KÖVARI, T., and CH. POMMERENKE
Math. Zeitschr. 99, 193—206 (1967)

On Faber Polynomials and Faber Expansions

T. KÖVARI and CH. POMMERENKE

Received October 8, 1966

1. Estimates for Faber Polynomials

Let

$$\psi(\zeta)=\zeta+b_0+b_1\zeta^{-1}+\cdots$$

be meromorphic and univalent in $|\zeta|>1$. Thus, $z=\psi(\zeta)$ maps $\{|\zeta|>1\}$ onto the complement of a continuum K of logarithmic capacity one. The Faber polynomials $F_n(z)=z^n+\cdots$ are defined by

$$\frac{\psi'(\zeta)}{\psi(\zeta)-z}=\sum_{n=0}^{\infty}\frac{F_n(z)}{\zeta^{n+1}}. \tag{1.1}$$

It is known [*9*] that

$$\log n<\max_{z\in K}\sum_{\nu=1}^{n}\frac{1}{\nu}|F_\nu(z)|^2<4\log n+8. \tag{1.2}$$

In particular [*13*, p. 134]

$$\max_{z\in K}|F_n(z)|<[n\log(n+1)]^{\frac{1}{2}}+e^{\frac{1}{2}}.$$

We shall give a sharper estimate.

Theorem 1. (i) *There are absolute constants A and $\alpha<\frac{1}{2}$ such that*

$$\max_{z\in K}|F_n(z)|\leqq A\,n^{\alpha}.$$

(ii) *There exists a function $\tilde{\psi}(\zeta)$ such that, for each fixed z,*

$$|\tilde{F}_n(z)|>n^{0.138}\qquad \textit{for infinitely many } n.$$

Proof of (i). This statement follows at once from the next lemma which we will also need later on.

Lemma 1. *For $z\in E$*

$$\int_0^{2\pi}\left|\frac{\psi'(r\,e^{i\vartheta})}{\psi(r\,e^{i\vartheta})-z}\right|d\vartheta<\frac{A_0}{(1-1/r)^{\alpha}} \tag{1.3}$$

where A_0 and $\alpha<\frac{1}{2}$ are absolute constants.

The proof will use the same method as in [3]. We may assume that $z=0$. Then $\psi(\zeta)\neq 0$ for $|\zeta|>1$. Let $0<\delta<1$. We write $t=re^{i\vartheta}$, $\rho=1/r$. Thus

$$\left\{\int_0^{2\pi}\left|\frac{\psi'(t)}{\psi(t)}\right|^{1+\delta}d\vartheta\right\}^2\leqq\int_0^{2\pi}\left|\frac{\psi'(t)}{\psi(t)}\right|^2 d\vartheta\cdot\int_0^{2\pi}\left|\frac{\psi'(t)}{\psi(t)}\right|^{2\delta}d\vartheta. \tag{1.4}$$

Using (1.1) and (1.2) we can easily show that

$$\int_0^{2\pi}\left|\frac{\psi'(t)}{\psi(t)}\right|^2 d\vartheta\leqq\frac{A_1}{1-\rho}\cdot\log\frac{1}{1-\rho}, \tag{1.5}$$

where $A_1, \ldots$ denote absolute constants. Let

$$J(\rho)=\int_0^{2\pi}\left|t\frac{\psi'(t)}{\psi(t)}\right|^{2\delta}d\vartheta.$$

From the power series expansion of $[t\psi'(t)/\psi(t)]^\delta$ and Parceval's formula we obtain

$$\begin{aligned}\rho J''(\rho)+J'(\rho)&=4\int_0^{2\pi}\left|\frac{d}{dt}\left[t\frac{\psi'(t)}{\psi(t)}\right]^\delta\right|^2 d\vartheta\\ &=4\delta^2\int_0^{2\pi}\left|\frac{1}{t}+\frac{\psi''(t)}{\psi'(t)}-\frac{\psi'(t)}{\psi(t)}\right|^2\left|t\frac{\psi'(t)}{\psi(t)}\right|^{2\delta}d\vartheta.\end{aligned}$$

The distortion theorems imply that

$$|\psi''(t)/\psi'(t)|\leqq A_2/(1-\rho),\qquad |\psi'(t)/\psi(t)|\leqq A_3/(1-\rho).$$

Hence

$$\rho J''(\rho)\leqq A_4\delta^2(1-\rho)^{-2}J(\rho).$$

It follows that

$$J(\rho)\leqq A_5(1-\rho)^{-\alpha_1\delta^2}.$$

Here $\alpha_1, \ldots$ denote absolute constants $<\frac{1}{2}$. Therefore we obtain from (1.4) and (1.5)

$$\int_0^{2\pi}\left|\frac{\psi'(t)}{\psi(t)}\right|^{1+\delta}d\vartheta\leqq A_6(1-\rho)^{-\frac{1}{2}-\alpha_2\delta^2}.$$

Hence the Hölder inequality gives

$$\int_0^{2\pi}\left|\frac{\psi'(t)}{\psi(t)}\right|d\vartheta\leqq A_7(1-\rho)^{-\frac{1}{2}+\frac{\delta}{2}-\alpha_3\delta^2}.$$

Choosing $\delta>0$ sufficiently small we obtain (1.3).

Proof of (ii). Let

$$\tilde{\psi}(\zeta)=\zeta+a_1\zeta^{-1}+\cdots\qquad(|\zeta|>1)$$

be the function constructed in [11]. Then

$$|a_n|>n^{0.139-1}\qquad\text{for infinitely many } n. \tag{1.6}$$

From (1.1), we can get the relation

$$(n+1)\,a_n = z\,\tilde{F}_n(z) - \tilde{F}_{n+1}(z) - \sum_{\nu=1}^{n-1} a_\nu \tilde{F}_{n-\nu}(z) \qquad (n=1,2,\ldots).$$

Suppose that, for some fixed z,

$$|\tilde{F}_\nu(z)| \leqq C_1\,\nu^\beta \qquad (\nu=1,2,\ldots)$$

where $\beta=0.138$ and $C_1, \ldots$ are certain constants. Then it follows that

$$(n+1)\,|a_n| \leqq C_2(n+1)^\beta + C_1 \sum_{\nu=1}^{n-1} |a_\nu|\,(n-\nu)^\beta.$$

SCHWARZ'S inequality and the area theorem give

$$\begin{aligned}(n+1)\,|a_n| &\leqq C_2(n+1)^\beta + C_1 \left(\sum_{\nu=1}^{n-1} \nu\,|a_\nu|^2\right)^{\frac{1}{2}} \left(\sum_{\nu=1}^{n-1} \frac{1}{\nu}\,(n-\nu)^{2\beta}\right)^{\frac{1}{2}} \\ &\leqq C_2(n+1)^\beta + C_1\,n^\beta (1+\log n)^{\frac{1}{2}} < C_3\,(n+1)^{0.1385},\end{aligned}$$

in contradiction to (1.6).

2. The Faber Polynomials of Convex Sets

We shall assume now that the set K is convex. Then the function

$$\begin{aligned}h(t,s) &= \frac{t\,\psi'(t)}{\psi(t)-\psi(s)} - \frac{1}{2}\,\frac{t+s}{t-s} = \frac{1}{2} + \sum_{n=1}^{\infty} [F_n(\psi(s)) - s^n]\,t^{-n} \\ &= \frac{1}{2} - \sum_{n=1}^{\infty} \left[\frac{t\,\psi'(t)}{n}\,F_n'(\psi(t)) - t^n\right] s^{-n}\end{aligned} \tag{2.1}$$

is analytic in $|s|>1$, $|t|>1$ and has positive real part [*4*, Lemma 2]. The second identity (2.1) and CARATHÉODORY'S coefficient estimate together with SCHWARZ' lemma imply

$$\left|\frac{t\,\psi'(t)}{n}\,F_n'(\psi(t)) - t^n\right| \leqq \frac{1}{|t|} \qquad (n=1,2,\ldots;\ |t|>1).$$

In particular, for $n=1$ we obtain a new proof of the inequality

$$|\psi'(t)-1| \leqq \frac{1}{|t|^2} \qquad (|t|>1),$$

due to GRÖTZSCH [*7*] and GOLUSIN [*5*].

Theorem 2. *If K is convex but not a segment then*

$$|F_n(\psi(\zeta)) - \zeta^n| < 1 \qquad (|\zeta| \geqq 1).$$

Hence all zeros of $F_n(z)$ lie in the interior of K.

Proof. The function $h(t, s)$ is continuous in $|t|>1, |s|\geqq 1$. Let s be fixed, $|s|\geqq 1$. Then $\operatorname{Re} h(t, s)>0, h(\infty, s)=\frac{1}{2}$. Hence the first identity (2.1) and Carathéodory's coefficient estimate show that

$$|F_n(\psi(s))-s^n|\leqq 1, \tag{2.2}$$

and equality can hold only if

$$h(t, s)=\frac{1}{2}\,\frac{\zeta^m+a}{\zeta^m-a} \qquad (m\geqq 1,\ |a|=1). \tag{2.3}$$

Suppose that $m\geqq 2$. Using (2.1) and integrating, we obtain

$$\psi(t)=\psi(s)+(t-s)\left[1-a\,t^{-m}+\cdots\right]^{1/m}$$

$$=t+\cdots+\frac{a\,s}{m}\,t^{-m}+\cdots$$

and therefore

$$t\,\psi'(t)=t+\cdots-a\,s\,t^{-m}+\cdots.$$

Since $t\psi'(t)$ is starlike in $|t|>1$ it follows [2] that $|as|\leqq 2/(m+1)\leqq 2/3$, in contradiction to $|as|\geqq 1$. Hence $m=1$. But in this case integration of (2.3) shows that K is a segment. Therefore equality cannot hold in (2.2).

3. Faber Expansions, General Remarks

Let first K be a closed Jordan domain, and let the function $f(z)$ be analytic in the interior of K and continuous on K. Let again

$$\psi(\zeta)=\zeta+b_0+b_1\,\zeta^{-1}+\cdots \tag{3.1}$$

map $|\zeta|>\rho$ onto the exterior of K. Then $\psi(\zeta)$ is continuous in $|\zeta|\geqq 1$.

The *Faber coefficients* of the function $f(z)$ are defined by

$$c_m=\frac{1}{2\pi i}\int_{|t|=1} f(\psi(t))\,\frac{dt}{t^{m+1}} \qquad (m=0, 1, \ldots). \tag{3.2}$$

The formal series

$$\sum_{m=0}^{\infty} c_m\,F_m(z)$$

is the *Faber series* or *Faber expansion* of the function $f(z)$.

One can easily extend this definition to the more general case when K is any continuum whose complement is connected. Using Fatou's theorem about the radial limits of bounded analytic function, one can define $\psi(\zeta)$ almost everywhere on $|\zeta|=1$ as a bounded integrable function. Therefore $f(\psi(\zeta))$ is also a bounded integrable function.

In the remaining sections of this paper we shall consider the following question: Under what conditions does the Faber expansion of $f(z)$ converge uniformly on K, and represent the function $f(z)$?

AL'PER has shown [1] that if the boundary Γ of K is a smooth rectifiable Jordan curve, which satisfies a certain smoothness condition[1], then, as far as uniform convergence is concerned, the Faber series behaves very much like a Fourier series.

It will be shown that the situation is rather similar if we assume that Γ is a Jordan curve of *bounded rotation*. We recall the definition of such curves [8].

Let $\Gamma: z(\tau)$ be a smooth Jordan curve, and let $\vartheta(\tau)$ denote the angle between the positive real axis and the tangent of Γ^* at the point $z(\tau)$. Then

$$\int_{\Gamma^*} |d\vartheta| = \int \left| \frac{d\vartheta}{d\tau} \right| d\tau$$

is the *total rotation* of Γ.

Now let Γ be an arbitrary Jordan curve, let (3.1) map $|\zeta|>\rho$ onto the exterior of Γ, and let Γ_r $(r>\rho)$ denote the level curves of the mapping. $V(r)$, the total rotation of Γ_r is a decreasing function of r. If $V(r)$ is bounded, Γ is said to be of *bounded rotation*, and

$$V = \sup_{r>\rho} V(r) = \lim_{r\to\rho} V(r)$$

is called the *total rotation* of Γ. A curve of bounded rotation has a right and left tangent at every point, and a proper tangent outside a countable set.

In section 4 we shall compare the approximation provided by the partial sums of the Faber-expansion of $f(z)$ with the best polynomial approximation of $f(z)$. In section 5 we shall give necessary conditions for the uniform convergence of the Faber expansion.

4. Faber Expansion and the Best Polynomial Approximation

It is known that if K is any continuum, and that if $f(z)$ is any function continuous on K and analytic in the interior of K, there exists a polynomial $\pi_n(z)$ of degree n (the polynomial of best uniform approximation) such that for every polynomial $P_n(z)$ of degree n

$$\max_{z\in K} |f(z)-P_n(z)| \geqq \max_{z\in K} |f(z)-\pi_n(z)| = \rho_n(f, K),$$

and $\rho_n(f, K)$ is the best (uniform) polynomial approximation of the function $f(z)$ on K.

Theorem 3. *If*

$$S_n(z) = \sum_{k=0}^{n} c_k F_k(z)$$

[1] Let s be the arc length parameter on Γ, and let $\vartheta(s)$ denote the angle between the positive real axis, and the tangent of the curve. Let $\Omega(h)$ denote the modulus of continuity of the function $\vartheta(s)$. Then Al'per requires that the condition

$$\int_0^c \frac{\Omega(h)}{h} |\log h| \, dh < \infty$$

should be satisfied for some $c>0$.

then for any continuum K whose complement is connected and for any function $f(z)$ analytic in the interior of K and continuous on K we have

$$|f(z)-S_n(z)| \leqq A\, n^{\alpha} \cdot \rho_n(f, K) \tag{4.1}$$

where A and $\alpha < \frac{1}{2}$ are absolute constants.

We shall need:

Lemma 2.

$$\frac{1}{2\pi i} \int_{|t|=1} \frac{F_n(\psi(t))}{t^{m+1}} = \begin{cases} 1 & \text{if } n=m \\ 0 & \text{if } n \neq m,\ m \geqq 0. \end{cases} \tag{4.2}$$

For $|\zeta|=1$, $\psi(\zeta)$ is defined by its radial limits.

Proof. It is known that $F_n(\psi(\zeta)) = \zeta^n + H_n(\zeta)$ where $H_n(\zeta)$ is regular and bounded in $|\zeta|>1$ and $H_n(\infty)=0$. Thus, it follows from the theorem of Golubev-Privalov [*12*, p. 144] that

$$\int_{|t|=1} \frac{H(t)}{t^{m+1}}\, dt = 0 \qquad (m=0,1,2,\ldots),$$

and (4.2) follows now immediately.

Proof of the theorem. We can write the polynomial of best approximation in the form

$$\pi_n(z) = \sum_{k=0}^{n} c_k^{(n)} F_k(z).$$

Using (4.2) we obtain

$$\frac{1}{2\pi i} \int_{|t|=1} \pi_n(\psi(t)) \frac{dt}{t^{m+1}} = \sum_{k=0}^{n} c_k^{(n)} \frac{1}{2\pi i} \int_{|t|=1} \frac{F_k(\psi(t))}{t^{m+1}}\, dt = c_m^{(n)}. \tag{4.3}$$

From (3.2) lnd (4.3) we obtain for $z \in K$

$$\begin{aligned}
|f(z)-S_n(z)| &= \left| f(z) - \sum_{k=0}^{n} c_k F_k(z) \right| \\
&\leqq |f(z)-\pi_n(z)| + \left| \pi_n(z) - \sum_{k=0}^{n} c_k F_k(z) \right| \\
&= |f(z)-\pi_n(z)| + \left| \sum_{k=0}^{n} (c_k^{(n)} - c_k) F_k(z) \right| \\
&= |f(z)-\pi_n(z)| + \left| \sum_{k=0}^{n} \frac{1}{2\pi i} \int_{|t|=1} \{f(\psi(t)) - \pi_n(\psi(t))\} F_k(z) \frac{dt}{t^{k+1}} \right| \\
&= |f(z)-\pi_n(z)| + \left| \frac{1}{2\pi i} \int_{|t|=1} \{f(\psi(t)) - \pi_n(\psi(t))\} \sum_{k=0}^{n} \frac{F_k(z)}{t^{k+1}}\, dt \right| \\
&\leqq \rho_n + \rho_n \frac{1}{2\pi} \int_{|t|=1} \left| \sum_{k=0}^{n} \frac{F_k(z)}{t^{k+1}} \right| |dt|;
\end{aligned}$$

$$|f(z)-S_n(z)| \leqq \rho_n(1+L_n) \tag{4.4}$$

where

$$L_n = \frac{1}{2\pi} \int_{|t|=1} \left| \sum_{k=0}^{n} \frac{F_k(z)}{t^{k+1}} \right| |dt| \tag{4.5}$$

is the "Lebesgue-constant" of the system of Faber polynomials.

Using (1.2) we could easily prove $L_n \leqq 2\sqrt{n(\log n+5)}$. To get a sharper estimate we shall use Lemma 1. By (1.1) and (4.5), with $1<r\leqq 2$,

$$L_n = \frac{1}{2\pi} \int_{|t|=1} \left| \sum_{k=0}^{n} \frac{1}{2\pi i} \cdot \int_{|s|=r} \frac{\psi'(s)}{\psi(s)-z} \left(\frac{s}{t}\right)^k ds \right| |dt|$$

$$\leqq \frac{1}{(2\pi)^2} \int_{|t|=1} \int_{|s|=r} \left| \frac{\psi'(s)}{\psi(s)-z} \right| \left| \frac{(s/t)^{n+1}-1}{s/t-1} \right| |ds|\,|dt|.$$

Exchanging the orders of integration, we obtain from Lemma 1

$$\begin{aligned} L_n &\leqq A_1 r^{n+1} \log \frac{1}{1-1/r} \int_{|s|=r} \left| \frac{\psi'(s)}{\psi(s)-z} \right| |ds| \\ &\leqq A_2 r^{n+1} \log \frac{1}{1-1/r} \cdot (1-r)^{-\alpha_0} < A_3 r^{n+1} (1-r)^{-\alpha} \end{aligned} \tag{4.6}$$

where $A_1, \ldots$ and $\alpha_0 < \alpha < \frac{1}{2}$ are absolute constants. Choosing $r = 1+1/n$ in (4.6) we obtain

$$L_n \leqq A_4 n^{\alpha}. \tag{4.7}$$

By (4.4), this implies (4.1).

Theorem 4. *Suppose that K is a closed Jordan domain whose boundary Γ is of bounded rotation. Then for any function $f(z)$ analytic in the interior of K and continuous on K we have that*

$$|f(z) - S_n(z)| \leqq (A \cdot \log n + B)\, \rho_n(f, K). \tag{4.8}$$

Here the constants A and B depend only on the domain K.

We shall need:

Lemma 3 [*10*, Lemma 1]. *Let Γ be a closed Jordan curve of bounded rotation, and let*

$$v(s, \vartheta) = \arg\left(\psi(e^{is}) - \psi(e^{i\vartheta})\right).$$

Then

$$\text{(i)} \quad \int_0^{2\pi} |d_\tau v(\tau, \vartheta)| \leqq V, \tag{4.9}$$

where V denotes the total rotation of Γ.

$$\text{(ii)} \quad F_K\left(\psi(e^{i\vartheta})\right) = \frac{1}{\pi} \int_0^{2\pi} e^{iks}\, d_s v(s, \vartheta). \tag{4.10}$$

Proof of the Theorem. Using (4.10), and (4.9), we estimate the constant L_n of (4.5):

$$L_n=\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=0}^{n} F_k(\varphi(e^{i\vartheta}))\,e^{-ik\vartheta}\right|d\vartheta$$

$$=\frac{1}{2\pi^2}\int_0^{2\pi}\left|\int_0^{2\pi}\left(\sum_{k=0}^{n} e^{ik(s-\vartheta)}\right)d_s v(s,\vartheta)\right|d\vartheta$$

$$\leqq\frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left|\sum_{k=0}^{n} e^{ik(s-\vartheta)}\right||d_s v(s,\vartheta)|\,d\vartheta$$

$$=\frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left|\frac{\sin\frac{n+1}{2}(s-\vartheta)}{\sin\frac{s-\vartheta}{2}}\right||d_s v(s,\vartheta)|\,d\vartheta \tag{4.11}$$

$$=\frac{1}{2\pi^2}\int_0^{2\pi}\left\{\int_0^{2\pi}\left|\frac{\sin\frac{n+1}{2}(s-\vartheta)}{\sin\frac{s-\vartheta}{2}}\right|d\vartheta\right\}|d_s v(s,\vartheta)|$$

$$\leqq\frac{2}{\pi^3}(\log n+C)\int_0^{2\pi}|d_s v(s,\vartheta)|\leqq\frac{2V}{\pi^3}(\log n+C).$$

From (4.4) and (4.11) we immediately obtain (4.8).

5. Uniform Convergence of Faber Expansions

Given any complex Fourier series:

$$\mathfrak{S}=\sum_{k=-\infty}^{+\infty} c_k e^{ik\vartheta},$$

its *conjugate* series is the series:

$$\tilde{\mathfrak{S}}=\sum_{k=-\infty}^{+\infty} \tilde{c}_k e^{ik\vartheta},$$

where

$$\tilde{c}_k=\begin{cases}-i\,c_k, & \text{if } k\geqq 0\\ +i\,c_k, & \text{if } k<0.\end{cases}$$

If both series converge uniformly, then so does the series:

$$\tfrac{1}{2}(\mathfrak{S}+i\,\tilde{\mathfrak{S}})=\sum_{k=0}^{\infty} c_k e^{ik\vartheta}.$$

Theorem 5. *Let K be a closed Jordan domain, whose boundary Γ is of bounded rotation. Let $f(z)$ be analytic in the interior of K and continuous on K. Suppose that the Fourier series of $f(\psi(e^{i\vartheta}))$ together with its conjugate series converges uniformly. Then the Faber expansion of $f(z)$ converges uniformly on K to $f(z)$.*

We shall need the following lemma:

Lemma 4. *Let K be a closed Jordan domain bounded by a rectifiable Jordan curve, and let the function $f(z)$ be analytic in the interior of K and continuous on K. If the Faber series of $f(z)$ converges uniformly on K, then its sum is $f(z)$.*

Proof. We write:

$$\sum_{k=0}^{\infty} c_k F_k(z) = f^*(z)$$

$f^*(z)$ is regular in the interior of K, and continuous on K. Let

$$c_m^* = \frac{1}{2\pi i} \int_{|t|=1} f^*(\psi(t)) \frac{dt}{t^{m+1}} = \frac{1}{2\pi i} \int_{|t|=1} \sum_{k=0}^{\infty} c_k \frac{F_k(\psi(t))}{t^{m+1}} \, dt.$$

Integrating term-by-term, and using Lemma 1, we obtain

$$c_m^* = \sum_{k=0}^{\infty} c_k \frac{1}{2\pi i} \int_{|t|=1} \frac{F_k(\psi(t))}{t^{m+1}} \, dt = c_m .$$

The function $g(z) = f(z) - f^*(z)$ is regular in the interior of K, continuous on K, and all its Faber coefficients are zero:

$$\text{(5.1)} \qquad \frac{1}{2\pi i} \int_{|t|=1} g(\psi(t)) \frac{dt}{t^{n+1}} = 0 \qquad (n = 0, 1, 2, \ldots).$$

Also, $g(\psi(\zeta))$ is continuous on $|\zeta| = 1$. By the theorem of Golubev-Privalov [*12*, p. 144] it follows from (5.1) that there exists an analytic function $G(\zeta)$, regular in $|\zeta| > 1$, continuous in $|\zeta| \geqq 1$, and such that

$$G(\infty) = 0,$$
$$G(\zeta) = g(\psi(\zeta)) \qquad \text{for } |\zeta| = 1.$$

If we denote the inverse function of $z = \psi(\zeta)$ by $\zeta = \varphi(z)$, then $G(\varphi(z))$ is regular in the exterior of Γ, continuous on the closed exterior, vanishes at $z = \infty$, and satisfies

$$G(\varphi(z)) = g(z) \qquad \text{for } z \in \Gamma.$$

Thus the function

$$g^*(z) = \begin{cases} g(z) & \text{for } z \in K \\ G(\varphi(z)) & \text{for } z \notin K \end{cases}$$

is continuous on the extended plane, and regular in the interior and exterior of Γ. Hence, since Γ is rectifiable, by a well-known theorem $g^*(z)$ is regular on Γ also. By Liouville's theorem: $g^*(z)$ is a constant, $g(z) \equiv g^*(\infty) = 0$. Hence $f(z) \equiv f^*(z)$ which was to be proved.

Proof of Theorem 5. We note that the Faber coefficients (3.1) of $f(z)$ are also Fourier coefficients of the function $F(\vartheta)=f(\psi(e^{i\vartheta}))$ for $k\geqq 0$. Applying (4.10):

$$(5.2)\quad \begin{aligned} S_n(\psi(e^{i\varphi})) &= \sum_{k=0}^{n} c_k F_k(\psi(e^{i\varphi})) = \sum_{k=0}^{n} c_k \frac{1}{\pi}\int_0^{2\pi} e^{ik\tau} d_\tau v(\tau,\varphi) \\ &= \frac{1}{\pi}\int_0^{2\pi}\left(\sum_{k=0}^{n} c_k e^{ik\tau}\right) d_\tau v(\tau,\varphi) = \frac{1}{\pi}\int_0^{2\pi} s_n^*(\tau)\, d_\tau v(\tau,\varphi) \end{aligned}$$

where

$$(5.3)\quad s_n^*(\vartheta)=\sum_{k=0}^{n} c_k e^{ik\vartheta}.$$

By the assumptions of the theorem and the remark made at the beginning of section 5, $s_n^*(\tau)$ converges uniformly to a continuous function $F^*(\vartheta)$:

$$(5.4)\quad \max_{\vartheta} |s_n^*(\vartheta)-F^*(\vartheta)|=\varepsilon_n \to 0.$$

From (5.2), (5.3) and (4.9), we obtain

$$\begin{aligned} &\left|S_n(\psi(e^{i\varphi}))-\frac{1}{\pi}\int_0^{2\pi} F^*(\tau)\, d_\tau v(\tau,\varphi)\right| \\ &= \left|\frac{1}{\pi}\int_0^{2\pi}\left(s_n^*(\tau)-F^*(\tau)\right) d_\tau v(\tau,\varphi)\right| \\ &\leqq \frac{1}{\pi}\int_0^{2\pi} |s_n^*(\tau)-F^*(\tau)|\,|d_\tau^\alpha v(\tau,\varphi)| \\ &\leqq \varepsilon_n\cdot\frac{1}{\pi}\int_0^{2\pi} |d_\tau v(\tau,\varphi)| \leqq \frac{V\varepsilon_n}{\pi} \to 0 \qquad (n\to\infty). \end{aligned}$$

Then $S_n(\psi(e^{i\varphi}))$ converges uniformly for $0\leqq\varphi\leqq 2\pi$, i.e. $S_n(z)$ converges uniformly on Γ. Hence, by the maximum principle, $S_n(z)$ converges uniformly on K. By Lemma 3 its sum is $f(z)$. This completes the proof of theorem 3.

Before stating the next theorem, we shall formulate some lemmas:

Lemma 5 [8, Nr. 28]. *Suppose that Γ is of bounded rotation. Then there exists a function $u(\vartheta)$ such that:*

$$(5.5)\quad \text{(i)}\quad \int_0^{2\pi} |du(\vartheta)|=V.$$

(ii) *At every point $\psi(e^{i\vartheta})$ of Γ where there is a tangent, $u(\vartheta)$ gives the angle between the positive real axis and the tangent.*

$$(5.6)\quad \text{(iii)}\quad \log\psi'(\zeta)=\frac{1}{\pi}\int_0^{2\pi}\log(1-e^{i\vartheta}/\zeta)\, du(\vartheta) \qquad \text{for } |\zeta|>1.$$

Lemma 6. *If Γ is of bounded rotation, and all its exterior angles are* $\geqq \pi\alpha$ $(0<\alpha\leqq 1)$ *then, for every* $\varepsilon>0$:

$$|\psi'(\zeta)|\leqq \frac{K_\varepsilon}{\left(1-\frac{1}{|\zeta|}\right)^{1-\alpha+\varepsilon}}. \tag{5.7}$$

Proof. Since $u(\vartheta)$ (cf. Lemma 5) is of bounded variation, we can write:

$$u(\vartheta)=u^+(\vartheta)-u^-(\vartheta),$$

$$\int_0^{2\pi}|du(\vartheta)|=\int_0^{2\pi}du^+(\vartheta)+\int_0^{2\pi}du^-(\vartheta)$$

where $u^+(\vartheta)$ and $u^-(\vartheta)$ are increasing functions. We also write:

$$u^+(\varphi+0)-u^+(\varphi-0)=h^+;\qquad u^-(\varphi+0)-u^-(\varphi-0)=h^-;$$
$$u(\varphi+0)-u(\varphi-0)=h=h^+-h^-.$$

If $h\geqq 0$ then $h^+=h$, $h^-=0$; if $h<0$, then $h^+=0$, $h^-=-h\leqq 1-\alpha$ by assumption.

Let $z=r\cdot e^{i\varphi}$. Applying Lemma 4:

$$\begin{aligned}\log|\psi'(re^{i\varphi})|&=\frac{1}{\pi}\int_0^{2\pi}\log\left|1-\frac{1}{r}e^{i(\vartheta-\varphi)}\right|du(\vartheta)\\ &=\frac{1}{\pi}\int_0^{2\pi}\log\left|1-\frac{1}{r}e^{i(\vartheta-\varphi)}\right|du^+(\vartheta)-\frac{1}{\pi}\int_0^{2\pi}\log\left|1-\frac{1}{r}e^{i(\vartheta-\varphi)}\right|du^-(\vartheta).\end{aligned} \tag{5.8}$$

For the first integral, we have the estimate

$$\frac{1}{\pi}\int_0^{2\pi}\log\left|1-\frac{1}{r}e^{i(\vartheta-\varphi)}\right|du^+(\vartheta)\leqq\frac{\log 2}{\pi}\int_0^{2\pi}du^+(\vartheta)\leqq\frac{\log 2}{\pi}V. \tag{5.9}$$

Using the monotonicity of u^- and the compactness of the unit circle it is easy to show that there exists a fixed $\delta>0$ such that

$$u^-(\varphi+\delta)-u^-(\varphi-\delta)=1-\alpha+\frac{\varepsilon}{2}$$

for every φ. Hence

$$\begin{aligned}&\frac{1}{\pi}\int_0^{2\pi}\log\left|1-\frac{1}{r}e^{i(\vartheta-\varphi)}\right|du^-(\vartheta)\\ &\quad=\frac{1}{\pi}\int_{\varphi-\delta}^{\varphi+\delta}\log\left|1-\frac{1}{r}e^{i(\vartheta-\varphi)}\right|du^-(\vartheta)+\frac{1}{\pi}\int_{\varphi+\delta}^{2\pi+\varphi-\delta}\log\left|1-\frac{1}{r}e^{i(\vartheta-\varphi)}\right|du^-(\vartheta)\\ &\quad\geqq\left(u^-(\varphi+\delta)-u^-(\varphi-\delta)\right)\log\left(1-\frac{1}{r}\right)+\log\left|1-\frac{1}{r}e^{i\delta}\right|\cdot\int_{\varphi+\delta}^{2\pi+\varphi-\delta}du^-(\vartheta)\\ &\quad\geqq\left(1-\alpha+\frac{\varepsilon}{2}\right)\log\left(1-\frac{1}{r}\right)+V\log\sin\delta.\end{aligned}$$

If

$$1-\frac{1}{r}\leqq(\sin\delta)^{\frac{2V}{\varepsilon}}=1-\frac{1}{r_\varepsilon}$$

then

$$\frac{\varepsilon}{2}\log\left(1-\frac{1}{r}\right)\leqq V\log\sin\delta$$

and

$$\frac{1}{\pi}\int_0^{2\pi}\log\left|1-\frac{1}{r}e^{i(\vartheta-\varphi)}\right|du^-(\vartheta)$$

$$\geqq\left(1-\alpha+\frac{\varepsilon}{2}\right)\log\left(1-\frac{1}{r}\right)+\frac{\varepsilon}{2}\log\left(1-\frac{1}{r}\right) \tag{5.10}$$

$$=(1-\alpha+\varepsilon)\log\left(1-\frac{1}{r}\right).$$

The inequalities (5.8), (5.9) and (5.10) yield:

$$\log|\psi'(r\cdot e^{i\varphi})|\leqq\frac{\log 2}{\pi}V-(1-\alpha+\varepsilon)\log\left(1-\frac{1}{r}\right),$$

$$|\psi'(r\cdot e^{i\varphi})|\leqq\frac{2^{\pi V}}{\left(1-\frac{1}{r}\right)^{1-\alpha+\varepsilon}}$$

for $r<r_\varepsilon$.

The class of the derivatives of functions (3.1) univalent in $|\zeta|>1$ is uniformly bounded in $|\zeta|\geqq r_\varepsilon>1$. Hence, with a suitable constant K_ε, (5.7) holds for every ζ $(|\zeta|>1)$.

Lemma 7 (Hardy-Littlewood, [6, p. 361]). *If $\psi(\zeta)$ is regular in $|\zeta|>1$, continuous in $|\zeta|\geqq 1$ and*

$$|\psi'(\zeta)|\leqq\frac{M}{\left(1-\frac{1}{|\zeta|}\right)^{1-\beta}}\qquad(|\zeta|>1)$$

then $\psi(\zeta)$ satisfies a Lipschitz-condition (with exponent β) on $|\zeta|=1$:

$$|\psi(e^{i\vartheta_1})-\psi(e^{i\vartheta_2})|\leqq K|\vartheta_1-\vartheta_2|^\beta.$$

Lemma 8. *If Γ is of bounded rotation and has no zero exterior angles, then $\psi(\zeta)$ satisfies on $|\zeta|=1$ a Lipschitz-condition*

$$|\psi(e^{i\vartheta_1})-\psi(e^{i\vartheta_2})|\leqq K|\vartheta_1-\vartheta_2|^\beta \tag{5.11}$$

for some $\beta>0$.

Proof. Since Γ is of bounded rotation, the number of exterior angles which are $\leqq\pi/2$ is finite. Let the smallest of these angles be equal to $\pi\alpha$. By hypothesis, $\alpha>0$. Applying lemma 6 with $\varepsilon=\alpha/2$, we obtain that

$$|\psi'(\zeta)|\leqq\frac{M}{\left(1-\frac{1}{|\zeta|}\right)^{1-\beta}}\qquad\left(\beta=\frac{\alpha}{2}\right),$$

and (5.11) is now a consequence of Lemma 7.

Theorem 6. *Let K be a closed Jordan domain, whose boundary Γ is of bounded rotation and has no zero exterior angles. Suppose that $f(z)$ is analytic in the interior of K, continuous on K, and moreover that it satisfies* DINI's *condition:*

$$\int_0^h \frac{\omega_f(x)}{x}\,dx<\infty \tag{5.12}$$

or some $h>0$. Here $\omega_f f$ is the modulus of continuity of $f(z)$ on K:

$$\omega_f(\delta)=\max_{\substack{z_1,z_2\in K\\ |z_1-z_2|\leqq\delta}}|f(z_1)-f(z_2)|.$$

Then the Faber expansion of $f(z)$ converges uniformly on K to $f(z)$.

Proof. By Lemma 8, there exists $\delta>0$, and $\eta>0$ such that:

$$|\psi(e^{i\vartheta_1})-\psi(e^{i\vartheta_2})|\leqq|\vartheta_1-\vartheta_2|^\eta$$

for $|\vartheta_1-\vartheta_2|\leqq\delta$. Hence, if $F(\vartheta)=f(\psi(e^{i\vartheta}))$:

$$\begin{aligned}|F(\vartheta_1)-F(\vartheta_2)|&=|f(\psi(e^{i\vartheta_1}))-f(\psi(e^{i\vartheta_2}))|\\ &\leqq\omega_f(|\psi(e^{i\vartheta_1})-\psi(e^{i\vartheta_2})|)\leqq\omega_f(|\vartheta_1-\vartheta_2|^\eta)\end{aligned}$$

i.e.

$$\omega_F(x)\leqq\omega_f(x^\eta)\qquad\text{for } x\leqq\delta.$$

Hence

$$\int_0^\delta\frac{\omega_F(x)}{x}\,dx\leqq\int_0^\delta\omega_f(x^\eta)\frac{dx}{x}=\frac{1}{\eta}\int_0^{\delta^\eta}\omega_f(y)\frac{dy}{y}<+\infty.$$

Hence $F(\vartheta)=f(\psi(e^{i\vartheta}))$ satisfies DINI's condition (5.12), and thus, by a well-known result [*14*, p. 54], the Fourier series of $f(\psi(e^{i\vartheta}))$ and its conjugate series converge uniformly. Theorem 6 is therefore a corollary of Theorem 5.

References

[*1*] AL'PER, S. YA.: On the uniform approximation to functions of a complex variable on closed domains [Russian]. Izv. Akad. Nauk SSSR, Ser. Mat. **19**, 423—444 (1955).

[*2*] CLUNIE, J.: On meromorphic schlicht functions. J. London Math. Soc. **34**, 215—216 (1959).

[*3*] —, and CH. POMMERENKE: On the coefficients of univalent functions. Mich. Math. J. (1967, to appear).

[*4*] DANZER, L., u. CH. POMMERENKE: Über die Diskriminante von Mengen gegebenen Durchmessers. Monatsh. Math. **71**, 100—113 (1967).

[5] GOLUSIN, G. M.: Some coefficient estimates for univalent functions. Mat. Sb. **3**, 321–330 (1938) [Russian, German Summary].
[6] — Geometrische Funktionentheorie. Berlin: Deutscher Verlag der Wissenschaften 1957.
[7] GRÖTZSCH, H.: Zur Theorie der Verschiebung bei schlichter konformer Abbildung. Comm. Math. Helv. **8**, 382–390 (1935/36).
[8] PAATERO, V.: Über die konforme Abbildung von Gebieten, deren Ränder von beschränkter Drehung sind. Ann. Acad. Sci. Fenn. Ser. A., **33**, No. 9, 1–77 (1931).
[9] POMMERENKE, CH.: Über die Faberschen Polynome schlichter Funktionen. Math. Z. **85**, 197–208 (1964).
[10] — Konforme Abbildung und Fekete-Punkte. Math. Z. **89**, 422–438 (1965).
[11] — On the coefficients of univalent functions. J. London Math. Soc. **42**, (1967, to appear).
[12] PRIVALOV, I. I.: Randeigenschaften analytischer Funktionen. Berlin: Deutscher Verlag der Wissenschaften 1956.
[13] SMIRNOV, V. J., and N. A. LEBEDEV: Constructive theory of functions of a complex variable. Moscow: Izd. Nauka 1964.
[14] ZYGMUND, A.: Trigonometric series, 2-nd ed. Cambridge: Univ. Press 1959.

Imperial College, Department of Mathematics, London S.W.7, England