# COMPUTATION OF CORRECTIONS TO THE FOCK ASYMPTOTICS FOR THE WAVE FIELD NEAR A CIRCULAR CYLINDER AND A SPHERE

V. I. Ivanov

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Corrections are computed to the asymptotics of V. A. Fock for diffraction fields in the region of semishadow near surfaces of constant curvature. Numerical computations are carried out to demonstrate the effectiveness of the corrections found in the computation of fields.

The shortwave asymptotics of a wave field in the zone of semishadow near an ideally reflecting, convex body were obtained by Fock [1, 2] more than 30 years ago. However, the question of refining the Fock asymptotics has so far not been solved, i.e., the question of computing additional terms of the asymptotic series. In the literature there are only a few results for the current on a reflecting surface. Thus, in [3] a graph is presented for the correction function to the Fock asymptotics of the current on a circular cylinder (without indication of a computational method). In [4] by means of an integral equation a correction (with a mistake) was computed for the Fock asymptotics of the field on the surface of an ideally rigid body.

In the present paper we compute corrections to the Fock asymptotics for the wave field near a circular cylinder and a sphere. A numerical comparison is made of the asymptotic expressions for the field with exact solutions which demonstrates the effectiveness of using the corrections in numerical computations of diffraction fields.

## 1. Computation of the Correction for the Field near a Circular Cylinder

We consider the problem of the diffraction of a plane wave $u_0 = e^{-ikx} = e^{-ikr\cos\varphi}$ by an ideally reflecting, circular cylinder of radius $a$ (Fig. 1). Suppose that the wave field $u = u_0 + v$ satisfies the equation $\Delta u + k^2 u = 0$ and the boundary condition $\Omega u|_{r=a} = 0$, where

$$\Omega = \begin{cases} 1 & \text{in the case of the Dirichlet problem,} \\ \dfrac{\partial}{\partial n} & \text{in the case of the Neumann problem.} \end{cases}$$

while the reflected field v satisfies the radiation principle.

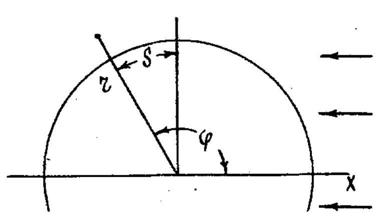


Fig. 1

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It is known (cf. [5, 6]) that the solution of the problem $u(r, \varphi)$ can be represented in the form

$$u = \sum_{n=-\infty}^{\infty} U(r, \varphi + 2\pi n), \tag{1}$$

where $U(r, \varphi)$ is a periodic solution of the Helmholtz equation on the infinitely sheeted plane $-\infty < \varphi < \infty$, $r \geq a$. For the function $U(r, \varphi)$ it is possible to obtain (cf. [6]) the expression

$$U(r, \varphi) = \int_{-\infty}^{+\infty} \left[ J_\nu(\rho) - \frac{\Omega J_\nu(\alpha)}{\Omega H_\nu^{(1)}(\alpha)} H_\nu^{(1)}(\rho) \right] e^{i\nu(\varphi - \pi/2)} d\nu, \tag{2}$$

where $\alpha = ka$, $\rho = kr$. We set $M = (\alpha/2)^{1/3}$ and seek the asymptotics of the integral (2) as $k \to \infty$ $(M \to \infty)$. We shall consider the field in the region $\rho - \alpha \leq M$ (a boundary layer near the cylinder), $\alpha(\varphi - \pi/2) \gtrsim -M^2$ (zones of semishadow and deep shadow).

It is known [6] that for the zone in question the essential interval of integration in (2) is a neighborhood of the point $\nu = \alpha$: $|\nu - \alpha| \lessapprox M$. To obtain the asymptotics of integral (2), the integrand should be expanded in this neighborhood in the small parameter $1/M$.

The asymptotics of the function $H_\nu^{(1)}(\alpha)$ and its derivative as $\alpha \to \infty$, $|\nu - \alpha| \lessapprox M$, are known [1, 7, 8]:

$$H_\nu^{(1)}(\alpha) = \frac{-i}{M\sqrt{\pi}} \left[ w(t) - \frac{1}{60M^2} w^{(5)}(t) + O\left(\frac{1}{M^4}\right) \right], \tag{3}$$

$$H_\nu^{(1)\prime}(\alpha) = \frac{i}{M^2\sqrt{\pi}} \left[ w'(t) + \frac{1}{60M^2}(10 w'''(t) - w^{(6)}(t) + O\left(\frac{1}{M^4}\right) \right],$$

where $t = (\nu - \alpha)/M$, and $w(t)$ is the Airy function in Fock's definition. The asymptotics of the functions $J_\nu(\alpha)$, $J'_\nu(\alpha)$ are expressed [because of the equality $J_\nu(\alpha) = \operatorname{Re} H_\nu^{(1)}(\alpha)$] by formulas analogous to (3) in terms of the Airy function $v(t) = \operatorname{Im} w(t)$.

For the function $H_\nu^{(1)}(\rho)$ we obtain an asymptotic expansion for $|\nu - \alpha| \lessapprox M$, $|\rho - \alpha| \lessapprox M$ from the integral representation

$$H_\nu^{(1)}(\rho) = \frac{1}{\pi i} \int_{-\infty}^{+\infty} e^{-\rho \operatorname{sh} z + \nu z} dz, \tag{4}$$

by expanding the exponent in a neighborhood of the saddle point $z = 0$ (cf. [7]). We set $y = (\rho - \alpha)/M$ and introduce in (4) the new variable of integration $\tau = Mz$; then

$$-\rho \operatorname{sh} z + \nu z = -(\alpha + My) \operatorname{sh} \frac{\tau}{M} + (\alpha + Mt)\frac{\tau}{M} = (t - y)\tau - \tau^3/3 - \frac{1}{60M^2}(\tau^6 + 10y\tau^3) + O\left(\frac{1}{M^4}\right).$$

We transform the contour of integration in (4) in the $\tau$ plane into a broken line $\bar{\Gamma}$ going from $\infty e^{4\pi i/3}$ to 0 and then to $\infty$. As a result, we obtain

$$H_\nu^{(1)}(\rho) = \frac{-i}{M\pi} \int_{\bar{\Gamma}} e^{(t-y)\tau - \tau^3/3} \left\{ 1 - \frac{1}{60M^2}(\tau^5 + 10\tau^3 y) + O\left(\frac{1}{M^4}\right) \right\} d\tau$$

$$= \frac{-i}{M\sqrt{\pi}} \left\{ w(t-y) - \frac{1}{60M^2}\left[ w^{(5)}(t-y) + 10y w'''(t-y) \right] + O\left(\frac{1}{M^4}\right) \right\}. \tag{5}$$

For the Bessel function $J_\nu(\rho) = \operatorname{Re} H_\nu^{(1)}(\rho)$ we obtain

$$J_\nu(\rho) = \frac{1}{M\sqrt{\pi}} \left\{ v(t-y) - \frac{1}{60M^2}\left[ v^{(5)}(t-y) + 10y v'''(t-y) \right] + O\left(\frac{1}{M^4}\right) \right\}. \tag{6}$$

We obtain the desired asymptotics of the wave field (in the zones of semishadow and shadow) by substituting (3), (5), (6) into (2), expanding the integrand in powers of $1/M$, and transforming the contour of integration (the real axis) into a broken line $\Gamma$ going from $\infty e^{2\pi i/3}$ to $\infty$. In the case of the Dirichlet problem we have

$$U=\frac{e^{iks}}{\sqrt{\pi}}\int_{\Gamma}e^{i\sigma t}\Big\{\Big[v(t-y)-\frac{v(t)}{w(t)}w(t-y)\Big]-\frac{1}{60M^2}\Big[v^{(5)}(t-y)+10yv'''(t-y)$$

$$-\frac{v(t)}{w(t)}\Big(w^{(5)}(t-y)+10yw'''(t-y)+\frac{t^2}{w^2(t)}w(t-y)\Big)\Big]\Big\}dt+O\Big(\frac{1}{M^4}\Big), \tag{7}$$

where $s = a(\varphi - \pi/2)$, $\sigma = M(\varphi - \pi/2)$. Here the first term

$$U_0^D=\frac{e^{iks}}{\sqrt{\pi}}\int_{\Gamma}e^{i\sigma t}\Big[v(t-y)-\frac{v(t)}{w(t)}w(t-y)\Big]dt$$

is the well-known asymptotic expression of Fock; the other terms are correction terms. The expression for the correction can be simplified by replacing the leading derivatives of the functions v(t) and w(t) (by Airy's equation) in terms of the functions v(t) and w(t) and their first derivatives:

$$U=U_0^D-\frac{e^{iks}}{60M^2\sqrt{\pi}}\int_{\Gamma}e^{i\sigma t}\Big\{(4t+6y)\varphi(t,y)+(t-y)(t+9y)\bar{\varphi}(t,y)+\frac{t^2}{w^2(t)}w(t-y)\Big\}dt+O\Big(\frac{1}{M^4}\Big), \tag{8}$$

where

$$\varphi(t,y)=v(t-y)-\frac{v(t)}{w(t)}w(t-y),\quad \bar{\varphi}(t,y)=-\frac{\partial\varphi}{\partial y}. \tag{9}$$

Similarly, for the Neumann problem we obtain

$$U=U_0^N-\frac{e^{iks}}{60M^2\sqrt{\pi}}\int_{\Gamma}e^{i\sigma t}\Big\{(4t+6y)\Psi(t,y)+(t-y)(t+9y)\bar{\Psi}(t,y)-\frac{t^3-6}{w'^2(t)}w(t-y)\Big\}dt+O\Big(\frac{1}{M^4}\Big), \tag{10}$$

$$\Psi(t,y)=v(t-y)-\frac{v'(t)}{w'(t)}w(t-y),\ \bar{\Psi}(t,y)=-\frac{\partial\Psi}{\partial y}; \tag{11}$$

where $U_0^N=\frac{e^{iks}}{\sqrt{\pi}}\int_{\Gamma}e^{i\sigma t}\Psi(t,y)dt$ is the asymptotic expression of Fock.

By the present method it is also possible to obtain subsequent corrections to the Fock asymptotics if we use subsequent terms of the asymptotic expansions (3) (see, e.g., [8]). Thus, for the current on the surface of the cylinder in the case of the Dirichlet problem it is possible to obtain

$$I_D=\frac{1}{k}\frac{\partial U}{\partial r}\Big|_{r=a}=\frac{e^{iks}}{M}\Big\{f_1^{00}+\frac{1}{60M^2}\big[f_2^{21}+4f_1^{10}\big]+\frac{1}{7200M^4}\big[2f_3^{42}-f_1^{50}\big]-\frac{1}{6300M^4}\big[45f_2^{01}+f_2^{31}+37f_1^{20}\big]+O\Big(\frac{1}{M^6}\Big)\Big\}, \tag{12}$$

where

$$f_k^{mn}=\frac{1}{\sqrt{\pi}}\int_{\Gamma}e^{i\sigma t}\frac{t^n[w'(t)]^m}{[w(t)]^k}dt \tag{13}$$

are the associated "current" functions of Fock. In the case of the Neumann problem the field on the surface of the cylinder (the current) is

$$I_N=U(a,\varphi)=e^{iks}\Big\{g_1^{00}+\frac{1}{60M^2}\big[g_2^{31}-6g_2^{01}-4g_1^{10}\big]$$

$$+\frac{1}{50400M^4}\big[504g_3^{02}-168g_3^{32}+14g_3^{62}+1512g_2^{11}-127g_2^{41}+604g_1^{20}-7g_1^{50}\big]+O\Big(\frac{1}{M^6}\Big)\Big\}, \tag{14}$$

where

$$g_k^{nm}=\frac{1}{\sqrt{\pi}}\int_{\Gamma}e^{i\sigma t}\frac{t^n[w(t)]^m}{[w'(t)]^k}dt. \tag{15}$$

The magnitude of the first correction for the current $I_N$ in the Neumann problem was computed in [4]. In the case of a circular cylinder, the result of [4] for the first correction has the form [in the notation of (15)]

$$\frac{1}{30M^2}\left[\overset{31}{g_2}-6\overset{01}{g_2}-4\overset{10}{g_1}\right],$$

i.e., it is exactly twice as large as the first correction in (14).

## 2. Computation of the Correction for the Field near a Sphere

Suppose that on an ideally conducting sphere of radius $a$ there is incident a plane wave $u_0 = e^{-ikz} = e^{-ikr\cos\theta}$. To obtain the shortwave asymptotics of the diffraction field it is more convenient to proceed from the integral representation of the field in the infinitely sheeted space $0 \le \theta < \infty$ [9]:

$$U(r,\theta)=\int_{-\infty}^{+\infty}(2\mu+1)Q_\mu^{(2)}(-\cos\theta)e^{i\pi\mu/2}\left[j_\mu(\rho)-\frac{\Omega j_\mu(\alpha)}{\Omega h_\mu(\alpha)}h_\mu(\rho)\right]d\mu, \tag{16}$$

where $\alpha = ka$, $\rho = kr$; $j_\mu(x)$, $h_\mu(x)$ are the spherical Bessel functions; $Q_\mu^{(2)}(x)$ is the Legendre function of second kind,

$$Q_\mu^{(2)}(x)=\frac{1}{2\sin\pi\mu}\left[e^{-i\pi\mu}P_\mu(x)-P_\mu(-x)\right].$$

As in the case of the circular cylinder, the asymptotic behavior of the integral (16) in the zones of semishadow and shadow (except for a neighborhood of the pole of the sphere) is determined by the behavior of the integrand in a neighborhood of the point $\mu = \alpha$. The asymptotic behavior of the Legendre function $Q_\mu^2$ as $\mu \to \infty$ is known [9]:

$$Q_\mu^{(2)}(-\cos\theta)=\left[\sqrt{\frac{i}{2\pi\mu\sin\theta}}+O(\mu^{-3/2})\right]\exp\left\{-i(\mu+1/2)(\pi-\theta)\right\}. \tag{17}$$

Asymptotic expressions for $h_\mu(\alpha)$, $h'_\mu(\alpha)$, $h_\mu(\rho)$ for $\alpha \to \infty$, $|\mu-\alpha| \le M$, $|\rho-\alpha| \lesssim M$ follow from (3), (5):

$$h_\mu(\alpha)=\frac{-i}{M\sqrt{2\alpha}}\left\{w(t)-\frac{1}{60M^2}w^{(5)}(t)+O\left(\frac{1}{M^4}\right)\right\}, \tag{18}$$

$$h'_\mu(\alpha)=\frac{i}{M^2\sqrt{2\alpha}}\left\{w'(t)+\frac{1}{60M^2}\left[(21-t^3)w(t)+4tw'(t)\right]+O\left(\frac{1}{M^4}\right)\right\} \tag{19}$$

$$h_\mu(\rho)=\frac{-i}{M\sqrt{2\alpha}}\left\{w(t-y)-\frac{1}{60M^2}\left[w^{(5)}(t-y)+10yw'''(t-y)+15yw(t-y)\right]+O\left(\frac{1}{M^4}\right)\right\}, \tag{20}$$

where $t = (\mu + 1/2 - \alpha)/M$, $y = (\rho - \alpha)/M$. For the functions $j_\mu(\alpha) = \operatorname{Re} h_\mu(\alpha)$ analogous formulas are obtained which contain the Airy function $v(t)$.

We obtain the desired correction to the Fock asymptotics for the wave field near the sphere as in the case of the cylinder by substituting the asymptotic expression (17)-(20) and expanding the integrand in powers of $1/M$. In the case of the Dirichlet problem after transformations we obtain

$$U=\frac{1}{\sqrt{\sin\theta}}\left\{U_0^D-\frac{e^{iks}}{60M^2\sqrt{\pi}}\int_\Gamma e^{i\sigma t}\left[(21y-11t)\varphi(t,y)+(t-y)(t+9y)\bar{\varphi}(t,y)+\frac{t^2}{w^2(t)}w(t-y)dt+O\left(\frac{1}{M^4}\right)\right\}. \tag{21}$$

Similarly, in the case of the Neumann problem we have

$$U=\frac{1}{\sqrt{\sin\theta}}\left\{U_0^N-\frac{e^{iks}}{60M^2\sqrt{\pi}}\int_\Gamma e^{i\sigma t}\left[(21y-11t)\Psi(t,y)+(t-y)(t+9y)\Psi(t,y)+\frac{21-t^3}{w'^2(t)}w(t-y)\right]dt+O\left(\frac{1}{M^4}\right). \tag{22}$$

TABLE 1. Amplitude of the Wave Field $|\mathcal{U}_{\mathcal{D}}(r,\varphi)|$ near the Cylinder (Dirichlet problem, k$a$ = 50, kr = 52)

| $\varphi$ | $\lvert\mathcal{U}\rvert$ of Fock | $\lvert\mathcal{U}\rvert$ with correction | $\lvert\mathcal{U}\rvert$ exact |
|---|---|---|---|
| 70° | I.366 | I.3239 | I.3248 |
| 80° | 0.937 | 0.9I82 | 0.9I89 |
| 90° | 0.549 | 0.54I6 | 0.54I9 |
| I00° | 0.274 | 0,2703 | 0.2704 |
| II0° | 0.II9 | 0.II64 | 0.II64 |
| I20° | 0.046 | 0.0449 | 0.0450 |

These expressions have the same form as formulas (8), (10) for the circular cylinder and differ from them only in the factor $1/\sqrt{\sin\theta}$ and numerical coefficients.

For the currents on the surface of the sphere we obtain expressions analogous to (12), (14):

$$I_{\mathcal{D}}=\frac{e^{iks}}{M\sqrt{\sin\theta}}\left\{\overset{00}{f_1}+\frac{1}{60M^2}\left[19\overset{10}{f_1}+\overset{21}{f_2}\right]+O\left(\frac{1}{M^4}\right)\right\}, \tag{23}$$

$$I_N=\frac{e^{iks}}{\sqrt{\sin\theta}}\left\{\overset{00}{g_1}+\frac{1}{60M^2}\left[11\overset{10}{g_1}+\overset{31}{g_2}-21\overset{01}{g_2}\right]+O\left(\frac{1}{M^4}\right)\right\}. \tag{24}$$

The magnitude of the correction obtained in [4] for the case of a sphere also differs from our correction (24) by a factor of two.

## 3. Numerical Comparison of the Asymptotic Formulas with the Exact Values of the Fields

The main purpose of the present work is to demonstrate the effectiveness of using the corrections found in numerical computations of fields in the semishadow zone. The considerable increase in the accuracy of the asymptotics formulas of V. A. Fock in using the corrections is clearly evident from the tables presented. In Tables 1 and 2 results are compared of computing the amplitude of the wave field near a cylinder according to the formulas of V. A. Fock without corrections (the first column) and according to formulas (8), (10) (second column) with the exact values of the field amplitude (third column). The computations were carried out on the BÉSM-6 computer. The programming of the correction functions reduced to a minor modification of the programs for computing the Fock functions described in [10].

TABLE 2. Amplitude of the Wave Field $|u_N(r,\varphi)|$ near the Cylinder (Neumann problem, k$a$ = 30, kr = 32)

| $\varphi$ | $\lvert\mathcal{U}\rvert$ of Fock | $\lvert\mathcal{U}\rvert$ with correction | $\lvert\mathcal{U}\rvert$ exact |
|---|---|---|---|
| 70° | I.246 | I.300 | I.296 |
| 80° | I.453 | I.46I | I.459 |
| 90° | I.4I6 | I.404 | I.402 |
| I00° | I.20I | I.I77 | I.I79 |
| II0° | 0.9I7 | 0.887 | 0.890 |
| I20° | 0.653 | 0.625 | 0.620 |

TABLE 3. Amplitude of the Current $|I_D|$ on the Surface of the Sphere (Dirichlet problem, $\theta = 90°$)

| $ka$ | $\lvert I_D \rvert$ of Fock | $\lvert I_D \rvert$ with correction | $\lvert I_D \rvert$ exact |
|---|---|---|---|
| 20 | 0.360 | 0.38176 | 0.38179 |
| 30 | 0.315 | 0.32878 | 0.32879 |
| 40 | 0.286 | 0.29636 | 0.29637 |
| 50 | 0.265 | 0.27371 | 0.27372 |

In Table 3 a similar comparison of asymptotic and precise results is made for the current on the surface of the sphere (23) (on the equator of the sphere) in the case of the Dirichlet problem.*

The effectiveness of using the corrections to the Fock asymptotics in the simplest problems indicates favorable prospects for investigation of corrections for surfaces of general form.

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