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Finite groups with invariant fourth maximal subgroups

By
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Let G be a finite group. Then a series of subgroups $G = X_0, X_1, \ldots, X_n$ of G is called a maximal series of G if X_i is a maximal subgroup of X_{i-1} $(1 \leqq i \leqq n)$. A subgroup H of G is called an n-th maximal subgroup of G if there exists at least one maximal series X_i $(0 \leqq i \leqq n)$ of G so that $H = X_n$. Naturally an n-th maximal subgroup H of G can be also an m-th $(m \neq n)$ maximal subgroup of G. If we speak about n-th maximal subgroups of a group G, then we suppose that they really exist.

The object of this paper is to generalize the following theorems of B. HUPPERT [6]:

I. If each second maximal subgroup of G is normal in G, then G is supersoluble. If the order of G is divisible by at least three different primes, G is nilpotent.

II. Let each third maximal subgroup of G be normal in G. Then G' is nilpotent and the order of each principal factor of G is divisible by at most two (equal) primes. If the order of G is divisible by at least three different primes, G is supersoluble.

Let G denote a finite group with the property that each fourth maximal subgroup of G is normal in G. Then the following theorems are valid:

Theorem 1. *If G is a simple group, then G is isomorphic to $LF(2, p)$, where $p = 5$ or p is such prime that $p - 1$ and $p + 1$ are products of at most three primes and $p \equiv \pm 3$ or ± 13* (mod 40).

Theorem 2. *If G is a non-soluble and non-simple group, then G is isomorphic to $SL(2, 5)$.*

Theorem 3. *If G is a soluble group, then G' is nilpotent or G is isomorphic to the holomorph of the elementary abelian group of order p^2 by the dihedral group of order $2 \cdot q$ (p, q are primes; q is odd, $p \neq q$) or G is one of two representation groups (in the sense of* SCHUR*) of the symmetric group S_4. If the order of G is divisible by at least four different primes, then G is supersoluble.*

Theorem 4. *If G is soluble, then the order of each principal factor of G is divisible by at most three (equal) primes.*

Professor MAHLER communicated to me that it is unknown whether we have infinitely many primes p which satisfy the conditions of Theorem 1.

Corollary 1. *Let G be a finite group with the property that each fourth maximal subgroup is invariant. If G has a principal factor of order p^3 (p prime), then G' is nilpotent.*

Corollary 2. *Let G be a finite group with the following properties:*

(a) *The order of G is divisible by exactly two different primes,*

(b) *G' is non-nilpotent,*

(c) *each fourth maximal subgroup of G is invariant.*

Then if G has more than one involution, G is isomorphic to S_4. If G has only one involution, G is the group of order 48 given by:

$$J^2 = [T_1, J] = [T_2, J] = [T_3, J] = 1,$$
$$T_1^2 = T_2^2 = T_3^2 = (T_1 T_2)^3 = (T_2 T_3)^3 = [T_1, T_3] = J.$$

Proof of Theorem 1. Let G be a (non-cyclic) simple group with the property that each fourth maximal subgroup of G is the unit group 1. Then by a celebrated theorem of FEIT-THOMPSON [*3*] $G_2 \neq 1$, where G_2 denotes a Sylow 2-subgroup of G. G_2 cannot be a maximal subgroup of G. If G_2 would be a maximal subgroup of G, then each third maximal subgroup of G_2 is 1 and the order $|G_2|$ of G_2 is less or equal 8. Hence the class of G_2 is $\leqq 2$. But then by a theorem of DESKINS [*1*] G would be a soluble group. Because G_2 cannot be a cyclic group, G_2 is the elementary abelian group of order 4. We can suppose that $C(G_2) = G_2$, where $C(G_2)$ denotes the centralizer of G_2 in G. If $C(G_2) \neq G_2$, then $C(G_2)$ is a maximal subgroup of G. Hence $C(G_2) = N(G_2)$, where $N(G_2)$ denotes the normalizer of G_2 in G. But this is impossible by a well known theorem of BURNSIDE. Hence G is a simple group whose 2-Sylow subgroup is the elementary abelian group of order 4, which is equal to its centralizer. We can apply a recent theorem of GORENSTEIN-WALTER [*5*] which shows that G is isomorphic to $LF(2, q)$, where q is an odd prime power $\neq 3$. But all subgroups of $LF(2, q)$ are known (DICKSON [*2*]) and we see that $q = 5$ or q is such prime that $q-1$ and $q+1$ are products of at most three primes and $q \equiv \pm 3$ or $\pm 13 \pmod{40}$.

Proof of Theorem 2. Let G be a non-soluble and non-simple group with the property that each fourth maximal subgroup is invariant. Let N be a maximal normal subgroup of G. Then the following is true:

(a) G/N is a simple group $LF(2, p)$ of theorem 1. Take a maximal subgroup H/N of G/N. By the HUPPERT'S theorem II, H is a soluble group. Hence N is a soluble group and if G/N would be soluble, then G would be soluble. Consequently G/N is a non-abelian simple group. Each fourth maximal subgroup of G/N is the unit group. Hence $G/N \cong LF(2, p)$ $(p > 3)$.

(b) N is a nilpotent group $\neq 1$. Take a third maximal subgroup K/N of G/N. Then K is nilpotent and so is N.

(c) $N = Z(G)$, where $Z(G)$ denotes the center of G. For each prime $p \neq 2$ we have $N_p \leqq Z(G)$. For let us take a third maximal subgroup H/N of G/N so that $|H/N| = 2$. Then H is a nilpotent group. Hence $C(N_p) \nleqq N$. N_p is normal in G $(N_p \lhd G)$ and so is $C(N_p)$. Assume that $C(N_p) \neq G$. Then $C(N_p)$ is a soluble group and $C(N_p)$. $N = G$ would be a soluble group. Hence $C(N_p) = G$ for each odd prime p. If $2 \nmid |N|$, then we have $N = Z(G)$. So we can suppose that $2 \mid |N|$. If N would not be a 2-group, then there exists an odd prime r so that $N_r \neq 1$. Let us take a prime q with the properties $q \neq 2$, $q \neq r$ and $q \mid |G/N|$. Consider a subgroup K/N of order q of G/N. The subgroup is a second or a third maximal subgroup of G. In any case, each proper subgroup of K is nilpotent. Hence by a theorem of IWASAWA [7] K is a nilpotent group because $|K|$ is divisible by at least three different primes. We get $C(N_2) \nleq N$ and, finally, $N = Z(G)$. Now we assume that N is a 2-group. Let us consider a subgroup L/N of G/N of order p $(p \neq 2)$. If L is a third maximal subgroup, then L is nilpotent. If L is a second maximal subgroup, then L is supersoluble by the HUPPERT's theorem I. Hence $L_p \lhd L$ $(p > 2)$ and L is again nilpotent. Consequently $C(N) \nleq N$ and we have $N = Z(G)$.

(d) $G' = G$. If $G' < G$, then $NG' = G$ and G would be a soluble group because G' is soluble by the HUPPERT's theorem II.

(e) G is isomorphic to $SL(2, p)$ and $|N| = 2$. This isomorphism we get by a theorem of SCHUR [*10*].

(f) Each third maximal subgroup K/N of G/N has order 1 or 2. Let K/N be a third maximal subgroup of G/N. If $|K/N| \neq 1$, then $|K/N| = q$ (prime). If $q \neq 2$, then $K_q \neq 1$ and K_q is a fourth maximal subgroup of G. Hence $K_q \lhd G$. But then $K = K_q N$ is a normal subgroup of G, which is impossible.

Finally, we have that each second maximal subgroup of G/N is nilpotent. Hence $G/N \cong LF(2, 5)$ (see JANKO [*8*]) and $G \cong SL(2, 5)$.

Proof of Theorem 3. Let G be a soluble group with the property that each fourth maximal subgroup is invariant. We consider three cases:

Case A. The order of G is $p^\alpha q^\beta$ (p, q primes; $\alpha, \beta > 0$).

We have to prove (by induction) that G' is nilpotent or G is isomorphic to S_4 or to the representation group T_4 of S_4, where T_4 has only one involution.

Let M be a minimal normal subgroup of G. Then $(G/M)'$ is nilpotent or $G/M \cong S_4$ or $G/M \cong T_4$. Representation group T_4 has the center $Z(T_4)$ of order 2 and $T_4/Z(T_4) \cong S_4$.

Suppose that G/M is isomorphic to S_4 or T_4. Then M is a 2-group or a 3-group. If M would be a 3-group, then we consider a subgroup K/M (of G/M) which has order 2 in the case $G/M \cong S_4$ and order 4 in the case $G/M \cong T_4$. We can choose K as a third maximal subgroup of G. Hence K is a nilpotent group. The 2-Sylow subgroup K_2 of K is contained in a fourth maximal subgroup B which is contained in K. Hence $K_2 \lhd G$ and $K = K_2 \cdot M \lhd G$. But this is impossible because S_4 has not a normal subgroup of order 2. It follows that M is a 2-group. Hence the Fitting subgroup $F = F(G)$ of G is a 2-group because the 3-Sylow subgroup G_3 of G is not normal in G. Namely, if G_3 would be normal in G, then the 3-Sylow subgroup of S_4 would be normal in S_4. (S_4 is a homomorphic image of G.)

Now we assume that the Fitting subgroup of G is not a p-group. Then we have two minimal normal subgroups M_1 and M_2 of G and $(G/M_1)'$ and $(G/M_2)'$ are nilpotent. Namely, if G/M_1 or G/M_2 would be isomorphic to S_4 or T_4, then $F(G)$ would be a 2-group. We can suppose that M_1 and M_2 are both contained in G'. Then G'/M_1 and G'/M_2 are nilpotent. Hence $G' = G'/(M_1 \cap M_2)$ is nilpotent.

Now we suppose that the Fitting subgroup $F(G)$ is a p-group ($\neq 1$). If $q^\beta \geqq q^4$, then it would exist a fourth maximal subgroup B (of G) whose order is divisible by q. B is nilpotent and normal in G. Hence $B_q \lhd G$. But this is impossible because F is a p-group. Hence $q^\beta \leqq q^3$.

Suppose, at first, that $F = G_p$. Then $|G_q| = q^3$ because, otherwise, G' is nilpotent. Each subgroup F_1 of F with $|F/F_1| = p$ is normal in G and $F_1 G_q$ is maximal subgroup (of G). Consequently $F_1 = 1$ and $|F| = p$. By a theorem of FITTING [4] we get $C(F) = F$. Hence G_q is a group of automorphisms of F and therefore cyclic. We have $G' \leqq F$ and G' is nilpotent.

Suppose now that $F \neq G_p$. Then we have $F G_q \neq G$ and $|G_q| \leqq q^2$. Assume that $|G_q| = q^2$. Then $G_q F$ must be a maximal subgroup of G. Let F_1 be a subgroup of index p of F. Then $F_1 \lhd G$ and $G_q F_1$ is a maximal subgroup of $G_q F$. If $F_1 \neq 1$, then again we would have a fourth maximal subgroup B of G with $B_q \neq 1$. Consequently $|F| = p$, and $C(F) = F$. G/F is a cyclic group and G' is nilpotent.

So we can assume that $|G_q| = q$. Consider a maximal normal subgroup M/F of G/F. Then we have $|G/M| \neq q$ because $F \neq G_p$. Hence $|G/M| = p$. If F is a minimal normal subgroup of M, then by the HUPPERT's theorem II, $|F| \leqq p^2$. If $|F| = p$, then G/F is cyclic and we are finished. So in this case we can suppose that $|F| = p^2$ and F is the elementary abelian group. G/F is an automorphism group of F. Let us take a principal series of G which contains M and F. First principal factor X/F between M and F must be a q-factor because F is a maximal normal nilpotent subgroup of G. Hence $F G_q$ is normal in G. If $M \neq F G_q$, then it must be $|M/F G_q| = p$ because, otherwise, G_q would be contained in a fourth maximal subgroup of G. But then F is a third maximal subgroup of G and each subgroup of order p of F must be normal in G and G is a supersoluble group. Hence G' is nilpotent. So we can suppose that $M = F G_q$, $|G| = p^3 q$ and F is a minimal normal subgroup of G. By a theorem of HUPPERT [6], a finite group is supersoluble if and only if every maximal subgroup has prime index. Because G is not a supersoluble group, there exists a maximal subgroup S of index p^δ $(2 \leqq \delta \leqq 3)$. But $\delta = 3$ is impossible because G_q is a proper subgroup of M. Hence $|S| = pq$. If $p > q$, then $S_p \lhd S$. Hence $S_p \lhd G$ because $N(S_p) \lneqq S$. On the other hand, $S_p < F$ because F contains every nilpotent normal subgroup of G. But then G would be a supersoluble group. So we can suppose that $q > p$ and, because G/F is an automorphism group of F, we have $q = p + 1$. Consequently $p = 2$, $q = 3$, and the order of G is 24. The subgroup S is not normal in G, also S_2 is not normal in S and S_3 is normal in S but not in G. Now we represent the group G by permutations of the cosets of S. This is a faithful representation of degree 4. Because the order of G is 24, G is isomorphic to S_4. On the other hand, every fourth maximal subgroup of S_4 is the unit group and the commutator subgroup of S_4 is not nilpotent. S_4 is also the holomorph of the elementary abelian group of order 2^2 by the dihedral group of order 2.3.

It remains to consider the case where F is not a minimal normal subgroup of M. If $|M/F| \neq q$, then $G_q F \neq M$ and there exists a fourth maximal subgroup (of G), whose order is divisible by q, which is impossible. Hence $|M/F| = q$ and $F = M_p$. Let F_1 be a maximal normal subgroup of M contained in F. Then by the HUPPERT's theorem II $|F/F_1| \leqq p^2$. $F_1 G_2$ is a maximal subgroup of M because, otherwise, G_q is contained in a fourth maximal subgroup of G. Also G_q is a maximal subgroup of $F_1 G_q$. Let F_2 be a maximal subgroup of F_1. Then F_2 (as a fourth maximal subgroup of G) is normal in G. And because G_q is maximal in $F_1 G_q$, the subgroup F_1 is a minimal normal subgroup of G. Hence $|F_2| = 1$ and $|F_1| = p$. If $|F/F_1| = p$, then G is a supersoluble group. So we can assume that $|F/F_1| = p^2$. Each subgroup of order p of F is normal in G. If F would have two subgroups of order p, then F_1 would not be a maximal normal subgroup of M contained in F. Hence F has only one subgroup of order p. If p is odd, then F is cyclic and G/F is cyclic. We can suppose that $p=2$ and that F is the quaternion group. Then $q=3$ and $G/F_1 \cong S_4$ is the automorphism group of the quaternion group F. The subgroup G' is not nilpotent because $(S_4)'$ is not nilpotent. It follows that G' must be a maximal subgroup of G because, otherwise, G' would be supersoluble, by the HUPPERT's theorem I, and $(G')_3 = G_3$ would be normal in G. On the other hand, we have $G' \leqq M$. Hence $G' = M$ and $G' \geqq F_1$. By a theorem of SCHUR [11], G is a representation group of S_4. But S_4 has exactly two representation groups T_4 and T_4^*, where T_4 has only one involution and T_4^* has more than one involution. Each fourth maximal subgroup of T_4^* is not normal in T_4^*. On the other hand, T_4 has the property that each fourth maximal subgroup is invariant because each fourth maximal subgroup of T_4 is either the unit group or the center of T_4. Hence in this case G is isomorphic to T_4.

Case B. The order of G is divisible by exactly three different primes.

We have to prove by induction that G' is nilpotent or G is isomorphic to the holomorph H of an elementary abelian group P of order p^2 by a dihedral group D of order $2q$, where $2, p, q$ are three different primes.

Let M be a minimal normal subgroup of G. Suppose that G/M is isomorphic to the holomorph H. P is a minimal normal subgroup of H. Hence the dihedral group D is a maximal subgroup of H. Consequently the unit group of H is a third maximal subgroup of H and M is a third maximal subgroup of G. It follows that each maximal subgroup of M is normal in G. Hence the order of M is a prime 2, p or q. If $|M| \neq p$, then we consider a subgroup K/M of order p of G/M. K_p is a fourth maximal subgroup of G. Hence $K_p \triangleleft G$ and $K = M K_p \triangleleft G$. But this is impossible because H is not a supersoluble group. If $|M| = p$, then we consider the subgroup L/M of order p^2 of G/M. Every subgroup of order p of L is normal in G. If L would have two subgroups of order p, then G/M would have a normal subgroup of order p, which is not true. Because p is odd, L must be a cyclic group. Hence P would be also a cyclic group. This is a contradiction. So we have proved that $(G/M)'$ is nilpotent.

If G would have two minimal normal subgroups M_1 and M_2, then $(G/M_1)'$ and $(G/M_2)'$ are nilpotent. Hence G' is nilpotent.

We assume that G has a unique minimal normal subgroup. Then the Fitting subgroup F of G is a p-group. Suppose that the order of G is $p^\alpha q^\beta r^\gamma$ (p, q, r are different primes).

At first we consider the case that F is not a p-Sylow subgroup of G. Then $F G_{q,r}$ is, by the HUPPERT's theorem II, a supersoluble group, where $G_{q,r}$ denotes a Hall subgroup of order $q^\beta r^\gamma$. We have $|G_{q,r}| = qr$ because, otherwise, there is a subgroup of order q or r which is contained in a fourth maximal subgroup of G, which is impossible (because $C(F) \leqq F$). Because of the same reason, $F G_{q,r}$ is a maximal subgroup of G. If F would not be a group of order p, then there exists a subgroup $\overline{F} \neq 1$ of F, with the properties $\overline{F} \lhd F G_{q,r}$ and $|F/\overline{F}| = p$. The subgroup $G_{q,r}\overline{F}$ is then a second maximal subgroup of G and G_q would be contained in a fourth maximal subgroup of G. Hence $|F| = p$ and $N(F)/C(F) = G/F$ is a cyclic group.

It remains to consider the case where F is a p-Sylow subgroup of G. We have $G = F G_{q,r}$. Suppose, at first, that $G_{q,r}$ is not a maximal subgroup of G. Then F is not a minimal normal subgroup of G and $G_{q,r}$ is a second maximal subgroup of G. Let $\overline{F}$ be a minimal normal subgroup (of G) contained in F. Then $G_{q,r}$ is maximal in $M = G_{q,r}\overline{F}$ and M is maximal in G. M is a supersoluble group, by the HUPPERT's theorem II, and $\overline{F}$ is a minimal normal subgroup of M. Hence $|\overline{F}| = p$. If $|F/\overline{F}| \geqq p^3$, then we consider the subgroup $G_q F$. By the HUPPERT's theorem II, there exists a subgroup F^* with the following properties: $\overline{F} < F^* < F$ and $F^* \lhd G_q F$. Then we have $\overline{F} G_q < F^* G_q < F G_q$ and G_q would be contained in a fourth maximal subgroup of G. So we have $|F/\overline{F}| \leqq p^2$ and also $|G_{q,r}| = q\,r$. We can suppose that $|F| = p^3$ because in the case $|F| = p^2$ the group G is supersoluble. Each subgroup of order p of F is normal in G. The group G has the unique minimal normal subgroup. Hence F is a cyclic group or the quaternion group. In the first case G' is nilpotent. In the second case $|N(F)/C(F)| = |G/\overline{F}| = 4\,q\,r$. This is impossible because the automorphism group of the quaternion group has order 24.

So we can suppose that F is a minimal normal subgroup of G. Then $G_{q,r}$ is maximal in G and $|G_{q,r}| < q^2 r^2$. Let us take a maximal normal subgroup M(of G) which contains F and suppose that $|G/M| = q$. Suppose that $q^2 \mid |G|$. Then we have $|G_r| = r$. By the HUPPERT's theorem II, M is a supersoluble group and F is a minimal normal subgroup of M. (Otherwise G_r would be contained in a fourth maximal subgroup.) Hence $|F| = p$ and G' is nilpotent. So we can suppose that $q^2 \nmid |G|$. If $r^2 \mid |G|$, then we consider a subgroup H/F of order r of M/F. The subgroup H is a second maximal subgroup of G and, by the HUPPERT's theorem I, H is a supersoluble group. F must be a minimal normal subgroup of H. Hence $|F| = p$ and G' is nilpotent. So we have $|G_{q,r}| = q\,r$ and $r > q$. If F is a minimal normal subgroup of M, then by the HUPPERT's theorem II we get $|F| \leqq p^2$. If F would not be a minimal normal subgroup of M, then there exists a subgroup $\overline{F} \neq 1$ with the properties: $\overline{F} < F$ and $\overline{F} \lhd M$. But then G_r is maximal in $G_r\overline{F}$ and $G_r \cdot \overline{F}$ is maximal in M.

By using HUPPERT's theorems I and II we get $|\bar{F}|=p$ and $|F/\bar{F}|\leqq p^2$. If $|F|=p^3$, then each subgroup of order p of F is normal in G. If F is cyclic, then G' is nilpotent. If F is a quaternion group, then $|N(F)/C(F)|=|G/Z(F)|=4r\,q$, which is impossible. Hence we can suppose that $|F|=p^2$. If $p=2$, then $r\,q=6$, which is not true. Hence we have $r|(p-1)$ and we can suppose that $G_{q,r}$ is a non-abelian group. Now we can apply the following theorem of NAHAMURA [9]: "Let G be a finite group of odd order n such that n does not contain any factor p^r, $(r\geqq 3)$, then G' is abelian." Hence we can suppose that G is a group of even order. Consequently $q=2$ and $G_{q,r}$ is a dihedral group of order $2r$. G is the holomorph H of the elementary abelian group of order p^2 by the dihedral group of order $2r$. On the other hand the holomorph H has non-nilpotent commutator subgroup and every fourth maximal subgroup of H is the unit group.

Case C. The order of G is divisible by more than three different primes.

We take a maximal subgroup M of G. $|M|$ is divisible by at least three different primes. Hence M is supersoluble by the HUPPERT'S theorem II. Every proper subgroup of G is supersoluble and the order of G is divisible by at least four different primes. Then by a theorem of HUPPERT [6], G is supersoluble and G' is nilpotent.

Proof of Theorem 4. The order of a principal factor of a soluble group G is p^n (p prime). The maximal n which occurs in the order p^n of any principal factor of G is called the rank of G. Let G be a soluble group with the property that each fourth maximal subgroup is invariant. We have to prove that the rank of G is less or equal 3. By theorem 3 we have to consider only the case where G' is nilpotent and the order of the group is divisible by exactly two or three different primes.

Case A. The order of G is divisible by exactly two different primes.

Let N be a minimal normal subgroup of order p^γ of G. If G has a normal subgroup R of order $q^\mu \neq 1$ $(q\neq p)$, then by induction, there exists a subgroup $\bar{N}\neq 1$ of N with the properties $R\bar{N}/R \trianglelefteq G/R$ and $|\bar{N}|\leqq p^3$. $\bar{N}=R\bar{N}\cap N$ is normal in G. Hence $\bar{N}=N$ and $|N|\leqq p^3$. By induction, the rank of G/N is $\leqq 3$ and we are finished. So we can suppose that the Fitting subgroup F of G is a p-group. The commutator subgroup of G is nilpotent. Hence $G_p \triangleleft G$. Let N be a minimal normal subgroup of G which is contained in the center $Z(G_p)$ of G_p. If $N\neq G_p$, then N is a minimal normal subgroup of NG_q $(q\neq p)$ and NG_q is a proper subgroup of G. Then by the HUPPERT's theorem II we get $|N|\leqq p^2$. If $N=G_p=F$, then G_q is a maximal subgroup of G and G_p is a minimal normal subgroup of G. We have $|G_q|\leqq q^3$ because, otherwise, we would have a non-trivial q-group contained in a fourth maximal subgroup of G. If $|G_q|=q^3$, then $|N|=p$; if $|G_q|=q^2$, then $|N|\leqq p^2$ and if $|G_q|=q$, then $|N|\leqq p^3$. By induction we have, finally, that the rank of G is less or equal 3.

Case B. The order of G is divisible by exactly three different primes. The order of the group is $p^\alpha q^\beta r^\gamma$.

In the same way as in the case A we see that we can suppose that $F=F(G)=G_p$ is a minimal normal subgroup of G. The Hall subgroup $G_{q,r}$ is a maximal subgroup of G and the order of $G_{q,r}$ is divisible by two or three primes. But then we have $|F|\leqq p^2$ or $|F|=p$ and by induction we are finished.

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