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Non-real zeros of real entire functions

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1. Let f be an entire function of finite genus q which is real on the real axis. According to a theorem of LAGUERRE and BOREL, [*1*, Ch. 2], if f and f' have respectively n and t non-real zeros, then

$$t \leqq q+n. \tag{1}$$

Denote by m the number of direct transcendental singularities of f^{-1} over the origin. The object of this note is to show that

$$t \leqq m+n. \tag{2}$$

For the cosine function, there is equality in (2), but not in (1). The exponential function shows that there need not be equality in (2).

2. Before proving (2), we show that (2) implies (1), by showing that $p \leqq q$, where p is the number of finite direct transcendental singularities of f^{-1}. This follows from the proof of the Denjoy-Carleman-Ahlfors theorem for entire functions [*2*, p. 313], according to which $p \leqq k$, where k is the order of f. For the relation between q and k, there are two possibilities: either $q=[k]$, or $q=k-1$, where, in the second case,

$$T(r)=o(r^k),$$

where T is the Nevanlinna characteristic function of f [*2*, pp. 234–235]. Since p and q are integers, we have $p \leqq q$ in the first case. In the second case, we use the last step in the proof of the Denjoy-Carleman-Ahlfors theorem [*2*, pp. 312–313],

$$T(r) \geqq C \cdot r^p \qquad (C>0),$$

to see that $p<k$, and hence that $p \leqq q$ in this case also.

3. For the proof of (2), we may suppose that n is finite.

If t is finite, denote by $w_1, \dots, w_r$ the images of the distinct non-real zeros $z_1, \dots, z_r$ of f'. (If $t=\infty$, consider at first only a finite number of zeros of f'.) There are half-lines l_i from w_i to ∞ such that

(a) l_i intersects the real axis only at w_i, if at all;

(b) l_i and l_j do not intersect if $w_i \neq w_j$;

(c) each branch of f^{-1} at (w_i, z_i) may be continued along l_i to ∞. For example, we may even suppose that the l_i all have the same slope, since by a simple extension of the theorem of GROSS [2, p. 292], the set of directions along which all branches of f^{-1} may not be continued to ∞ has measure zero.

Denote by s_i the component of $f^{-1}(l_i)$ in the z-plane which contains z_i. The sets s_i are disjoint, and together with the point at infinity form the edges and vertices of a connected graph on the z-sphere. There are $r+1$ vertices. To a non-real zero z_i of order ν_i of f' correspond ν_i+1 edges, so there are altogether $\sum(\nu_i+1)=t+r$ edges. By EULER's theorem, there are therefore $t+1$ faces.

The real axis intersects no edge, by (a), and is therefore contained in the interior of one of the faces. In the interior of each of the other faces none of whose vertices is a zero of f, either f has a zero, of f tends to zero along an asymptotic path, by (b) and a simple extension of IVERSEN's theorem [2, p. 291]. To each of the other faces with a non-real multiple zero z_i of f of order ν_i+1 as vertex, one $(\nu_i+1)\,st$ part of the zero is assigned. In this way, to each of the t faces not containing the real axis is associated at least one "simple" non-real zero of f, or asymptotic path along which f tends to zero, so that no zero or path is associated with more than one face.

Asymptotic paths associated with distinct faces determine distinct transcendental singularities of f^{-1}, since they are separated by edges, along which f tends to ∞, by (c). Each associated asymptotic path determines a direct transcendental singularity, since the real axis is separated from the path by edges, and there are only a finite number of non-real zeros.

This completes the proof of (2) in case t is finite. (If $t=\infty$, this argument shows that $t_1 \leqq m+n$, for arbitrarily large t_1, which implies that $m+n=\infty$.)

4. We remark that if $n<t$, then each non-zero complex value is taken infinitely often off the real axis. In fact, if $n<t$, there is a face not containing the real axis with an associated asymptotic path along which f tends to zero. Between this path and each neighboring edge, f takes each complex value with at most one exception infinitely often, by a theorem of LINDELÖF [2, p. 67].

References

[1] BOREL, É.: Leçons sur les fonctions entières. Paris: Gauthier-Villars 1900.

[2] NEVANLINNA, R.: Eindeutige Analytische Funktionen. Zweite Auflage. Berlin-Göttingen-Heidelberg: Springer 1953.

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