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Asymptotic values of normal subharmonic functions

By
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1. The purpose of this paper is to prove, for a class of functions subharmonic in the open unit disk in the complex plane, a variant of a theorem of O. LEHTO and K. I. VIRTANEN. According to LEHTO and VIRTANEN [5, p. 53] a meromorphic function $f(z)$ defined in a simply connected domain G in the plane is said to be a normal function if the family $\{f(S(z))\}$, where $z'=S(z)$ denotes an arbitrary one-one conformal mapping of G onto G is a normal family in the sense of Montel.

It is shown that if $f(z)$ is a non-constant normal meromorphic function in G then G must be of hyperbolic type. Hence for brevity we state the above mentioned theorem in the case in which G is the unit disk $D=\{z \,|\, |z|<1\}$. Set $C=\{z \,|\, |z|=1\}$.

If γ is a Jordan arc in D with one endpoint $\tau \in C$; and $f(z)$ is a function defined in D taking values on the Riemann sphere W such that $f(z)$ tends to a value $\alpha \in W$ as z approaches τ along γ then α is called an asymptotic value for $f(z)$ at τ.

Theorem (LEHTO and VIRTANEN). *Let $f(z)$ be a normal function in D with asymptotic value α at $\tau \in C$. In this case $f(z)$ tends uniformly to α as z approaches τ within any Stolz domain at τ* [5, p. 53].

The notion of function shall be limited to finite valued complex functions in this paper unless otherwise noted.

If the definition of a normal function as given by LEHTO and VIRTANEN is restricted to holomorphic function, HURWITZ's theorem gives the following formulation which will define in the sequel a normal (not necessarily holomorphic) function.

Definition 1. A function $U(z)$ defined in D is said to be a normal function if the family $F=\{U(S(z))\}$, where $z'=S(z)$ is an arbitrary one-one conformal map of D onto D, has the property that for every sequence $\{f_n\}$, $f_n \in F$, there exists a subsequence which either converges uniformly on every compact subset of D, or else converges uniformily to infinity on every compact subset of D. For the notion of a normal family used in this definition see e.g. [1, p. 168].

That this definition is not equivalent to, and in fact is more restrictive than, the corresponding definition obtained from the Lehto-Virtanen definition by replacing the word "meromorphic" by "complex-valued" in their definition is fairly obvious and we remark on this in § 3. We call such a complex-valued function satisfying this definition *a normal function in the sense of* LEHTO *and* VIRTANEN.

Consider the case of a non-constant normal holomorphic function $f(z)$ which has a finite asymptotic value α at some point of C. The normality $f(z)-\alpha$ clearly implies the normality of $|f(z)-\alpha|$. Further $|f(z)-\alpha|$ has the additional property that $\log|f(z)-\alpha|$ is a subharmonic function in D. (For the definition and discussion of subharmonic functions see e.g. [7].) Thus the theorem of LEHTO and VIRTANEN can be interpreted that if 0 (or ∞) is an asymptotic value of $U_\alpha(z)=|f(z)-\alpha|$ at $\tau\in C$ then $U_\alpha(z)$ tends to 0 (or ∞) within any Stolz domain at τ.

It seems natural to ask if this property holds for a normal function $U(z)$ in D such that $\log U(z)$ is subharmonic. We answer in the affirmative, at least in the case in which the asymptotic value 0 (or ∞) is associated with a arc approaching $\tau\in C$ within some Stolz domain at τ.

2. If M is a simply connected domain in the plane bounded by a Jordan curve Γ and if α is any open arc on Γ let $\omega(z,\alpha,M)$ equal the harmonic measure of α at $z\in M$ with respect to M. (For details see e.g. [6, p.26ff].)

Consider also in D the non-Euclidean hyperbolic distance

$$\rho(z_1,z_2)=\frac{1}{2}\log\frac{|1-z_1\bar{z}_2|+|z_1-z_2|}{|1-z_1\bar{z}_2|-|z_1-z_2|}.$$

For further details see [3, Chap. II, IV].

For $\tau\in C$ let $S(\tau,\beta)=S(\beta)$, $0<\beta<\pi/2$, denote the open set bounded by the two hypercycles from τ to $-\tau$ making angle β and $-\beta$ respectively with the diameter between τ and $-\tau$. If a non-empty set E, $E\subset D$, is such that the closure of E intersects C only at τ and E is contained in some $S(\tau,\beta)$ we say E approaches τ in a non-tangential manner.

Finally, for any function $f(z)$ defined in D and any set $E\subset D$ such that the closure of E meets $\tau\in C$, $C_E(f,\tau)$ will be the set of all values $w\in W$ such that there is a sequence $\{z_n\}$, $z_n\in E$, and $z_n\to\tau$, $n\to\infty$, with $f(z_n)\to w$ as $n\to\infty$. If the equation $C_{S(\tau,\beta)}(f,\tau)=w$ is satisfied for some $w\in W$ and all $0\leqq\beta<\pi/2$, we say $f(z)$ has angular limit w at τ.

With these preliminaries we now give

Theorem 1. *Let $U(z)$ be a non-negative normal function in D such that $\log U(z)$ is subharmonic. If γ is a Jordan arc in D approaching $\tau\in C$ in a non-tangential manner and $U(z)$ has asymptotic value* 0 *(or ∞) along γ then $U(z)$ has angular limit* 0 *(or ∞) at τ.*

Proof. We may suppose without loss of generality that $\tau=1$. We first consider the case in which the asymptotic value is 0. Let $0<\beta<\pi/2$ be chosen so that $\gamma\subset S(1,\beta_0)=S(\beta_0)$. We need only show that $C_{S(\beta_0)}(U,0)=\{0\}$, and we suppose the contrary.

Let $\{z_n\}$, $z_n\in S(\beta_0)$, $z_n\to1$, $n\to\infty$, while $U(z_n)\to a$, $a>0$. Consider the non-Euclidean straight line E_n through z_n perpendicular to the real axis. Let r_n be the point of intersection of E_n and the real axis. Since $z_n\in S(\beta_0)$, $n=1,2,\dots$

$$\rho(r_n,z_n)<\tfrac{1}{2}\log\cot(\pi/4-\beta_0/2)\equiv K,\qquad n=1,2,\dots. \tag{2.0}$$

By the normality of $U(z)$ there is a subsequence of

$$\{U(S_n(\zeta))\}, \quad S_n(\zeta)=\frac{\zeta+r_n}{1+\zeta r_n}, \quad |\zeta|<1,$$

which converges uniformly on each compact subset of $D_\zeta=\{\zeta \,|\, |\zeta|<1\}$. To simplify the notation we assume $\{U(S_n(\zeta))\}$ is the desired subsequence. Now $\{U(S_n(\zeta))\}$ cannot tend uniformly to infinity on the compact subset

$$|\zeta|\leq\frac{e^{2K}-1}{e^{2K}+1}\equiv K'$$

because of the presence of points (corresponding to arcs of γ under S_n^{-1}) on which $\{U(S_n(\zeta))\}$ tends to zero. Assume therefore that $\{U(S_n(\zeta))\}$ converges

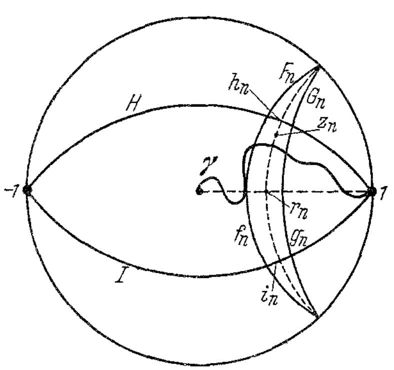

Fig. 1

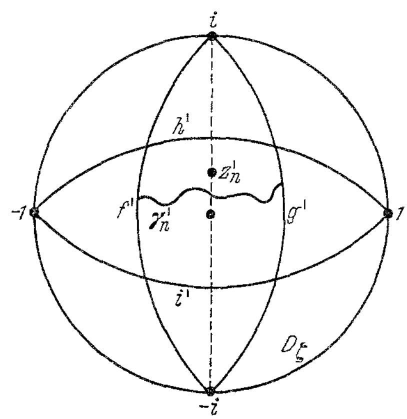

Fig. 2

to a limit function $V(\zeta)$ which is uppersemicontinuous in D. This property follows from the hypothesis of the theorem which implies $U(z)$ is uppersemicontinuous in D, and so also is each $U(S_n(\zeta))$, $n=1, 2, \ldots$, in D; as well as the uniform limit $V(\zeta)$.

For $n=1, 2, \ldots$ let F_n and G_n denote the two curves in D whose non-Euclidean distance from E_n is one. Let F_n be that curve closest to the origin. Let H and I be the hypercycles from -1 to 1 with non-Euclidean distance one from $S(\beta_0)$; and J_n be the "quadrilateral" bounded by the curves F_n, G_n, H and I. Label the boundary curves of J_n, f_n, g_n, h_n and i_n where $f_n\subset F_n$, etc. To conclude, set γ_n equal to any arc of the curve γ which joins the sides f_n and g_n and otherwise is entirely contained in J_n (see Fig. 1).

γ_n splits J_n into two components and we suppose that z_n is contained in that component O_n bounded by h_n. The proof is similar if there is a subsequence of $\{z_n\}$ with each term contained in the "lower" component of the corresponding set J_n.

For any fixed $n=1, 2, \ldots$, let the prime superscript indicate the image in D_ζ by $\zeta=S_n^{-1}(z)$ of the appropriate point set in D, so that $h_n'=S_n^{-1}(h_n)$ and so on.

It is easily verified that each point set f'_n, g'_n, h'_n, i'_n and thus J'_n is identical for every value of $n=1, 2, \ldots$; set $f'_n \equiv f'$, $g'_n \equiv g'$, $h'_n \equiv h'$, $i'_n \equiv i'$ and $J'_n \equiv J'$. In fact f' and g' are subarcs of a pair of symmetric hypercycles from i to $-i$ and h' and i' are subarcs of another symmetric pair of hypercycles joining 1 to -1. Observe that $S_n^{-1}(z_n)=z'_n$ lies on the imaginary axis with $|z'_n| \leq K'$, $n=1, 2, \ldots$. See Fig. 2.

CARLEMAN's principle of Gebietserweiterung [6, p. 69] implies, for $\zeta \in O'_n$, $n=1, 2, \ldots$,

$$\omega(\zeta, \gamma'_n, O'_n) \geqq \omega(\zeta, i', J'). \tag{2.1}$$

Since that segment of the imaginary axis in J' from i' to the point iK' is bounded away from that part of the boundary of J' on which $\omega=0$, (2.1) and the properties of harmonic measure yield, for a suitably small $0<\omega_0<1$,

$$\omega(z'_n, \gamma'_n, O'_n) \geqq \omega_0, \qquad n=1, 2, \ldots. \tag{2.2}$$

If

$$\varepsilon_n = \max_{\zeta \in \gamma'_n} U(S_n(\zeta)), \tag{2.3}$$

the hypothesis implies $\varepsilon_n \to 0$ as $n \to \infty$.

The upper semicontinuity of $V(\zeta)$ in D_ζ guarantees the existence of a positive constant T such that $V(\zeta) \leqq T$, $\zeta \in J'$; and the uniform convergence of $\{U(S_n(\zeta))\}$ on J' gives

$$U(S_n(\zeta)) \leqq T+1, \qquad \zeta \in J', \qquad n>N_0. \tag{2.4}$$

We can apply the two-constant theorem of F. and R. NEVANLINNA [6, p. 42] (based on the Phragmen-Lindelöf maximal principle for subharmonic functions) to the domain O'_n if we observe that $\log U(S_n(\zeta))$ is subharmonic and by (2.4) bounded above in O'_n. In this manner for $n > N_0$, we obtain, after referring to (2.2),

$$\begin{cases} \log U(S_n(z'_n)) \leqq \omega(z'_n, \gamma'_n, O_n) \log \varepsilon_n + \\ \qquad\qquad + (1-\omega(z'_n, \gamma'_n, O'_n)) \log (T+1) \\ \qquad \leqq \omega_0 \log \varepsilon_n + (1-\omega_0) \log (T+1). \end{cases} \tag{2.5}$$

With $\varepsilon_n \to 0$ as $n \to \infty$, this implies $\log U(S_n(z'_n)) \to -\infty$ or $U(z_n) \to 0$ as $n \to \infty$, which is the desired contradiction.

For the case in which the sequence $\{z_n\}$ is such that each z_n belongs to the "lower" component of J_n we simply apply the two-constant theorem to the domain bounded by γ'_n, f', g' and i' with the harmonic minorant $\omega(\zeta, h', J')$.

To conclude the theorem we consider the situation in which $U(z)$ tends to $+\infty$ on γ. The limit function $V(\zeta)$ must be identically infinite. Otherwise $U(S_n(\zeta))$ is uniformly bounded in J' for $n>N_0$. But J' contains each γ'_n and $U(S_n(\zeta))$ is certainly not uniformly bounded on γ'_n for all values n. This concludes the proof.

If the function $U(z)$ is bounded in D we may dispense with the hypothesis of normality, which was only utilized to insure the existence of an upper bound for $U(S_n(\zeta))$ in J', and obtain

Theorem 2. *Let $U(z)$ be a non-negative function in D such that* $\log U(z)$ *is subharmonic in D and further let $U(z) \leqq M$ in D. Suppose γ is a Jordan arc in D which tends non-tangentially to a point $\tau \in C$ with*

$$\overline{\lim}\, U(z) \leqq K < M\,, \quad z \to \tau, \quad z \in \gamma\,.$$

If $\alpha \in C_{S(\tau,\beta)}(U, \tau)$, $0 < \beta < \pi/2$, there is a real number $\lambda = \lambda(\beta)$, $0 < \lambda < 1$, with $\lambda(\beta) \to 0$ as $\beta \to \pi/2$, such that $\alpha \leqq K^\lambda M^{1-\lambda}$.

Proof. Let $\{z_n\}$ be a sequence in $S(\tau, \beta)$ for which $f(z_n) \to \alpha$. Proceed as in Theorem 1 except in (2.3) replace ε_n by $K + \varepsilon_n$ and in (2.4) replace $T+1$ by M, obtaining as in (2.5),

$$\log U(z_n) \leqq \omega_0 \log (K + \varepsilon_n) + (1 - \omega_0) \log M\,, \qquad n = 1, 2, \ldots,$$

or

$$U(z_n) \leqq (K + \varepsilon_n)^{\omega_0} M^{(1-\omega_0)}.$$

As $n \to \infty$, $U(z_n) \to \alpha$ and $\varepsilon_n \to 0$ so $\alpha \leqq K^{\omega_0} M^{1-\omega_0}$.

The value ω_0 depends upon the domain J' and the point iK' both of which are functions only of the angle β. Thus we may set $\omega_0 = \lambda = \lambda(\beta)$, and a further analysis shows that $\lambda(\beta) \to 0$ as $\beta \to \pi/2$.

Briefly Theorem 2 states that if Ω_τ^* is the set of all Jordan arcs approaching τ in a non-tangential manner either

$$C_\gamma(U, \tau) \cap \{|w| = M\} \neq \emptyset, \quad \text{all} \quad \gamma \in \Omega_\tau^*;$$

or

$$C_\gamma(U, \tau) \cap \{|w| = M\} = \emptyset, \quad \text{all} \quad \gamma \in \Omega_\tau^*.$$

Clearly the Theorem also remains true if we require, instead of $U(z) \leqq M$, $z \in D$, that for any $0 < \beta < \pi/2$ there is a constant M, independent of β, such that

$$\overline{\lim}\, U(z) \leqq M\,, \quad z \to \tau, \quad z \in S(\tau, \beta)\,.$$

3. The condition of normality is obviously necessary in Theorem 1. Select any holomorphic function $f(z)$ in D which tends to, say 0, along some rectilinear segment terminating at a point of C, but which does not have angular limit 0 at this point. The $U(z) = |f(z)|$ is the desired Gegenbeispiel. (There are many such examples of holomorphic functions of the above type. One interesting example is given in [*2*, pp. 287–288].)

Before investigating the requirement that $\log U(z)$ be subharmonic we contrast the two definitions of a normal function. To this end we recall a criterion that a continuous function $f(z)$ from D into the extended plane be normal in the sense of Lehto and Virtanen. This criterion was noted by Lappan in [*4*], and states that $f(z)$ is normal on D if and only if given any two sequences $\{z_n\}$, $\{z'_n\}$, in D such that $\rho(z_n, z'_n)\to 0$ as $n\to\infty$ then $\chi(f(z_n), f(z_n))\to 0$ as $n\to\infty$. Here $\chi(w_1, w_2)$ is the chordal distance between w_1 and w_2 in the extended plane.

Of course a function $f(z)$ in D, which is normal according to Definition 1, is also normal in the sense of LEHTO and VIRTANEN while the converse statement is not true. While this is fairly obvious we construct such a function which also serves to investigate the necessity of the subharmonic condition in Theorem 1.

For $z=x+iy$, $x>0$, define

$$\arg^* z = \begin{cases} \arg z, & y\geqq 0, \quad 0\leqq \arg z<\pi/2; \\ -\arg z, & y<0, \quad -\pi/2<\arg z\leqq 0; \end{cases}$$

and for $z\in D$ set $U(z)=\arg^*(z+1)+(x+1)$. $U(z)$ is a positive, bounded, continuous function in D. Employing the criterion of LAPPAN let $\{z_n\}$ and $\{z'_n\}$ be two sequences in D with

$$\rho(z_n, z'_n)\to 0, \qquad n\to\infty. \tag{3.0}$$

Suppose $|U(z_n)-U(z'_n)|$ does not tend to zero as $n\to\infty$. Without loss of generality we assume, for $\varepsilon>0$,

$$|U(z_n)-U(z'_n)|\geqq\varepsilon, \qquad \text{all } n. \tag{3.1}$$

Let a subsequence $\{z_{n_k}\}$ be chosen so that $z_{n_k}\to b_0$, $k\to\infty$. Necessarily $z'_{n_k}\to b_0$ as $k\to\infty$. We consider several cases.

Case I. $b_0\in D\cup C$, $b_0\neq -1$. In this circumstance (3.1) is not compatible with the continuity of $U(z)$ at b_0.

Case II. $b_0=-1$. Setting $\vartheta_k=\arg^*(z_{n_k}+1)$, $\vartheta'_k=\arg^*(z'_{n_k}+1)$, we claim (3.0) implies $|\vartheta_k-\vartheta'_k|\to 0$ as $k\to\infty$, which would again show (3.1) is untenable. Indeed, if $|\vartheta_k-\vartheta'_k|$ did not tend to zero as $k\to\infty$, we could find subsequences $\{\vartheta_{k_i}\}$ and $\{\vartheta_{k'_i}\}$ with $\vartheta_{k_i}\to\alpha$, $\vartheta_{k'_i}\to\beta$, $0\leqq\alpha\leqq\pi/2$, $0\leqq\beta\leqq\pi/2$, $\alpha\neq\beta$. If say $\alpha=\pi/2$, the corresponding subsequence of $\{z_{n_k}\}$ approaches -1 in a tangential manner while its companion subsequence of $\{z'_{n_k}\}$ approaches -1 non-tangentially, and this clearly contradicts (3.0). The case $\beta=\pi/2$ is similar. For the remaining possibility, (3.0) is also not satisfied for the appropriate subsequences and in fact the non-Euclidean distance between corresponding terms is greater than

$$\tfrac{1}{4}\,|\log\cot(\pi/2-\beta/2)-\log\cot(\pi/2-\alpha/2)|$$

for all but a finite number of terms.

A definition of normality according to LEHTO and VIRTANEN implies $V(z)=(U(z))^{-1}$ is also normal whereas $V(z)$ is not normal according to Definition 1. Indeed if $V(z)$ were normal the family

$$\left\{V_n(\zeta)=V\left(\frac{\zeta-\frac{1}{n}}{1-\frac{\zeta}{n}}\right)\right\}$$

would contain a uniformly convergent subsequence which would converge to the identically infinite function on account of $V_n(0) \to \infty$ as $n \to \infty$. But each compact subset of D contains points for which $V_n(\zeta)$ is uniformly bounded for all n.

To return to the normal function $U(z)$ we observe that it is subharmonic in D. Hence $U(z)$ is a normal, positive, subharmonic (and even bounded), function which tends to 0 as $z \to -1$ along the real axis yet does not have angular limit 0 at -1. Thus Theorem 1 is not valid if the condition that $\log U(z)$ be subharmonic is omitted and is replaced by the weaker condition that $U(z)$ itself be subharmonic.

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