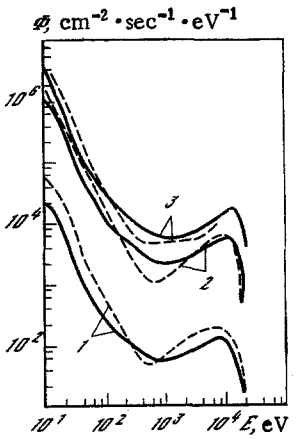

Fig. 6

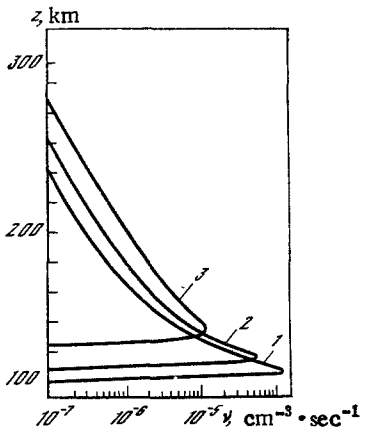

Fig. 7

Fig. 5. Angular dependence of electron flux. The solid curves were taken from [10] and the dashed from [12]. Curves 1-5 correspond to heights (in km): 90(1); 95(2); 100(3); 141(4); 250(5).

Fig. 6. Energy dependence of the electron flux. The solid curves were taken from [10] and the dashed from [12]. Curves 1-3 correspond to heights (in km): 95(1); 105(2); 148(3).

Fig. 7. Ionization rate [$cm^{-3} \cdot sec^{-1}$] at various electron-beam energies. Curves 1-3 correspond to energies (in keV): 10(1); 5(2); 2(3).

iterations were needed in [10] to obtain a solution that preserves the energy accurate to 1%.

We see thus that solutions of (2.9) by various schemes have a number of features in common. First, iteration over the spatial coordinate, and second, a connection between the forward flux and the backscatter near $\mu = 0$. This tie-in of the fluxes imposes a stringent requirement on the difference scheme, viz., high accuracy of the approximation of Eq. (2.9a) of the difference scheme. Unfortunately, the cited papers contain no analysis of the quality of the difference schemes, and this prevents us in turn from assessing the solutions obtained. Moreover, the difference between the values obtained for the total flux in [9] and [10] is apparently due to just this effect.

If we now compare the "eigensolution" method with the algorithms based on difference schemes, we can point to the following advantages of the former. First, in the eigensolutions method there are no iteractions. Second, there are no restrictions on the approximation of the differential equation near $\mu = 0$. Thus, the quality of the numerical analog of the eigensolution method depends only on the quality of the procedure used to find the eigenvalues and the eigenvectors of a real matrix.

Mention must be made of one more aspect of the numerical solution of the Green's function of a stationary problem. The numerical solution is made difficult by the presence of numerical dispersion due to the unilateral approximation of the derivative with respect to energy (to the residual range). This difficulty was first pointed out in [8]. The dispersion on the difference grid leads to an appreciable distortion of the solution, and this calls for testing the difference schemes. Unfortunately, the authors of [9, 10] do not mention this effect at all. Our calculations show that the influence of dispersion as well as of numerical diffusion is particularly substantial near breaks or abrupt changes of the electron distribution function.

It follows from all the foregoing that further work must be performed both to develop new high-efficiency numerical algorithms and to investigate those already developed.

4.2. Some Results of the Calculations. We cite in this subsection calculation results obtained by several workers [9, 10, 12]. Even though these calculations were obtained for definite initial conditions (isotropic distribution for $\mu > 0$ [10, 12] and Gaussian distribution in E) and are not strictly speaking Green's function calculations, they help understand the qualitative aspects of the solution of the stationary problem. The notation on the figures is the same as in the papers from which they are taken.

Figure 5 shows the angular dependence of an electron flux with initial energy 10 keV at various heights. It can be seen from the figure that the angular distribution in depth tends to be isotropic, although at a height of 90 km, equal to the total mean free path, the electron flux does not become isotropic.

Figure 6 shows the energy dependence of an electron flux at various heights. The increase of the flux at energies lower than 1 keV is due to the increase of the cross section for formation of secondary electrons.

Figure 7 shows the ionization rate at various initial electron energies. It can be seen from the figure that the ionization rate is a maximum at the end of the free path. The reason is that at the end of the deceleration path the beam electrons have a much lower energy than at the start of the trajectory. Since the ionziation cross section is 1/E, the ionization rate increases as the energy is lowered.

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