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Multiply connected domains of normality in iteration theory

By

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1. Introduction

The theory of the iteration of a rational or entire function $f(z)$ of the complex variable z treats the sequence of "iterates" $\{f_n(z)\}$ defined by

$$f_0(z) = z, \qquad f_1(z) = f(z), \qquad f_{n+1}(z) = f_1(f_n(z)), \qquad n = 0, 1, 2, \ldots.$$

In the theory developed by FATOU [1, 2] and JULIA [3], a fundamental role is played by the set $\mathfrak{F}(f)$ of those points of the complex plane where $\{f_n(z)\}$ is not normal in the sense of Montel. $\mathfrak{F}(f)$ is a non-empty perfect set whose complement $\mathfrak{C}(f)$ consists of an at most countably infinite set of components G_i, each of which is a maximal domain where $\{f_n(z)\}$ is normal.

The determination of $\mathfrak{F}(f)$ and the G_i corresponding to a given $f(z)$ is a problem of some difficulty, especially in the case where $f(z)$ is entire and transcendental. So it is that despite the need, urged by FATOU [2], to establish by numerous examples the various ways in which $\mathfrak{F}(f)$ can divide the plane, rather few such examples have been worked out. It is remarkable that in the two cases of transcendental functions given by FATOU [2], as also in the cases $\sin z$ and $\cos z$ investigated by TÖPFER [4], all the domains G_i are simply connected. So far as I know, no example has yet been given of an entire transcendental function with some of its domains G_i multiply connected. The object of this paper is to provide such an example:

(A). *The function* $g(z)$ *of* (1) (see p. 207) *has at least one multiply connected domain* G i.e. *the complement* $\mathfrak{C}(g)$ *of* $\mathfrak{F}(g)$ *has at least one multiply connected component.*

A set A in the complex plane is *completely invariant* with respect to the iteration of $f(z)$, if $\alpha \in A$ implies (i) $f(\alpha) \in A$ and (ii) $\beta \in A$ whenever $f(\beta) = \alpha$; in other words if $f(A) = A = f_{-1}(A)$. It is well known ([1, 2, 3]) that $\mathfrak{F}(f)$ and its complement are completely invariant in this sense.

From now on $f(z)$ shall always denote a non-linear entire function. Töpfer's paper [4] contains the following remarks about the components G_i of $\mathfrak{C}(f)$:

(B). *If* G_1 *is multiply connected, then* $\lim_{n\to\infty} f_n(z) = \infty$ *in* G_1.

(C). *If* G_1 *is unbounded, then all* G_i *other than* G_1 *are simply connected; if in addition* G_1 *is multiply connected, then it is completely invariant.*

If $f(z)$ is a non-linear polynomial, there is clearly a neighbourhood of ∞ in which $f_n(z) \to \infty$ uniformly as $n \to \infty$, and this is contained in an unbounded

G_1. G_1 is multiply connected in the plane, although not necessarily on the sphere as the example $f(z)=z^2$, $G_1=\{z/|z|>1\}$ shows. It is not difficult to provide examples of multiple connectivity on the sphere (see Section 3).

If $f(z)$ is transcendental it seems possible that G_1 should be multiply connected and all G_i, $i=1, 2, \ldots$ bounded. Suppose this is the case. If at a point of G_1 we fix a branch of $f_{-1}(z)$ and continue this branch throughout G_1, it follows from the complete invariance of $\mathfrak{F}(f)$ and $\mathfrak{C}(f)$ that for the chosen branch and its continuations within G_1:

$$f_{-1}(G_1) < G_p$$

for a single component G_p. Moreover G_1 contains at most algebraic singularities of $f_{-1}(z)$, since if $z=z_0$ were a non-algebraic singularity one would have $f_{-1}(z)\to\infty$ as $z\to z_0$ in G_1, with $f_{-1}(z)\in G_p$, which would make G_p unbounded. Also $f(G_p)$ lies in a single component G, which must be G_1 and we have

$$f(G_p)=G_1.$$

If w is a boundary point of G_p, there is a sequence $\{z_n\}$, $n=1, 2, \ldots$ such that $z_n\in G_p$, $z_n\to w$ as $n\to\infty$. Then $f(z_n)\in G_1$ and $f(z_n)\to f(w)$, so that $f(w)\in\overline{G}_1$. But $w\in\mathfrak{F}(f)$ and hence $f(w)\in\mathfrak{F}(f)$, so that $f(w)\notin G_1$ and in fact $f(w)$ lies in the boundary of G_1. Similarly, if t is a boundary point of G_1, there is a sequence $\{t_n\}$, $n=1, 2, \ldots$ such that $t_n\in G_1$, $t_n\to t$ as $n\to\infty$. Take u_n to be any of the values of $f_{-1}(t_n)$ which lies in G_p. Then there is a convergent subsequence of $\{u_n\}$ which we may assume to be $\{u_n\}$ itself, with limit (say) v in $\overline{G}_p$. Clearly $f(v)=t\in\mathfrak{F}(f)$ so that v is in $\mathfrak{F}(f)$ and hence in the boundary of G_p. To sum up: the boundary of G_1 is the image of the boundary of G_p under the continuous mapping $z\to f(z)$.

If G_p is simply connected, then since it is also bounded, its boundary is a continuum and the boundary of G_1 as the continuous image of a continuum is also a continuum. Then G_1 is simply connected, against our original assumption. Thus G_p is multiply connected. By taking the different determinations of $f_{-1}(z)$ at a point z_0 in G_1, such that $f_{-1}(z_0)$ has infinitely many determinations z_n with $|z_n|\to\infty$ as $n\to\infty$, we get for each determination a bounded multiply connected domain of the type G_p and altogether an infinite set of such domains.

Thus *if G_1 is a multiply connected component of $\mathfrak{C}(f)$, then either*

(D). G_1 is unbounded and completely invariant, and all other G_n are simply connected, or

(E). all G_n (including G_1) are bounded and there exist infinitely many multiply connected G_n.

2. Construction of $g(z)$ satisfying (A)

Lemma 1. *There is an entire function $g(z)$ given by the canonical product*

$$g(z)=Cz^2\cdot\prod_{n=1}^{\infty}\left(1+\frac{z}{r_n}\right), \qquad 1<r_1<r_2<\cdots, C>0, \tag{1}$$

which satisfies

(2) $$|g(e^{i\vartheta})| < \tfrac{1}{4}, \qquad 0 \leqq \vartheta \leqq 2\pi,$$

(3) $$r_{n+1} < g(r_n) < 2r_{n+1} \quad \textit{for all} \quad n = 1, 2, \ldots.$$

Proof. Choose r_1 and $C > 0$ so that

(4) $$C \exp\left(\frac{2}{r_1}\right) < \frac{1}{4}; \quad C r_1 > 1; \quad r_1 > 1; \quad \text{e.g.} \quad C = \frac{1}{4e}, \quad r_1 > 4e.$$

Define the sequence $\{r_n\}$ inductively by $r_2 = C r_1^2\left(1 + \frac{r_1}{r_1}\right) = 2C r_1^2$ and in general

(5) $$r_{n+1} = C r_n^2 \left(1 + \frac{r_n}{r_1}\right)\left(1 + \frac{r_n}{r_2}\right) \cdots \left(1 + \frac{r_n}{r_n}\right), \qquad n = 1, 2, \ldots.$$

Then $r_2 = 2Cr_1 \cdot r_1 > 2r_1$ by (4), and inductively: $r_{n+1} > 2r_n$, since from (5) $r_{n+1} \geqq 2Cr_n^2 > 2Cr_1 r_n > 2r_n$.

Thus $1 < r_1 < r_2 < \cdots$ holds and, moreover,

(6) $$r_n > 2^{n-1} r_1, \qquad n = 2, 3, \ldots; \qquad r_{n+k} > 2^k r_n, \qquad n = 1, 2, \ldots$$

so that (1) is an entire function.

Now by (1), (6), (4)

$$|g(e^{i\vartheta})| \leqq C \prod_{n=1}^{\infty} \left(1 + \frac{1}{r_n}\right) < C \prod_{n=1}^{\infty} (1 + 2^{1-n} r_1^{-1}) < C \exp\left(\frac{2}{r_1}\right) < \frac{1}{4},$$

which establishes (2).

Further:

$$r_{n+1} = C r_n^2 \prod_{k=1}^{n} \left(1 + \frac{r_n}{r_k}\right) < g(r_n) = r_{n+1} \cdot \prod_{k=n+1}^{\infty} \left(1 + \frac{r_n}{r_k}\right).$$

But from the second part of (6):

$$\prod_{n+1}^{\infty} \left(1 + \frac{r_n}{r_k}\right) < \prod_{k=1}^{\infty} \left(1 + \frac{1}{2^k}\right) = 2$$

and (3) is proved

Lemma 2. *If $g(z)$ is the function of Lemma 1, then*

(7) $$g(r_n^{\frac{1}{2}}) < r_{n+1}^{\frac{1}{2}}, \qquad n = 1, 2, \ldots \quad \textit{and}$$

(8) $$\tfrac{1}{4} g(r_n^2) > r_{n+1}^2, \qquad n = 1, 2, \ldots.$$

Proof. $g(r)$ is the maximum modulus of $g(z)$ for $|z| = r$. Applying Hadamard's convexity theorem to $V(s) = \log\big(g(e^s)\big)$ we obtain for $s > 0$

$$V(2s) - V(0) > 2\{V(s) - V(0)\}$$

or

$$V(2s) > 2V(s) - V(0),$$

so that

$$g(r^2) > \frac{g(r)^2}{g(1)} > 4g(r)^2. \tag{9}$$

Putting $r = r_n^{\frac{1}{2}}$ in (9) and using (3) gives

$$4g(r_n^{\frac{1}{2}})^2 < g(r_n) < 2r_{n+1},$$

which proves (7). Putting $r = r_n$ in (9) and using (3) gives

$$g(r_n^2) > 4g(r_n)^2 > 4r_{n+1}^2,$$

which proves (8).

Lemma 3. *If $g(z)$ is the function of Lemma 1, then*

$$g(r) < 4|g(-r)| \tag{10}$$

holds in the region

$$B_n\colon\ 4r_n < r < \tfrac{1}{4}r_{n+1} \tag{11}$$

for all large enough n.

Proof. We recall (c.f. (5)) that $\frac{r_{n+1}}{r_n} > Cr_n \to \infty$ as $n \to \infty$, so that B_n is non-empty for all large enough n. We note that

$$\log(1+x) < x \quad \text{for} \quad x > 0,$$

$$-\log(1-x) < 2x \quad \text{for} \quad 0 < x < \tfrac{1}{2},$$

so that

$$\log\left(\frac{1+x}{1-x}\right) < 3x \quad \text{for} \quad 0 < x < \frac{1}{2}. \tag{12}$$

Now

$$\log\left|\frac{g(r)}{g(-r)}\right| = \sum_{n=1}^{\infty} I_n = \sum_{k=1}^{n-1} I_k + I_n + I_{n+1} + \sum_{k=n+2}^{\infty} I_k \tag{13}$$

where

$$I_n = \log\left|\frac{1+\frac{r}{r_n}}{1-\frac{r}{r_n}}\right|. \tag{13'}$$

For r satisfying (11) and for $k \leqq n-1$, we have $0 < \frac{r_k}{r} < \frac{r_k}{4r_n} < \frac{1}{8}$, and by (13'), (12)

$$0 < I_k = \log\left|\frac{1+\frac{r_k}{r}}{1-\frac{r}{r_k}}\right| < 3\frac{r_k}{r} < \frac{3r_k}{4r_n}.$$

Hence

$$\left\{\begin{aligned} \sum_{k=1}^{n-1} I_k &< \frac{3}{4}\sum_{k=1}^{n-1}\frac{r_k}{r_n} = \frac{3}{4}\frac{r_{n-1}}{r_n}\left\{1 + \frac{r_{n-2}}{r_{n-1}} + \frac{r_{n-2}}{r_{n-1}}\cdot\frac{r_{n-3}}{r_{n-2}} + \cdots\right\} \\ &< \frac{3}{4}\frac{r_{n-1}}{r_n}\left\{1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right\} < \frac{3}{2}\frac{r_{n-1}}{r_n}. \end{aligned}\right. \tag{14}$$

For r satisfying (11) and for $k \geqq n+2$ we have $0 < \frac{r}{r_k} < \frac{r_{n+1}}{4 r_{n+2}} < \frac{1}{8}$, and by (13'), (12)

$$0 < I_k = \log \left(\frac{1 + \frac{r}{r_k}}{1 - \frac{r}{r_k}} \right) < \frac{3r}{r_k} < \frac{3}{4} \frac{r_{n+1}}{r_k}.$$

Hence

$$\sum_{k=n+2}^{\infty} I_k < \frac{3}{4} \sum_{k=n+2}^{\infty} \frac{r_{n+1}}{r_k} < \frac{3}{4} \sum_{k+n=2}^{\infty} \frac{r_{n+1}}{r_{n+2}} \cdot 2^{n+2-k} = \frac{3}{2} \frac{r_{n+1}}{r_{n+2}}. \tag{15}$$

From (13), (14) and (15) it follows that for r satisfying (11)

$$\left\{ \begin{aligned} \log \left| \frac{g(r)}{g(-r)} \right| &< \frac{3}{2} \frac{r_{n-1}}{r_n} + \frac{3}{2} \frac{r_{n+1}}{r_{n+2}} + \log \left(\frac{1 + \frac{r_n}{r}}{1 - \frac{r_n}{r}} \right) + \log \left(\frac{1 + \frac{r}{r_{n+1}}}{1 - \frac{r}{r_{n+1}}} \right) \\ &< \frac{3}{4} \frac{r_{n-1}}{r_n} + \frac{3}{2} \frac{r_{n+1}}{r_{n+2}} + 2 \log \frac{5}{3}; \end{aligned} \right. \tag{16}$$

but as remarked at the beginning of the proof of this lemma, $\frac{r_{n+1}}{r_n} \to \infty$ as $n \to \infty$, so that for all large enough n the right hand side of (16) is less than $\log 4$ and (10) holds.

Theorem 1. *(Proof of statement (A) of the introduction). If $g(z)$ is the function of Lemma 1 and A_n is the annulus*

$$A_n\colon\ r_n^2 < |z| = r < r_{n+1}^{\frac{1}{2}}, \tag{17}$$

then there is an integer $N > 0$, such that for all $n > N$ the mapping $z \to g(z)$ maps A_n into A_{n+1} and $g_n(z) \to \infty$ uniformly in A_n. For each $n > N$, A_n belongs to a multiply connected component of $\mathfrak{C}(g)$.

Proof. We note that (by (5)), for any fixed m, $r_{n+1}/r_n^m \to \infty$ as $n \to \infty$. Thus the annuli A_n are non-empty for sufficiently large n. If, moreover, $r_n > 4$ and $r_{n+1} > 16$, then the annulus A_n of (17) lies in the annulus B_n of (11). For all sufficiently large n, say for $n > N$ and $z \in A_n$ we have from (7)

$$|g(z)| \leqq g(|z|) < g(r_{n+1}^{\frac{1}{2}}) < r_{n+2}^{\frac{1}{2}}, \tag{18}$$

while by (8) and, since $A_n \subset B_n$, by (10) we have

$$|g(z)| \geqq g(-|z|) > \tfrac{1}{4} g(|z|) > \tfrac{1}{4} g(r_n^2) > r_{n+1}^2. \tag{19}$$

Together (18) and (19) show that A_n is mapped into A_{n+1} by $g(z)$. Then A_n is mapped into A_{n+p} by $g_p(z)$ and, since the minimum distance of A_{n+p} from $z=0$ is r_{n+p}^2, which tends to infinity with p, we have $\lim_{p \to \infty} g(z) = \infty$ uniformly in A_n.

In the unit circle $|z| \leqq 1$ one has $|g(z)| < \frac{1}{4}|z|$ by (2) and Schwarz' lemma; hence by iteration $|g_n(z)| < 4^{-n}|z|$ and $\lim_{n \to \infty} g_n(z) = 0$ uniformly in the unit circle, which belongs to some domain of normality G_1 of $\{g_n(z)\}$. Similarly A_n belongs to some domain of normality G_n which must be different from G_1 and thus contains no point of G_1. Thus G_n is multiply connected.

The proof of the theorem is now complete. One might ask whether the G_n are different or not; some discussion on this point is given in Section 4.

3. The construction applied to polynomials

Let k be an integer greater than the N of the theorem of Section 2. Let $P(z)$ be the partial product of (1) given by

$$P(z) = C z^2 \prod_{n=1}^{k} \left(1 + \frac{z}{r_n}\right). \tag{20}$$

For all z with $|z| < r_{k+1}$ we have

$$|g(-|z|)| \leqq |P(-|z|)| \leqq |P(z)| \leqq P(|z|) \leqq g(|z|),$$

whence it follows that for $n = N+1, N+2, \ldots, k$ the annulus A_n of (17) is mapped by $z \to P(z)$ into A_{n+1}. In particular A_k is mapped into a region where $|z| > r_{k+1}^2$. On the boundary circle $|z| = r_k^2$ of A_k one has

$$|P(z)| \geqq r_{k+1}^2$$

and

$$\left|\frac{P(z)}{z^2}\right| \geqq \left(\frac{r_{k+1}}{r_k}\right)^2 \geqq 4.$$

Since the zeros of $P(z)$ have modulus at most $r_k < r_k^2$, we can conclude that for all $|z| > r_k^2$, and in particular for $|z| > r_{k+1}^2$:

$$|P(z)| \geqq 4|z|^2 \geqq 4|z|$$

and

$$|P_n(z)| > 4^n |z|.$$

Thus as $n \to \infty$, $P_n(z) \to \infty$ uniformly in $|z| > r_{k+1}^2$ and also in A_n for $n = N+1, N+2, \ldots, k$.

Now $P(r)/r$ is an increasing function for $r \geqq 0$, so that there is a unique $R > 0$ for which $P(R) = R$ holds, while for any $r > R$ one has $P(r) > r$. Thus if $r > R$ the sequence $P_n(r)$ is increasing and divergent (since its convergence to s would imply $P(s) = s, s > R$). From a certain value of n onwards $P_n(r) > r_{k+1}^2$. Thus since $\{z \,|\, |z| > r_{k+1}^2\} = K$ is in $\mathfrak{C}(P)$, it follows from the complete invariance of $\mathfrak{C}(P)$ that the ray $r > R$ of the real axis belongs to $\mathfrak{C}(P)$, and indeed to the same component as K. Comparing (20) with (5) we see that

$$P(r_1) \geqq 2C r_1^2 > 2r_1$$

so that $R < r_1$. Then A_n, $n = N+1, N+2, \ldots, k$, must all belong to the same component G_1 of $\mathfrak{C}(P)$ as K, being connected by the ray $r > R$. As in the case of $g(z)$ we have $|P(e^{i\vartheta})| < \frac{1}{4}$ and the unit circle belongs to a region of normality $G_0 \neq G_1$. Each of the zeros $-r_n$ of $P(z)$ is contained in a region of normality G'_n where $\lim_{n\to\infty} P_n(z)=0$, i.e. in a region other than G_1. Thus G_1 is multiply connected, in fact at least $(k-N)$-fold connected; for the regions $G'_{N+2}, G'_{N+3}, \ldots, G'_k, G_0$ are all different. The region G_1 is completely invariant and it follows from results of [1] that its boundary is the whole set $\mathfrak{F}(P)$.

We conclude this section by remarking that the connectivity of G_1 is infinite. Suppose this is not the case: then the boundary of G_1 consists of a finite number of disjoint components, each compact and connected. Let $d>0$ be the minimal distance between different components. Since $\mathfrak{F}(P)$ is a perfect set, each of its finite number of components contains an infinity of points. Let C_1 be any such boundary component. Now it is shown in [1] that for any $s\in\mathfrak{F}(P)$, any disc $D\colon |z-s|<\varrho$, $\varrho>0$, and any bounded set E of the plane not meeting neighbourhoods of two possibly exceptional points which depend on $P(z)$, there is an n_0 such that $P_n(D)$ contains E for $n>n_0$. We take E to be C_1 with neighbourhoods N_1, N_2 of the two exceptional points subtracted if necessary to give $E=C_1-(N_1\cup N_2)\neq\emptyset$. We take the radius ϱ of D so small that $\varrho<d$. Then for $n>n_0$, $P_n(D)$ meets C_1 so that D contains points of $P_{-n}(C_1)$, and these, belonging to the boundary $\mathfrak{F}(P)$, must belong to the same boundary component C_2 as s. Thus $P_n(C_2)$ meets C_1 for $n>n_0$, and since $P_n(C_2)$ is a connected subset of $\mathfrak{F}(P)$ we have $P_n(C_2)$ contained in C_1 for every $n>n_0$. Now, since s is arbitrary and G_1 is multiply connected we may assume $C_1\neq C_2$. A result of [1] or [3] states that every $s\in\mathfrak{F}(P)$ is a point of accumulation of fixpoints in $\mathfrak{F}(P)$, i.e. points $z\in\mathfrak{F}(P)$ such that $P_m(z)=z$ for some integer m. But this implies that $P_m(C_2)=C_2$ for some arbitrarily large m, which contradicts $P_n(C_2)=C_1$, $n>n_0$, $C_1\neq C_2$. Thus the connectivity of G_1 is not finite.

4. A difference between the transcendental and polynomial cases

It is interesting to note that, in contrast to the case of $P(z)$ in (20), it is not true for the $g(z)$ of (1) that the annuli A_n (17) are connected by a segment of the real positive axis belonging to $\mathfrak{C}(g)$.

Theorem 2. *There is a unique $R>0$ such that $g(R)=R$, and for $r>R$ we have $g(r)>r$. There is also $R'>0$ such that $|g(re^{i\frac{\pi}{2}})|>2r$ for $r>R'$. Then for any $r_1>\operatorname{Max}(R,R')$, the interval $[r_1, g(r_1)]$ contains a point of $\mathfrak{F}(g)$.*

Proof. The function $\frac{1}{r}\left|g\left(re^{i\frac{\pi}{2}}\right)\right|$ increases monotonely from 0 to ∞ as r increases from 0 to ∞. This establishes the existence of R'. Similarly for R.

Define the function $\varphi(\vartheta)$ to be

$$\varphi(r,\vartheta)=\arg g(re^{i\vartheta})=2\vartheta+\sum_{n=1}^{\infty}\arg\left(1+\frac{re^{i\vartheta}}{r_n}\right)>2\vartheta$$

with all the arg functions normalized to zero at $\vartheta=0$. For fixed r, $\varphi(r,\vartheta)$ is monotone increasing in $0\leqq\vartheta\leqq\frac{\pi}{2}$, while for fixed ϑ in $0\leqq\vartheta\leqq\frac{\pi}{2}$, $\varphi(r,\vartheta)$

increases steadily to ∞ as r increases to ∞. For fixed $\alpha > 0$ put $\vartheta(r) = \vartheta(\alpha, r)$ equal to the smallest positive solution ϑ of $\varphi(r, \vartheta) = \alpha$. Then $\vartheta(r)$ is defined for all sufficiently large r and decreases steadily to 0 as $r \to \infty$.

We note also that $|g(re^{i\vartheta(r)})| \to \infty$ monotonely as $r \to \infty$, since if $r_1 < r_2$, $\vartheta < \frac{\pi}{2}$, then

$$|g(r_1 e^{i\vartheta(r_1)})| < |g(r_2 e^{i\vartheta(r_1)})| < |g(r_2 e^{i\vartheta(r_2)})|.$$

We make two applications of the function $\vartheta(\alpha, r)$:

(i) $\alpha = \pi$. As $r \to \infty$, $g(re^{i\vartheta(r)})$ runs to ∞ along the negative real axis and runs through the values $-r_n$ for all large enough n. Since $\vartheta(r) \to 0$ as $r \to \infty$, we conclude that for any $\varepsilon > 0$ the angle:

$$\{0 \leqq \arg z < \varepsilon,\ 0 \leqq r < \infty\},$$

which we denote by A_ε, contains the curve $z = re^{i\vartheta(r)}$ for all sufficiently large r; hence A_ε contains points w_n (with $|w_n| \to \infty$ as $n \to \infty$) such that $g(w_n) = -r_n$, $g_m(w_n) = 0$ for all $m \geqq 2$.

(ii) We now suppose that for some $r_1 > \mathrm{Max}(R, R')$, the interval $[r_1, g(r_1)]$ belongs to $\mathfrak{C}(g)$. Since $\mathfrak{C}(g)$ is open, it must contain the set

$$W\{r_1, g(r_1), \varepsilon\} = \{z \mid r_1 \leqq |z| \leqq g(r_1),\ 0 \leqq \arg z < \varepsilon\},$$

where $\varepsilon > 0$. Take $\alpha = \varepsilon < \frac{\pi}{2}$ in defining $\vartheta(r) = \vartheta(\varepsilon, r)$. Now $\varphi(r, \vartheta) > 2\vartheta$, so that $\vartheta(\varepsilon, r) < \frac{1}{2}\varepsilon$. Thus for $r_1 < r < g(r_1)$ the segment S_r: $0 \leqq \arg z < \vartheta(r)$ of the circle $|z| = r$ belongs to $W\{r, g(r_1), \varepsilon\}$. Further, since $|g(re^{i\vartheta})|$ is a decreasing function of ϑ, we see that $g(S_{r_1}) \subset W\{r_1, g(r_1), \varepsilon\}$. For $r_1 \leqq r \leqq g(r_1)$ $g(S_r)$ is a simple arc, whose minimum distance from the origin occurs at the upper end-point; for $r = g(r_1)$ the minimum occurs at

$$g\{g(r_1) \exp(i\vartheta(g(r_1)))\}$$

and has a value $> 2g(r_1)$ since $\vartheta(g(r_1)) < \frac{\pi}{2}$. As r increases from r_1 to $g(r_1)$ the arc $g(S_r)$ sweeps out a region which includes

$$W\{g(r_1), 2g(r_1), \varepsilon\},$$

and since S_r is contained in $\mathfrak{C}(g)$ the same is true of the region swept out by $g(S_r)$. Thus $\mathfrak{C}(g)$ includes

$$W\{r_1, 2g(r_1), \varepsilon\}.$$

By an inductive repetition of the above argument we find that $\mathfrak{C}(g)$ includes the whole angle A_ε. Therefore the angle A_ε combines with the A_n of the Theorem 1 to form part of the multiply connected completely invariant domain of normality G in which $\lim_{n \to \infty} g_n(z) = \infty$ holds. But by (i) A_ε contains the points w_m at which $\lim_{n \to \infty} g_n(w_m) = 0$. This contradiction shows in fact the interval $[r_1, g(r_1)]$ must contain points of $\mathfrak{F}(g)$.

The proof of Theorem 2 is now complete. It shows that if the domains A_n of Theorem 1 do belong to a single multiply connected domain G, then this domain is connected in a more complicated way than in the polynomial case. One might conjecture that the A_n belong to different components G_n so that alternative (E) of the introduction applies. However, I have not been able to prove this.

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